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Image restoration and classification by topological asymptotic expansion.

Mathematics and Image Analysis 2006,

MIA'06, Paris, France

September 18-21, 2006

Talk overview

1. Introduction to topological asymptotic expansion
2. Image restoration
   (a) Linear approach
   (b) Nonlinear variational approach
   (c) Topological gradient approach
3. Regularized image classification
   (a) Topological gradient applied to the classical approach
   (b) Restoration-based preprocessing method
4. A few more numerical results
5. Conclusion

Introduction

Image restoration problem : how to de-noise the image ? In order to prevent the image becoming blurred, we must identify its contours in order to preserve them.

To find an optimal shape (the set of contours) is equivalent to find its characteristic function ($0-1$ optimization problem).

The goal of image classification is to find a partition of the image in subsets with a constant color level in each subset.

Main problem in both cases : the non differentiability.

Asymptotic Analysis and Topological Gradient

Topological gradient

General formulation : Let Ω be a bounded open set, and let us consider a PDE on this set. We denote by u_Ω the solution of this PDE. We also consider a cost function $J(u_\Omega)$:

$$\Omega \longrightarrow u_\Omega \longrightarrow j(\Omega) := J(u_\Omega).$$

The topological asymptotic measures the impact of creating a hole around a point x in the domain Ω on the cost function :

$$j(\Omega \backslash \varepsilon B_x) - j(\Omega) = f(\varepsilon) g(x) + o(f(\varepsilon))$$

where $f(\varepsilon) > 0$ and $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$.

g is called the topological gradient, and one should create holes where the topological gradient is negative.

Topological gradient computation

We consider the variational formulation of the PDE :

$$a_\varepsilon(u_\varepsilon, w) = l_\varepsilon(w), \ \forall w \in V.$$

We assume that V is a Hilbert space, a_ε and l_ε are continuous functions, and that a_ε is coercive. **Hypothesis :**

$$\begin{cases} J_\varepsilon(u_\varepsilon) - J_\varepsilon(u_0) = L_\varepsilon(u_\varepsilon - u_0) + f(\varepsilon)\delta J_1 + o(f(\varepsilon)), \\ J_\varepsilon(u_0) - J_0(u_0) = f(\varepsilon)\delta J_2 + o(f(\varepsilon)), \\ (a_\varepsilon - a_0)(u_0, v_\varepsilon) = f(\varepsilon)\delta a + o(f(\varepsilon)), \\ (l_\varepsilon - l_0)(v_\varepsilon) = f(\varepsilon)\delta l + o(f(\varepsilon)), \end{cases}$$

where v_ε is solution of the adjoint equation :

$$a_\varepsilon(w, v_\varepsilon) = -L_\varepsilon(w), \forall w \in V.$$

Then, the topological gradient is given by $\delta a - \delta l + \delta J_1 + \delta J_2$.

IMAGE RESTORATION

Image restoration

Problem : how to restore a noised image ?

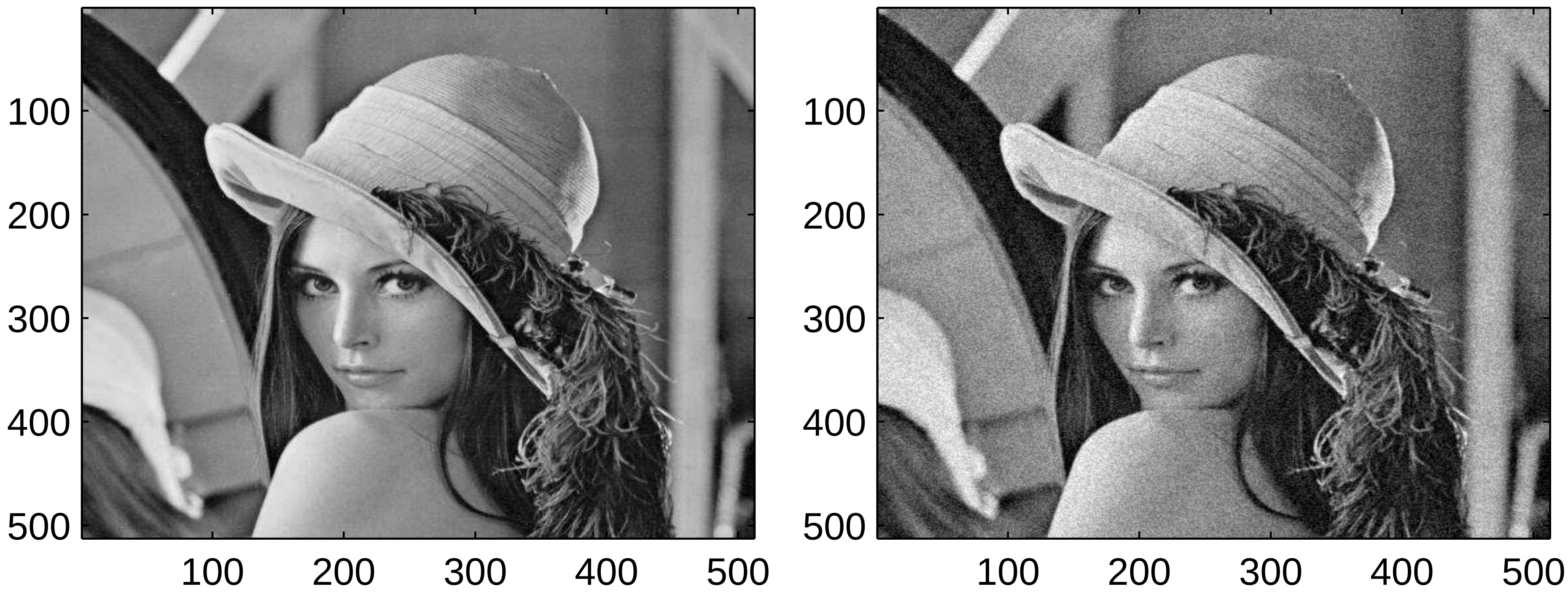


FIG. 1 – Original image (left) and noised image (right).

Linear approach

We consider the following operator

$$K : H^1(\Omega) \rightarrow L^2(\Omega), \quad u \mapsto u$$

and we have to solve the following problem

$$Ku = v.$$

A necessary optimality condition is

$$K^*Ku = K^*v,$$

which is an ill-posed problem. The regularization of Tikhonov gives

$$K^*Ku + cu = K^*v.$$

The weak formulation is

$$\langle K^*Ku + cu, w\rangle_{H^1(\Omega)} = \langle K^*v, w\rangle_{H^1(\Omega)}, \ \forall w \in H^1(\Omega).$$

Linear approach

By definition of K, the previous equation is equivalent to the following one :

$$\langle u, w\rangle_{L^2(\Omega)} + \langle cu, w\rangle_{H^1(\Omega)} = \langle v, Kw\rangle_{L^2(\Omega)},\ \forall w \in H^1(\Omega).$$

The classical variational formulation of image restoration is then the following :

$\Omega \subset \mathbb{R}^2$, $v \in L^2(\Omega)$ is the noised image, and we have to find the solution $u \in H^1(\Omega)$ of

$$\begin{cases} -div(c\nabla u) + u = v & in\ \Omega, \\ \partial_n u = 0 & in\ \Gamma = \partial\Omega. \end{cases}$$

Nonlinear approach

$$E(u) = \frac{1}{2}\int_\Omega |v-u|^2 dx + \lambda \int_\Omega \psi(|\nabla u|)dx.$$

By choosing $\psi(|\nabla u|) = |\nabla u|^2$, we obtain the linear approach. If $E(u)$ has a minimum u, u must satisfy the Euler-Lagrange equation :

$$-\lambda\, div\left(\psi'(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) + u = v.$$

Hypothesis on ψ :

- $\psi'(0) = 0$ and $\lim_{t\to 0^+} \frac{\psi'(t)}{t} = \psi''(0) > 0$: isotropic regularization where the gradient is weak.
- $\lim_{t\to +\infty} \frac{\psi'(t)}{t} = \lim_{t\to +\infty} \psi''(t) = 0$ and $\lim_{t\to +\infty} \frac{\psi''(t)}{\psi'(t)/t} = 0$: anisotropic regularization where the gradient is strong.
- $\lim_{t\to +\infty} \psi(t) = +\infty$: well posed problem.

