Image Decomposition

Using Bounded Variation and Homogeneous Besov Spaces

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AMS Eastern Sectional Meetting #1009
Annandale-on-Hudson, NY, October 8-9, 2005

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   - Vese-Osher's approximation of Meyer G-model.
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   - Approximation to Meyer F-model.
3. Modeling oscillatory components with Besov spaces.
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Variational image decomposition

Let f be periodic with the fundamental domain $\Omega = [-\frac{1}{2}, \frac{1}{2}]^2 \subset \mathbb{R}^2$. For short notation, we write X for $X(\Omega)$. A variational method for decomposing f into $u + v$ is given by an energy minimization problem

$$\inf_{(u,v)\in X_1 \times X_2} \{\mathcal{K}(u,v) = F_1(u) + \lambda F_2(v) : f = u + v\}, \text{ where}$$

- $F_1, F_2 \geq 0$ are functionals on spaces of functions or distributions X_1, X_2, respectively.
- $\lambda > 0$ is a tuning parameter.

A good model for $\mathcal{K}$ is given by a choice of X_1 and X_2 so that $F_1(u)$ and $F_2(v)$ are small.

Mumford-Shah (1989)

$$\inf_{(u,v)\in SBV\times L^2}\left\{\int_{\Omega\setminus J_u}|\nabla u|^2dx+\alpha\mathcal{H}^1(J_u)+\beta\|v\|_{L^2}^2,\ f=u+v\right\}.$$

- $f\in L^\infty\subset L^2$ is split into $u\in SBV$, a piecewise-smooth function with its discontinuity set J_u composed of a union of curves, and $v=f-u\in L^2$ representing noise or texture.
- $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure,
- $\alpha,\beta>0$ are tuning parameters.

With the above notations, the first two terms in the above energy compose $F_1(u)$, while the third term makes $F_2(v)$.

Rudin-Osher-Fatemi (1992)

$$\inf_{(u,v)\in BV\times L^2}\left\{\int |\nabla u|\, dx + \lambda \|v\|_{L^2}^2,\quad f = u+v\right\},$$

- $\int |\nabla u|\, dx$ denotes $|u|_{BV}$,
- $f \in L^2$ is split into $u \in BV$, a piecewise-smooth function and $v = f - u \in L^2$ representing noise or texture.
- $\lambda > 0$ is a tuning parameter.

With the above notation, $F_1(u) = |u|_{BV}$, and $F_2(v) = \|v\|_{L^2}^2$. Replacing $\|v\|_{L^2}^2$ with $\|v\|_{L^1}$ was proposed by Cheon, Paranjpye, Vese and Osher as a Summer project, and further analysis by Chan and Esedoglu, Esedoglu and Vixie, and Allard.

Meyer models (2001)

Remark: Oscillatory functions do not have small norms in L^2. In 2001, Y. Meyer proposed

$$\inf_{(u,v)\in BV\times X_2} \left\{ |u|_{BV} + \lambda \|v\|_{X_2}, \quad f = u + v \right\}.$$

Here X_2 is either G, F, or E.

- The space G consists of distributions T which can be written as

$$T = \operatorname{div}(\vec{g}), \quad \vec{g} = (g_1, g_2) \in \left(L^\infty\right)^2, \text{ with}$$

$$\|T\|_G = \inf \left\{ \left\| \sqrt{(g_1)^2 + (g_2)^2} \right\|_{L^\infty} : \ T = \operatorname{div}(\vec{g}), \ \vec{g} \in (L^\infty)^2 \right\}.$$

Meyer (cont.)

- The space F consists of distributions T which can be written as

$$T = \operatorname{div}(\vec{g}), \quad \vec{g} = (g_1, g_2) \in (BMO)^2, \text{ with}$$

$$\|T\|_F = \inf \left\{ \|g_1\|_{BMO} + \|g_2\|_{BMO} : \; T = \operatorname{div}(\vec{g}), \; \vec{g} \in (BMO)^2 \right\}.$$

We say that f belongs to BMO, if

$$\|f\|_{BMO} = \sup_{Q \subset \Omega} \frac{1}{|Q|} \int_Q |f - f_Q| < \infty,$$

where $Q \subset \Omega$ is a square (with sides parallel with the axis). Here $f_Q = |Q|^{-1} \int_Q f(x, y)$ denotes the mean value of f over the square Q.

Meyer (cont.)

- We say a generalize function T belongs to the space E if it can be written as $T = \Delta g$, such that

$$\sup_{|y|>0} \frac{\|g(.+y) - 2g(.) + g(.-y)\|_{L^\infty}}{|y|} < \infty.$$

- Both $G = \text{div}(L^\infty)$ and $F = \text{div}(BMO)$ (as defined previously) consist of first order differences of vector fields in L^∞ and BMO, respectively.
- E (as defined above) consists of second order differences of functions satisfying the Zygmund condition.

Approximating Meyer's G-model

Vese-Osher (2003): model oscillatory components as first order differences of vector fields in L^p, for $1 \leq p < \infty$.

$$\inf_{u,\vec{g}} \left\{ |u|_{BV} + \mu \|f - u - \partial_x g_1 - \partial_y g_2\|_{L^2}^2 + \lambda \left\| \sqrt{g_1^2 + g_2^2} \right\|_{L^p} \right\}.$$

- $f \in L^2$ is decomposed into $u + v + r$, such that $u \in BV$, $v = \text{div}(\vec{g}) \in \text{div}(L^p)$, and $r = f - u - v \in L^2$ is a residual which is negligible numerically for large μ.
- $\mu,\ \lambda > 0$ are tuning parameters.
- Other motivating work on the G space includes Aujol et al, Aubert and Aujol, S. Osher and O. Scherzer, among others.

Osher-Sole-Vese (2003)

- From the standpoint of view of PDE, sometimes second order differences are much more useful than first order differences (a remark made by Zygmund).
- From the point of view of image processing in the PDE/variational approach, S. Osher, A. Sole, and L. Vese were among the first to consider second order differences. They model oscillatory components as $v = \Delta g$, where $g \in \dot{H}_2^1$. I.e. $v \in \dot{H}_2^{-1}$.

$$\inf_{u,v} \left\{ |u|_{BV} + \lambda \|\nabla(\Delta^{-1} v)\|_{L^2}^2, \; f = u + v \right\}.$$

- L. Linh and L. Vese (2005) recently considered modeling oscillatory components as $v \in H_2^s$, $s \in \mathbb{R}^-$.

Approximating Meyer's F-model

- (Joint work with **L. Vese**), we considered a strictly convex variational problem (motivated from **Vese-Osher**):