Topological gradient

We consider the classical restoration PDE :

$$\begin{cases} -div(c\nabla u) + u = v & in\ \Omega, \\ \partial_n u = 0 & in\ \Gamma = \partial\Omega. \end{cases}$$

Let $x_0 \in \Omega$ and $\varepsilon > 0$ *small*, we denote by $\Omega_\varepsilon = \Omega \backslash \bar{\sigma}_\varepsilon$ the perturbed domain by the insertion of a crack $\sigma_\varepsilon = x_0 + \varepsilon\sigma(n)$, where n is a unit vector normal to the crack.

The solution $u_\varepsilon \in H^1(\Omega_\varepsilon)$ of the perturbed problem satisfies

$$\begin{cases} -div(c\nabla u_\varepsilon) + u_\varepsilon = v & in\ \Omega_\varepsilon, \\ \partial_n u_\varepsilon = 0 & in\ \Gamma_\varepsilon = \partial\Omega_\varepsilon. \end{cases}$$

Topological gradient

Variational formulation :

$$a_\varepsilon(u_\varepsilon, w) = l_\varepsilon(w),\ \forall w \in H^1(\Omega_\varepsilon)$$

where $a_\varepsilon(u,w) = \int_{\Omega_\varepsilon} (c\nabla u \nabla w + uw)dx$ and $l_\varepsilon(w) = \int_{\Omega_\varepsilon} vw\ dx.$

To find the contours of the image is equivalent to find a subset of Ω in which the energy is weak $\Rightarrow$ minimize the energy out of the contours :

$$j(\varepsilon) = J_\varepsilon(u_\varepsilon) = \int_{\Omega_\varepsilon} \|\nabla u_\varepsilon\|^2.$$

Topological gradient

We have then the following asymptotic expansion :

$$j(\varepsilon) - j(0) = \varepsilon^2 G(x_0, n) + o(\varepsilon^2)$$

$$G(x_0, n) = -\pi(\nabla u_0(x_0).n)(\nabla v_0(x_0).n) - \pi|\nabla u_0(x_0).n|^2$$

where v_0 is solution of the adjoint problem

$$\begin{cases} -div(c\nabla v_0) + v_0 = -\partial_u J(u) & in\ \Omega, \\ \partial_n v_0 = 0 & in\ \Gamma = \partial\Omega. \end{cases}$$

Topological gradient

The topological gradient can be written as

$$G(x,n) = \langle M(x)n, n\rangle$$

where $M(x)$ is the symmetric matrix defined by

$$M(x) = -\pi \frac{\nabla u_0(x)\nabla v_0(x)^T + \nabla v_0(x)\nabla u_0(x)^T}{2} - \pi \nabla u_0(x)\nabla u_0(x)^T.$$

For a given x, $G(x,n)$ takes its minimal value when n is the eigenvector associated to the lowest eigenvalue λ_{min} of M. This value will be considered as the topological gradient associated to the optimal orientation of the crack $\sigma_\varepsilon(n)$.

Topological gradient

Algorithm :

- Initialization : $c = c_0$.
- Computation of u_0 and v_0, solutions of the direct and adjoint problems.
- Computation of the 2×2 matrix M and its lowest eigenvalue λ_{min} at each point of the domain.
- $$c_1 = \begin{cases} \varepsilon_c & if \quad x \in \Omega \quad such \quad that \quad \lambda_{min} < \alpha < 0, \quad \varepsilon_c > 0 \\ c_0 & elsewhere. \end{cases}$$
- Calculation of u_1 solution to the perturbed problem with $c = c_1$.