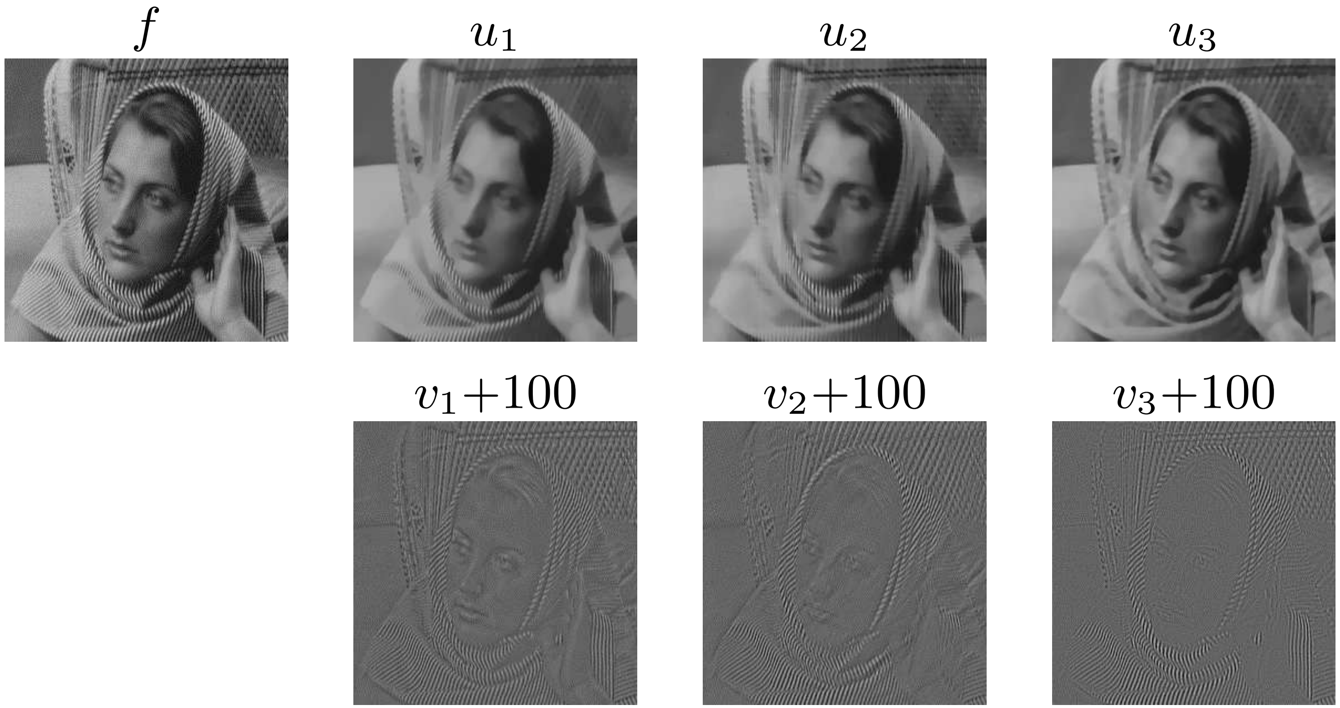
$$\inf_{u,\vec{g}} \left\{ |u|_{BV} + \mu \|f - u - \partial_x g_1 - \partial_y g_2\|_{L^2}^2 + \lambda \left[\|g_1\|_{BMO} + \|g_2\|_{BMO} \right] \right\}$$

- An equivalent isotropic problem by setting $\vec{g} = \nabla \cdot g$, i.e. $v = \Delta g$ (motivated from **Osher-Sole-Vese**),

$$\inf_{u,g} \left\{ |u|_{BV} + \mu \|f - u - \Delta g\|_{L^2}^2 + \lambda \left[\|g_x\|_{BMO} + \|g_y\|_{BMO} \right] \right\}$$

- Here, $f = u + v + r$, where $u \in BV$, $v = \mathrm{div}(\vec{g}) = \Delta g \in F$, and $r = f - u - v \in L^2$ is a residual. As $\mu \to \infty$, These models approach Meyer's F-model.

$v = \Delta g$ is more preferable



1) R-O-F decomposition (u_1, v_1), 2) Meyer's F decomposition (u_2, v_2) with $v_2 = \text{div}(\vec{g}),\ g_i \in BMO$, 3) Meyer's F decomposition (u_3, v_3) with $v_3 = \Delta g$, $\nabla g \in (BMO)^2$.

Homogeneous Besov spaces

Consider the Cauchy-Poisson semi-group

$$P_t g(x) = (e^{-2\pi t|\xi|}\hat{g}(\xi))^{\vee}(x),\ t > 0,\ \text{and } P_0 = I.$$

Let $\alpha \in \mathbb{R}$, $k \in \mathbb{N}_0 \ni k > \alpha$ $1 \leq p \leq \infty$. We say $g \in \dot{B}^{\alpha}_{p,q}$ if

$$\|g\|_{\dot{B}^{\alpha}_{p,q}} = \left(\int \left| t^{k-\alpha} \left\| \frac{\partial^k P_t}{\partial t^k} g \right\|_{L^p} \right|^q \frac{dt}{t} \right)^{1/q} < \infty,\ \text{for } q < \infty,$$

$$\|g\|_{\dot{B}^{\alpha}_{p,\infty}} = \sup_{t \geq 0} \left\{ t^{k-\alpha} \left\| \frac{\partial^k P_t}{\partial t^k} g \right\|_{L^p} \right\} < \infty,\ \text{for } q = \infty,$$

For $-2 < \alpha < 0$ we choose $k = 0$, and $k = 2$ for $0 < \alpha < 2$.