Image restoration

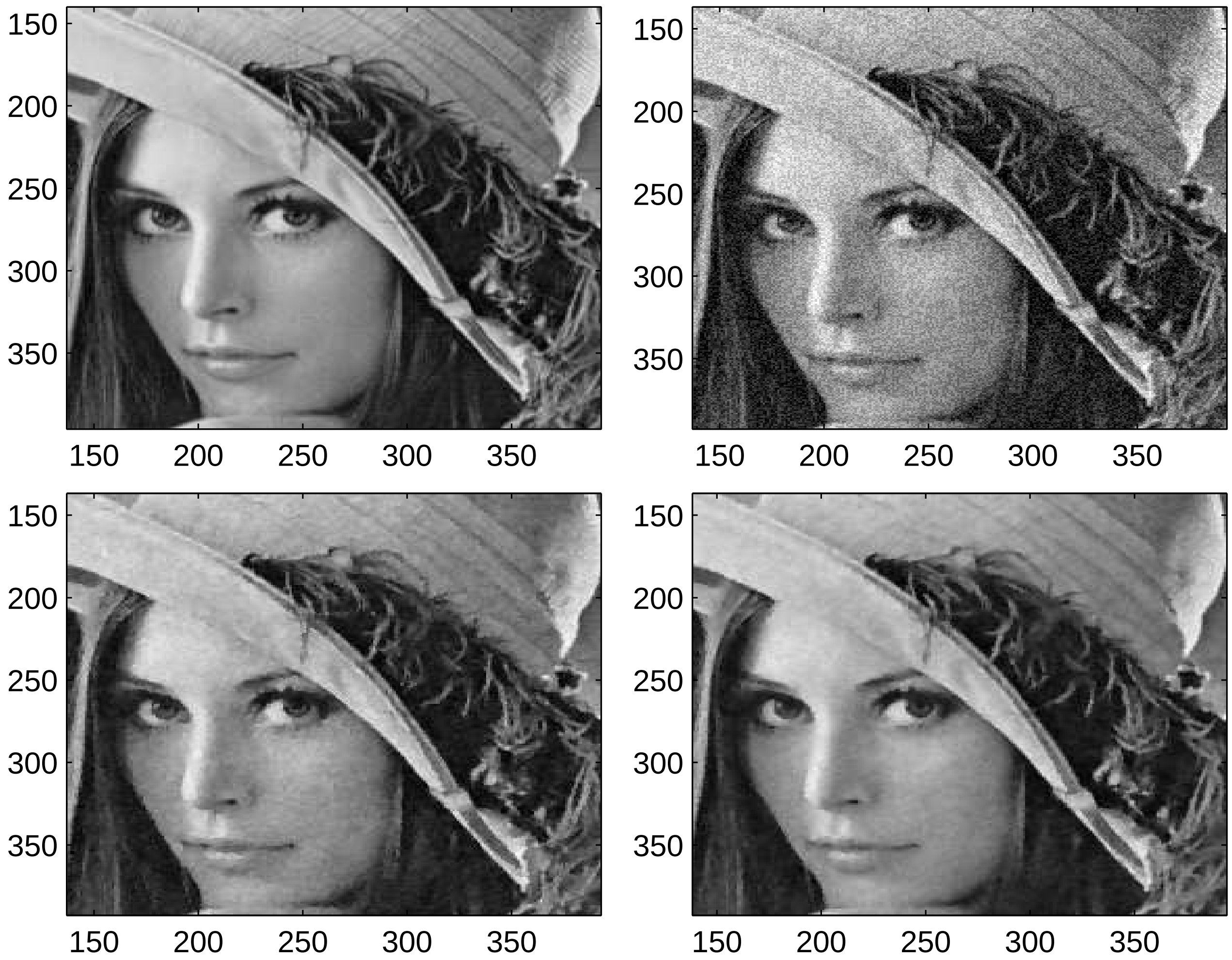


FIG. 2 – Top : Original image (left) and noised image (SNR=17) (right) ; Bottom : restored image by nonlinear diffusion (SNR=27) (left) and restored image by topological gradient (SNR=29) (right).

Color images

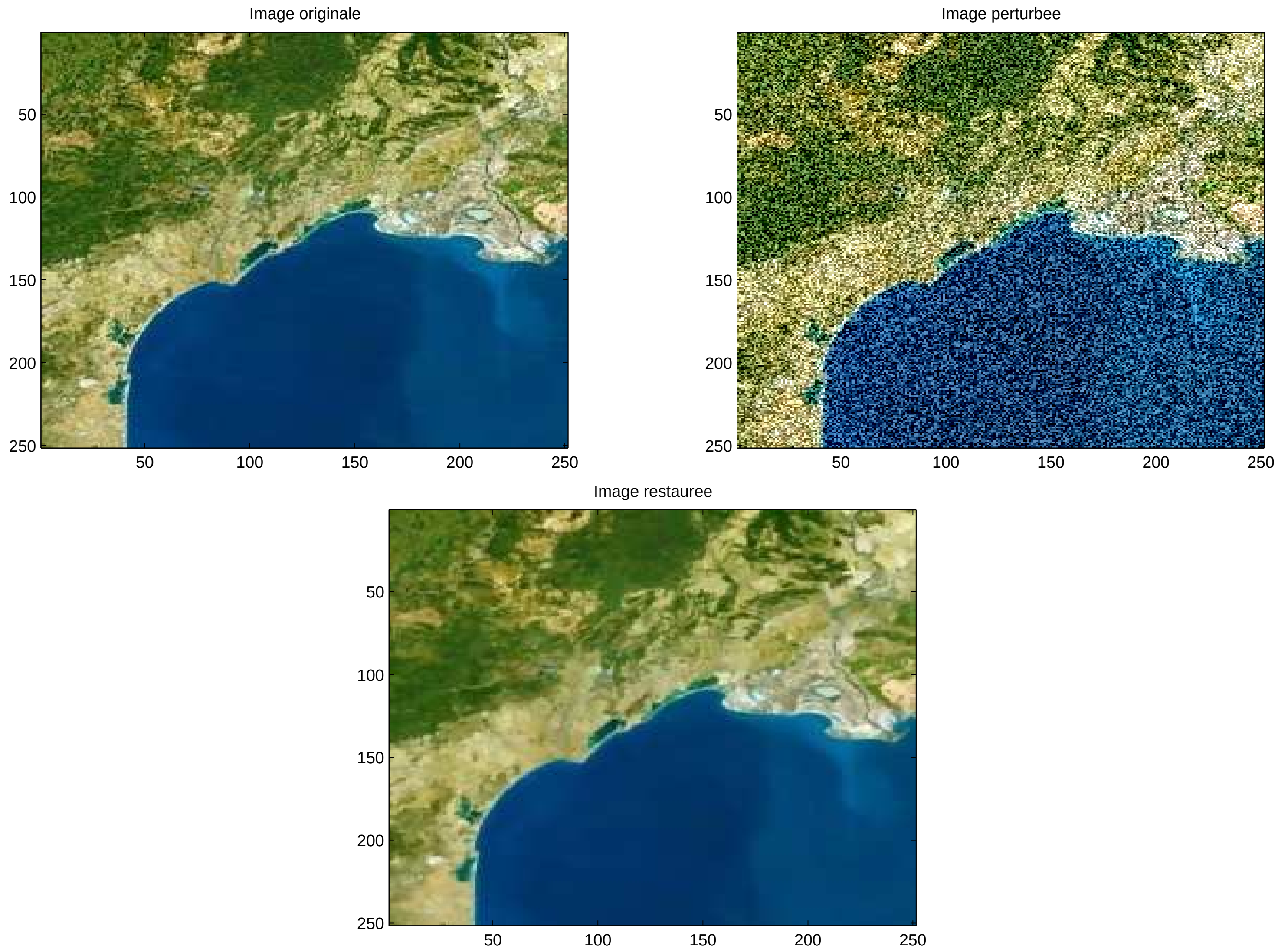


FIG. 3 – Top : Original image (left) and noised image (SNR=10) (right) ; Bottom : restored image by topological gradient (SNR=23).

Color images

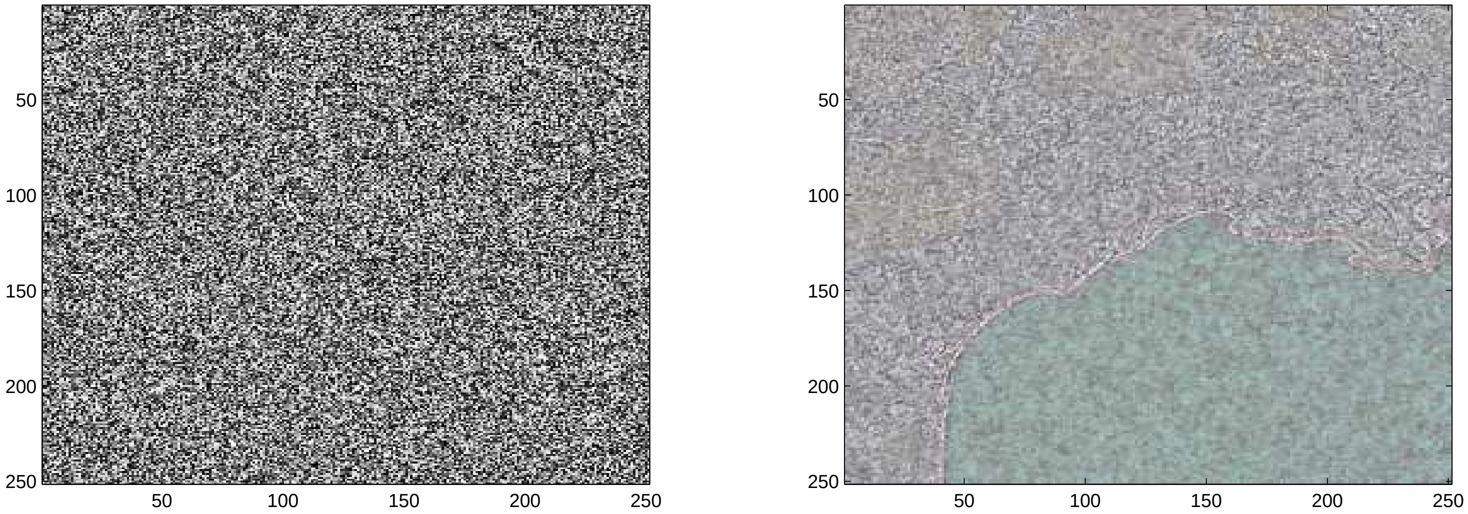


FIG. 4 – Difference between the original and noised images (left) ; Error on the restored image by topological gradient (right).

DCT algorithm - c constant

Let us consider the Fourier basis

$$\phi_{m,n} = \delta_{m,n} \cos(m\pi x) \cos(n\pi y)$$

where $\delta_{m,n}$ are normalisation coefficients.
If c is constant,

$$-c\,\Delta u + u = v$$

is equivalent to

$$\sum_{m,n} \left(1 + c(m\pi)^2 + c(n\pi)^2\right) u_{m,n}\phi_{m,n} = \sum_{m,n} v_{m,n}\phi_{m,n}.$$

DCT algorithm

Dicrete Cosine Transform algorithm :

– calculate $v_{m,n}$ the DCT of v,
– the DCT of u is then

$$u_{m,n} = \frac{v_{m,n}}{1 + c(m\pi)^2 + c(n\pi)^2}$$

– $u = \sum_{m,n} u_{m,n} \phi_{m,n}$ (inverse DCT).

Complexity : $\mathcal{O}(n \log(n))$ operations.

DCT preconditioner

When c is close to a constant :
we have to solve the linear system

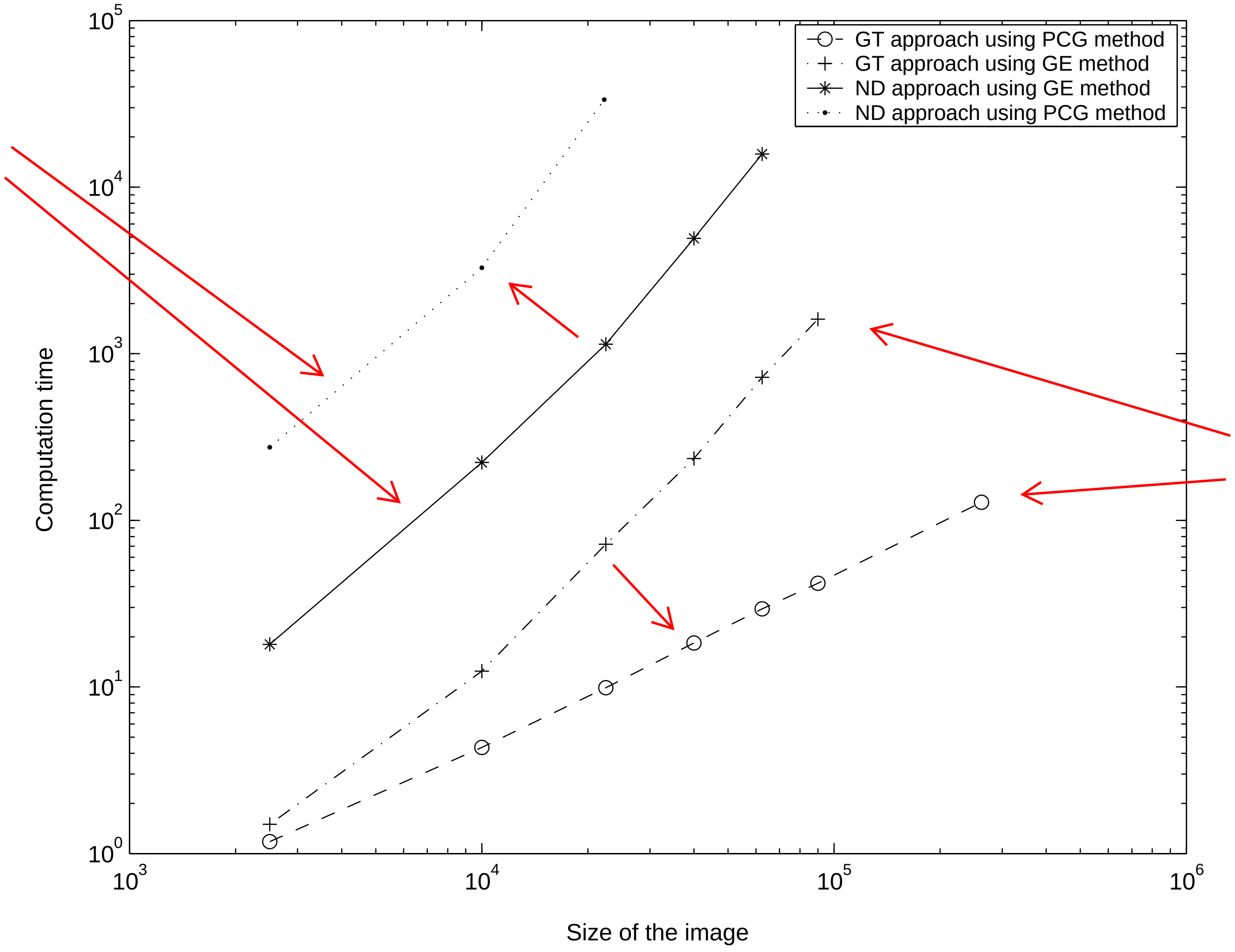
$$A(c)u = B.$$

We use the Preconditioned Conjugate Gradient method :

$$A(c_0)^{-1}A(c)u = A(c_0)^{-1}B.$$

Image restoration

Nonlinear diffusion



Topological gradient

FIG. 5 – Computation time versus the size of the image for the topological gradient (GT) and nonlinear diffusion (ND) approaches. GE : Gauss elimination method, PCG : preconditioned conjugate gradient using a discrete cosine transform.

Image Classification

Image classification

Data : $\Omega =]0,1[\times]0,1[$, w an image, $(\mu_i)_{i=1..K}$ color (or grey level) classes.

Problem : find a partition of Ω in subsets $(\Omega_i)_{i=1..K}$ such that :

– w is close to μ_i in Ω_i,

– the length of interfaces between the different subsets Ω_i is minimum.

We first assume that the number and values of classes are known.