Homogeneous Besov space (cont.)

Denote $I_s v = (-\Delta)^{s/2}(v) = ((2\pi|\xi|)^s \hat{v}(\xi))^{\vee}$, We have

$$I_s : \dot{B}^{\alpha}_{p,q} \to \dot{B}^{\alpha-s}_{p,q},$$ isometrically (injectively).

Define $\tau_\delta f(x) = f(\delta x),\ \delta > 0$. We have

$$\|\tau_\delta f\|_{L^p(\mathbb{R}^n)} = \delta^{-\frac{n}{p}} \|f\|_{L^p(\mathbb{R}^n)},$$ and

$$\|\tau_\delta f\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)} = \delta^{-\frac{n}{p}+\alpha} \|f\|_{\dot{B}^{\alpha}_{p,q}(\mathbb{R}^n)},$$ for all $1 \leq p, q < \infty$.

The following embedding holds,

$$\dot{B}^{\alpha_1}_{p,q_1}(\mathbb{R}^n) \subset \dot{B}^{\alpha_2}_{p,q_2}(\mathbb{R}^n),$$

if either $0 < \alpha_2 \leq \alpha_1 < 2$, or $\alpha_1 = \alpha_2$ and $1 \leq q_1 \leq q_2 \leq \infty$.

Besov spaces for oscillatory components

- Meyer's E-model corresponds to modeling

$$u \in BV, \text{ and } v = \Delta g,\ g \in \dot{B}^1_{\infty,\infty}. \text{ I.e. } v \in \dot{B}^{-1}_{\infty,\infty}.$$

- (Joint work with **J. Garnett** and **L. Vese**) we consider decomposing $f = u + v$, such that

$$u \in BV, \text{ and } v = \Delta g \in \dot{B}^{\alpha-2}_{p,\infty},\ g \in \dot{B}^{\alpha}_{p,\infty},\ 0 < \alpha < 2, 1 \le p \le \infty,$$

with the minimization problems

- $\inf_{u,g} \left\{ \mathcal{J}_a(u,g) = |u|_{BV} + \mu \|f - u - \Delta g\|^2_{L^2} + \lambda \|g\|_{\dot{B}^{\alpha}_{p,\infty}} \right\}$
- $\inf_{u} \left\{ \mathcal{J}_e(u) = |u|_{BV} + \lambda \|f - u\|_{\dot{B}^{\alpha-2}_{p,\infty}} \right\}$

Numerical computation of $\mathcal{J}_a,\ p < \infty$

$$\mathcal{J}_a(u,g) = |u|_{BV} + \mu\,\|f - u - \Delta g\|^2_{L^2} + \lambda \|g\|_{\dot{B}^{\alpha}_{p,\infty}},$$
$$= \int_\Omega |\nabla u| + \mu \int_\Omega |f - u - \Delta g|^2 + \lambda \sup_{t>0} \|K_t^\alpha * g\|_{L^p}\,,$$

where $K_t^\alpha = t^{2-\alpha}\frac{\partial^2 P_t}{\partial t^2} = t^{2-\alpha}\left((2\pi|\xi|)^2 e^{-2\pi t|\xi|}\right)^\vee$.
In practice, we consider only a discrete set

$$\left\{t_i = 2.5\tau^i :\ \tau = 0.9,\ i = 1, ..., N = 150\right\}.$$

These t_i's are chosen so that discretely $P_{t_1}(x)$ is a constant and $P_{t_N}(x)$ approximates the Dirac delta function.