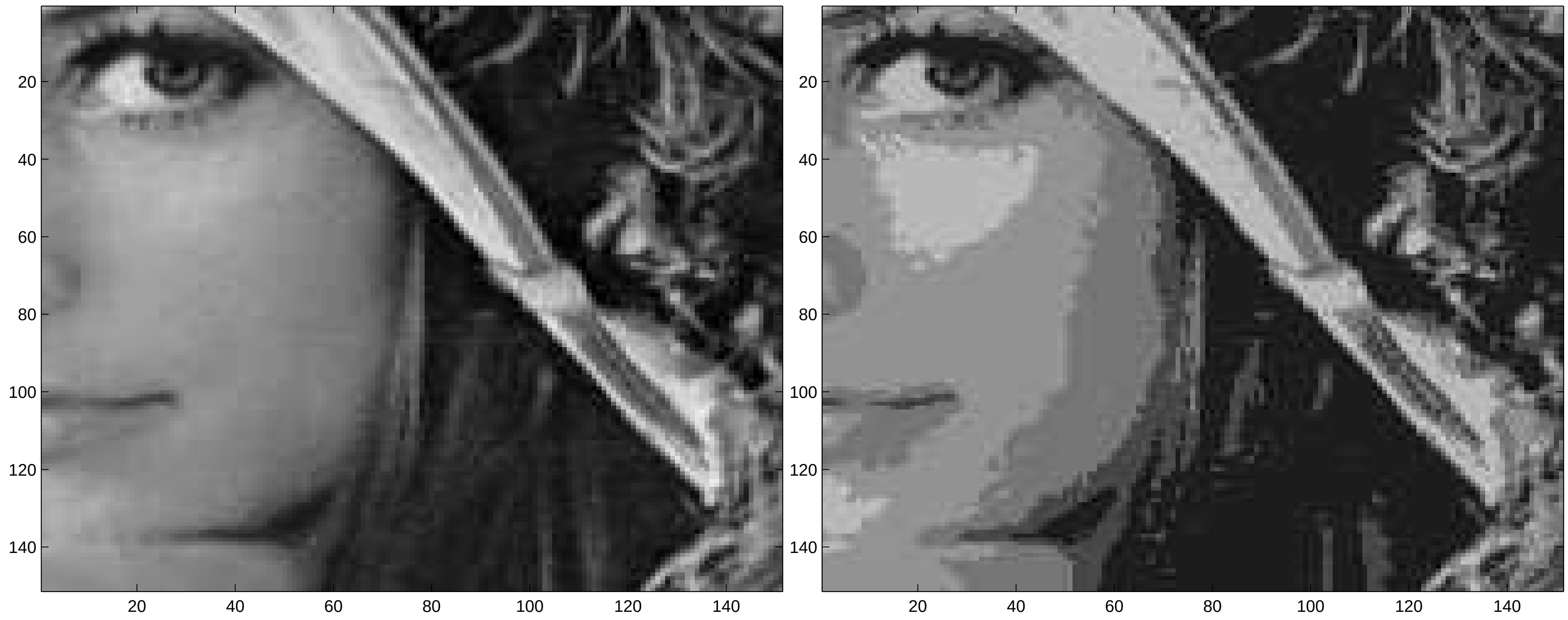


FIG. 6 – Original image (left) and classified image (without regularization) (right).

Image classification

We have to minimize with respect to Ω_i

$$J_1 = \sum_{i=1}^{K} \int_{\Omega_i} (w - \mu_i)^2 dx$$

and

$$J_2 = \sum_{i \neq j} |\Gamma_{ij}|.$$

The main difficulty comes from the fact that the unknowns are sets and not variables $\Longrightarrow$ define a topological gradient for each class.

For each pixel, the most negative topological gradient gives the subset and the class to which it should be reassigned in order to minimize the cost function.

Remark : in the present case, the asymptotic expansion is in fact an exact variation.

Image classification

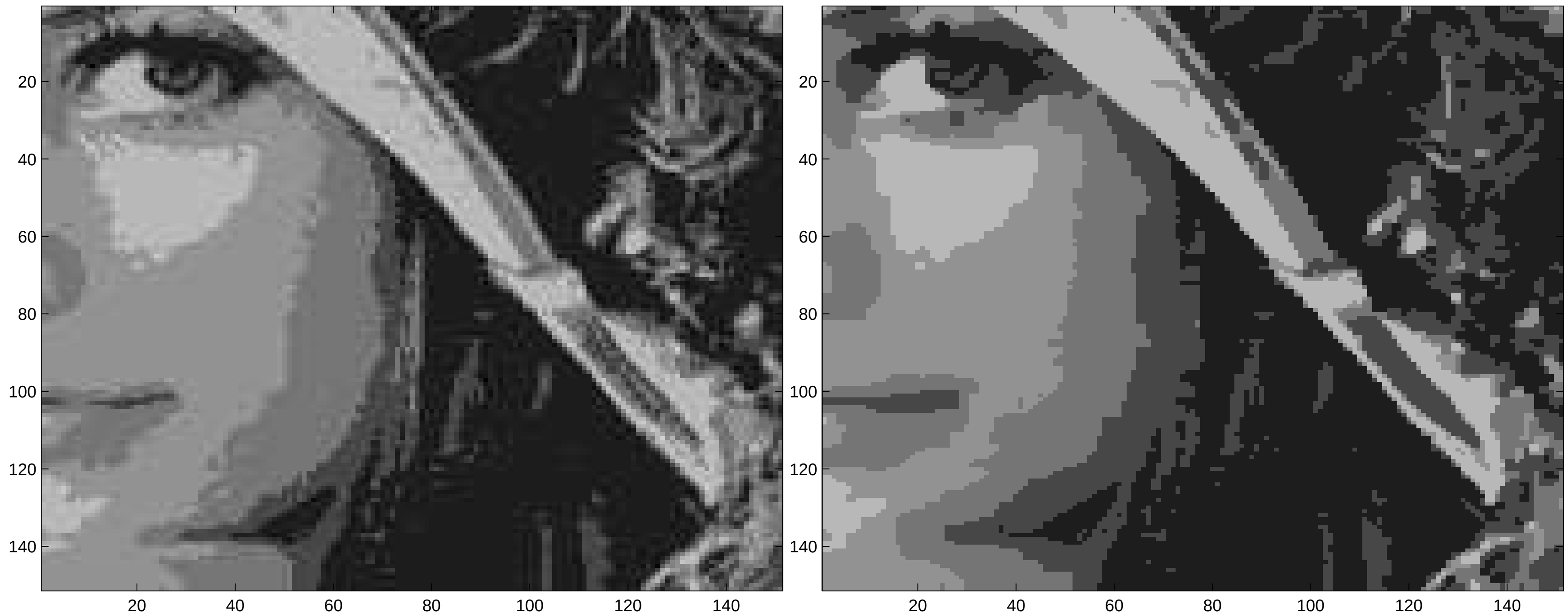


FIG. 7 – Classified images, without regularization (left) and with regularization (right).

Image classification

Another way to use the topological gradient : we consider the following *PDE*

$$\begin{cases} -div(c\nabla u) + u = w & in\ \Omega, \\ \partial_n u = 0 & in\ \Gamma = \partial\Omega, \end{cases}$$

where $c = \frac{1}{\varepsilon_c}.\chi_{\Omega_1} + \varepsilon_c.\chi_{\Omega_\varepsilon}$.

– if the pixel is on a contour ($c = \varepsilon_c$), the PDE is nearly equivalent to $u = w$,
– if it is not on a contour ($c = \dfrac{1}{\varepsilon_c}$), the PDE is nearly equivalent to $\Delta u = 0$ and we smooth the image.

We apply then the topological gradient method to this PDE.

Image classification

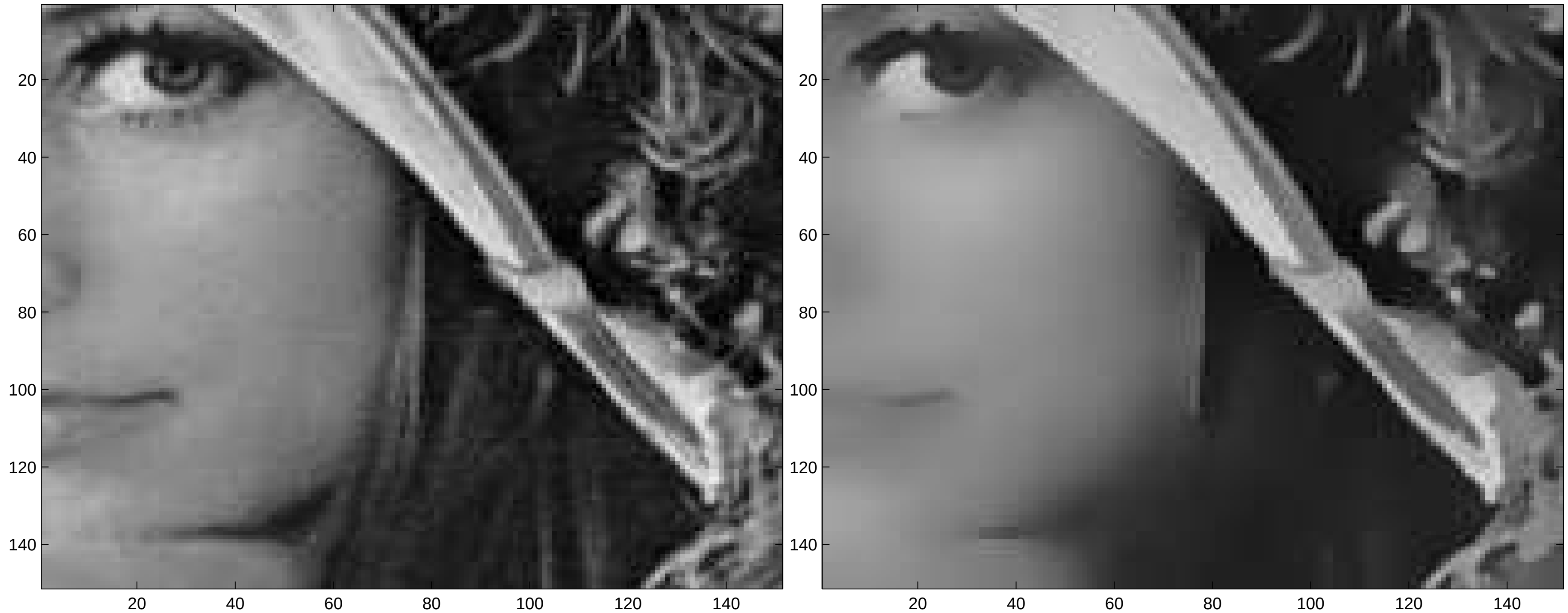


FIG. 8 – Original image (left) and smooth image (right).

Image classification

If we assume that the color classes are given, we only have to apply the previous unregularized classification method : each pixel is reassigned to its closest class.

If the classes are not given, it is possible to determine them in an optimal way, still by using the topological gradient method. The idea is to study the impact of changing a class $\mu_i := \mu_i + 1$ or $\mu_i - 1$ on the cost function.

If the number of classes is not given, we can add a penalization term in the cost function, measuring the number of classes and the previous algorithm provides the *optimal* number of classes, and their *optimal* values.

Image classification

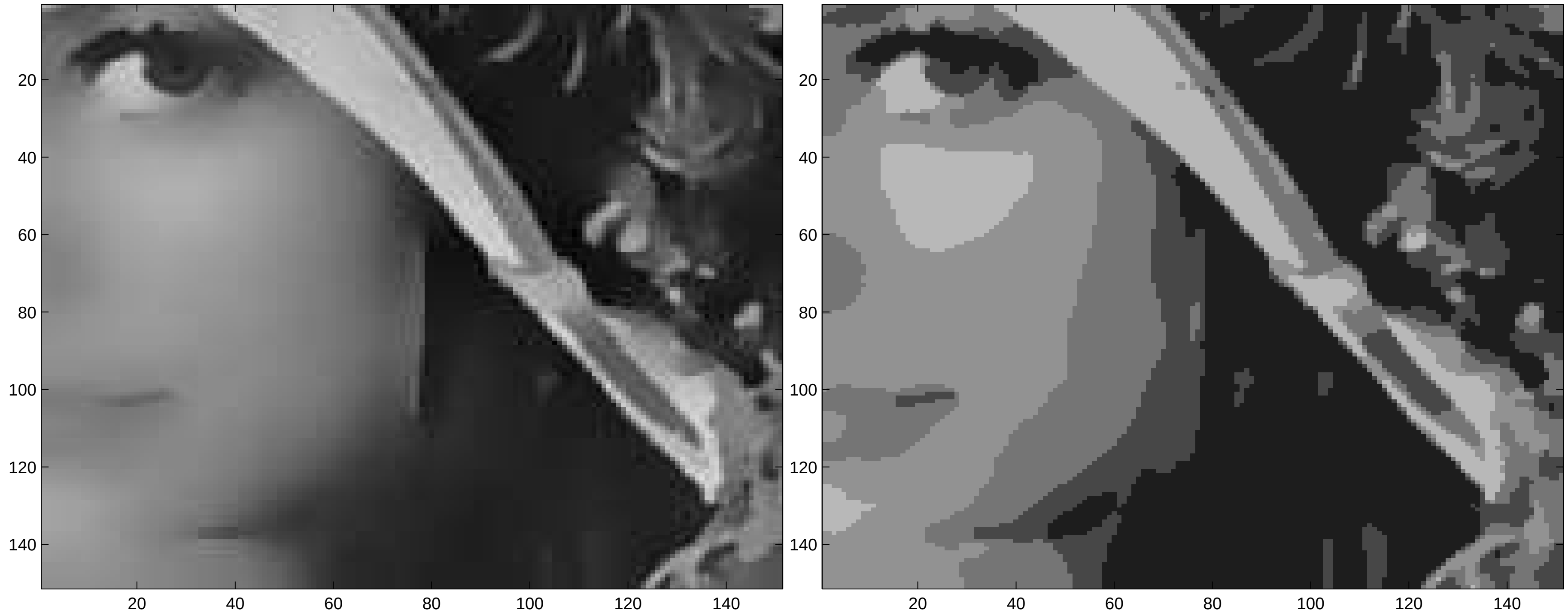


FIG. 9 – Smooth image (left) and classified image (right).

Image classification

FIG. 10 – Top : original image (left), unregularized classified image (right) ; Bottom : regularized classified image (left), smooth and classified image (right).

Image classification

Tab. 1 – Computational cost (time in seconds) and length of the interfaces for the different algorithms and number of classes.