Algorithm

- Given an initial guess (u_0, g_0).
- Compute $\bar{t}_0 = \text{argmax}_{t \in \{t_1,\ldots,t_N\}} \|K_t^\alpha * g_0\|_{L^p}$.
- Suppose $(u_n, g_n, \bar{t}_n)$ is known. Compute (u_{n+1}, g_{n+1}) via

$$\left(\frac{\partial \mathcal{J}_a}{\partial u} =\right), \; 0 = -\nabla \cdot \left(\frac{\nabla u_{n+1}}{|\nabla u_n|}\right) - 2\mu(f - u_{n+1} - \Delta g_n)$$

$$\left(\frac{\partial \mathcal{J}_a}{\partial g} =\right), \; 0 = -2\mu\Delta(f - u_{n+1} - \Delta g_{n+1}) +$$

$$\lambda \left\|K_{\bar{t}_n}^\alpha * g_n\right\|_{L^p}^{1-p} K_{\bar{t}_n}^\alpha * \left(\left|K_{\bar{t}_n}^\alpha * g_n\right|^{p-2} K_{\bar{t}}^\alpha * g_n\right)$$

- Suppose $\bar{t}_n = t_k$. Compute $\bar{t}_{n+1} = \text{argmax}_{t \in \{t_{k-1},t_k,t_{k+1}\}} \|K_t^\alpha * g_{n+1}\|_{L^p}$. Continue...

Numerical computation of $\mathcal{J}_a,\ p = \infty$

$$\mathcal{J}_a(u,g) = |u|_{BV} + \mu\,\|f - u - \Delta g\|^2_{L^2} + \lambda \|g\|_{\dot{B}^{\alpha}_{\infty,\infty}},$$
$$= \int_\Omega |\nabla u| + \mu \int_\Omega |f - u - \Delta g|^2 + \lambda \sup_{t>0, h \in L^1} \frac{\langle K^\alpha_t * g, h\rangle}{\|h\|_{L^1}}.$$

• **Algorithm:** The steps are the same as in the previous case, but now at each iteration we need to compute

$$\bar{h}_n = \text{argmax}_{h \in L^1} \frac{\left\langle K^\alpha_{\bar{t}_n} * g_n, h \right\rangle}{\|h\|_{L^1}}, \text{ via}$$

$$h_\tau = \frac{K^\alpha_{\bar{t}} * g}{\|h\|_{L^1}} - \frac{\left\langle K^\alpha_{\bar{t}} * g, h\right\rangle}{\|h\|^2_{L^1}} \frac{h}{|h|}.$$

Numerical computation of $\mathcal{J}_e,\ p < \infty$

$$\mathcal{J}_e(u) = |u|_{BV} + \lambda \|f-u\|_{\dot{B}^{\alpha-2}_{p,\infty}}$$
$$= \int_\Omega |\nabla u| + \lambda \sup_{t>0} \|H_t^\alpha * (f-u)\|_{L^p},$$

where $H_t^\alpha = t^{2-\alpha} P_t = t^{2-\alpha}\left(e^{-2\pi t|\xi|}\right)^\vee$.

Suppose $(u_n, \bar{t}_n)$ is known. Compute (u_{n+1}, t_{n+1}) via

- $\left(\frac{\partial \mathcal{J}_e}{\partial u} = \right),\ \frac{u_{n+1}-u_n}{\Delta\tau} = \nabla \cdot \left(\frac{\nabla u_{n+1}}{|\nabla u_n|}\right) +$
$\lambda \left\|H^\alpha_{\bar{t}_n} * (f-u_n)\right\|^{1-p}_{L^p} H^\alpha_{\bar{t}_n} * \left(|H^\alpha_{\bar{t}_n} * (f-u_n)|^{p-2} H^\alpha_{\bar{t}_n} * (f-u_n)\right).$
- $t_{n+1} = \text{argmax}_{t\in\{t_{k-1}, t_k=\bar{t}_n, t_{k+1}\}} \|H_t^\alpha * (f-u_{n+1})\|_{L^p}.$

Numerical results

f

Numerical results (cont.)