Number of classes / Algorithm	$n = 2$ $C = \{0; 255\}$	$n = 3$ $C = \{34; 112; 165\}$	$n = 5$ $C = \{29; 71; 117; 146; 184\}$
Closest class (unregularized)	$t = 0.02$ $\|\Gamma\| = 2358$	$t = 0.06$ $\|\Gamma\| = 4513$	$t = 0.05$ $\|\Gamma\| = 7913$
Topological gradient (regularized)	$t = 12.67$ $\|\Gamma\| = 2069$	$t = 45.63$ $\|\Gamma\| = 3872$	$t = 81.78$ $\|\Gamma\| = 5770$
Smothering - closest class (+ topological gradient)	$t = 37.16$ $\|\Gamma\| = 1566$	$t = 37.17$ $\|\Gamma\| = 2870$	$t = 37.17$ $\|\Gamma\| = 4839$

Image classification

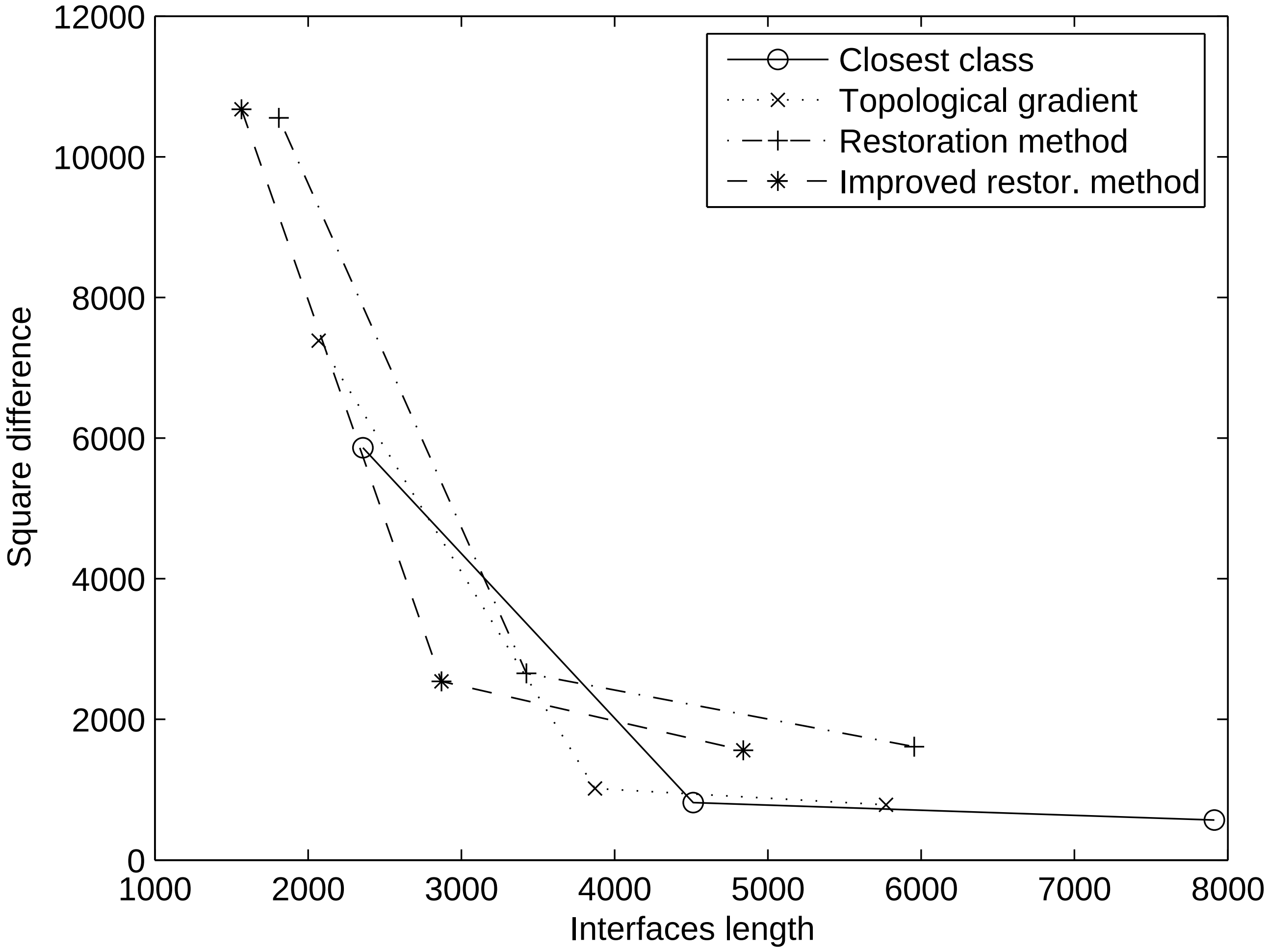


FIG. 11 – Square difference between the classified images and the original image versus the length of the interfaces for the different algorithms, and for 2, 3 and 5 classes.

Color image classification

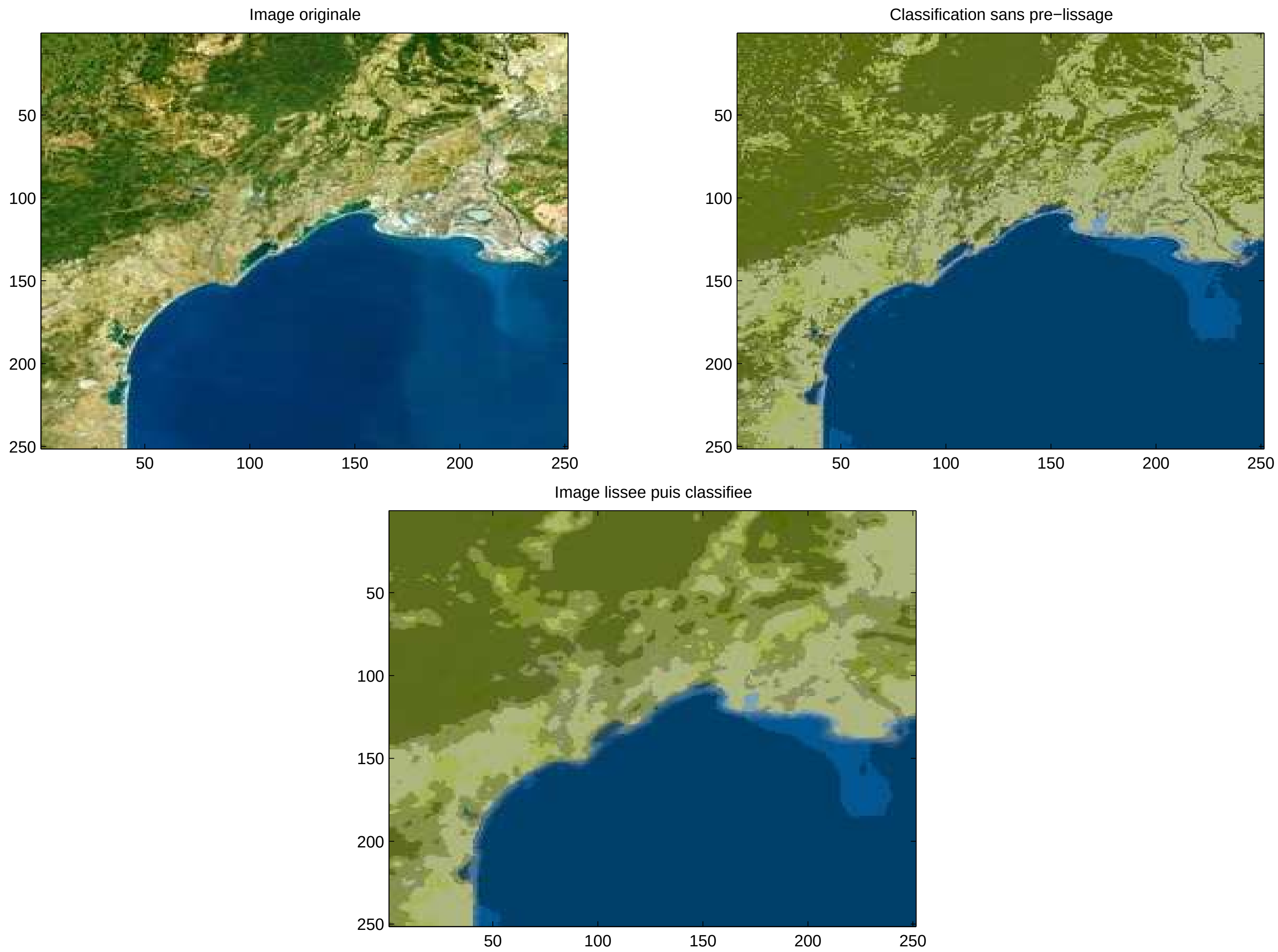


FIG. 12 – Top : original image (left), unregularized classified image (right) ; Bottom : regularized and classified image.

Restoration of 3d-images, or Movies

Algorithm

- Initialization : $c = c_0$.

- Computation of u_0 and v_0, solutions of the direct and adjoint problems, using a preconditioned conjugate gradient method (preconditioner = discrete cosine transform).

- Computation of the 3×3 matrix M and its lowest eigenvalue λ_{min} at each point of the domain.

- $c_1 = \begin{cases} \varepsilon_c & if \quad x \in \Omega \quad such \quad that \quad \lambda_{min} < \alpha < 0, \quad \varepsilon_c > 0 \\ c_0 & elsewhere. \end{cases}$

- Calculation of u_1 solution to the perturbed problem with $c = c_1$, still using a PCG method (preconditioned by a DCT).

- u_1 is the restored movie.

Small resolution movies

- Original movie
- Noised movie
- Restored movie
- All together

Size of the movie : 180×72 pixels, 30 frames.
$\rightsquigarrow$ 388.000 points, 5 minutes.

Small resolution movies

- Original movie
- Noised movie
- Restored movie
- All together

Size of the movie : 288×176 pixels, 52 frames.
$\rightsquigarrow$ 2.6 million points, 1 hour.

Small resolution movies

- Original movie
- Noised movie
- Restored movie
- All together

Size of the movie : 320×144 pixels, 100 frames.

$\rightsquigarrow$ 4.5 million points, 2 hours.

Large resolution movies

- Original movie
- Noised movie
- Restored movie
- All together

Size of the movie : 512×288 pixels, 110 frames (24 fps).
$\rightsquigarrow$ more than 16 million points, 12 hours.

Computation time

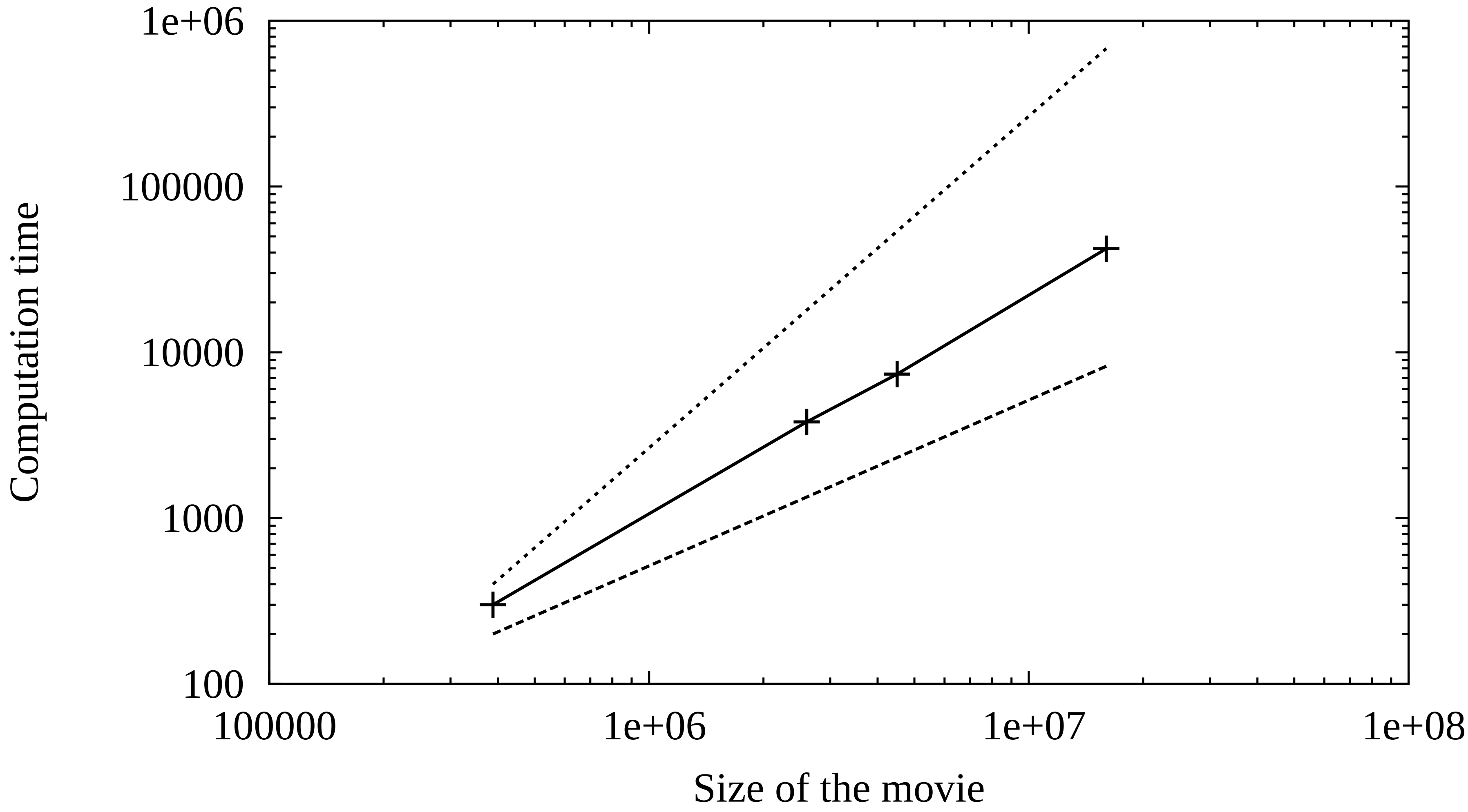


FIG. 13 – Computation time versus the size of the movie ; Topological gradient : $\mathcal{O}(n^{1.3})$.

Conclusion

– The topological gradient is very efficient.

– The image restoration (and classification) is performed in only one iteration : only 3 resolutions of a PDE are performed.

– The quality of the obtained images is good.

– Next step : color movies, restoration of a missing frame, ...

One missing image

- Original movie
- Restored movie
- All together

THANK YOU FOR YOUR ATTENTION