u

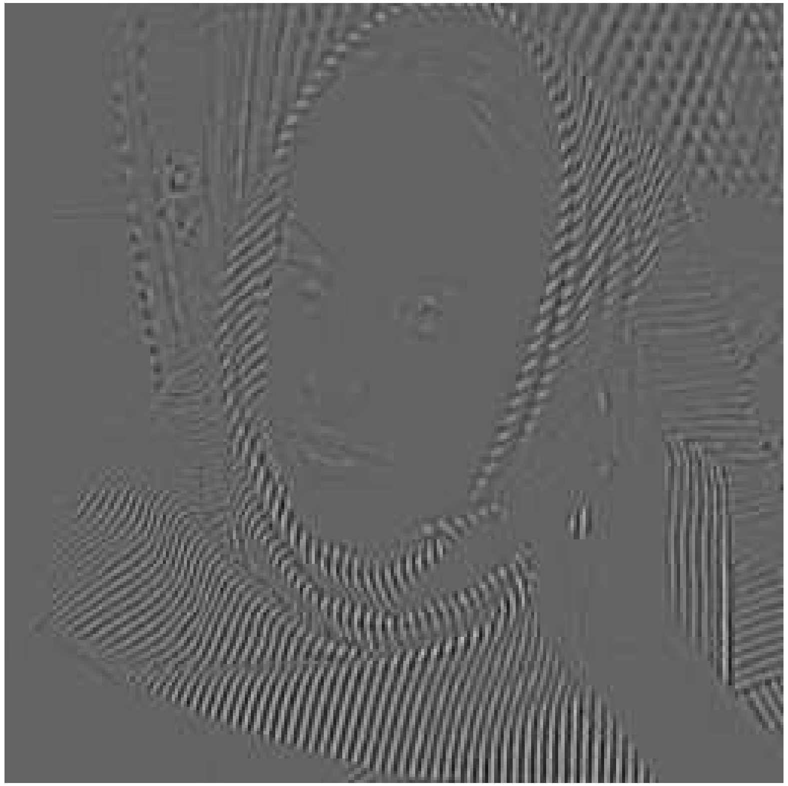
$f\!-\!u\!+\!100$

A decomposition using $\mathcal{J}_a$ with $\alpha = 1.5$, $p = 1$, $\mu = 1$, and $\lambda = 1e - 04$.

Numerical results (cont.)

u

$f - u + 100$

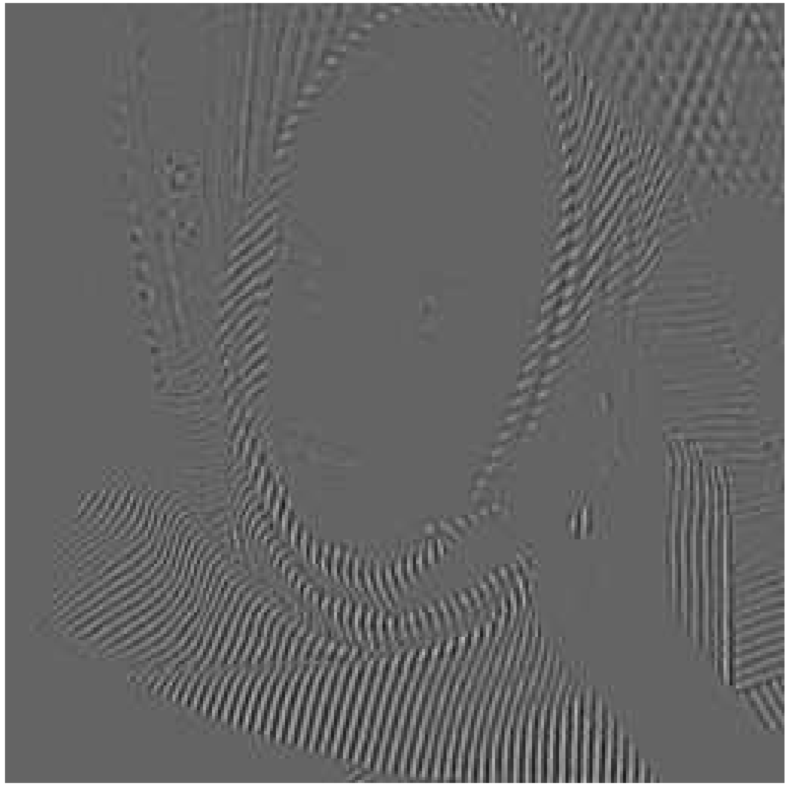


A decomposition using $\mathcal{J}_a$ with $\alpha = 1.0$, $p = 1$, $\mu = 1$, and $\lambda = 3e - 03$.

Numerical results (cont.)

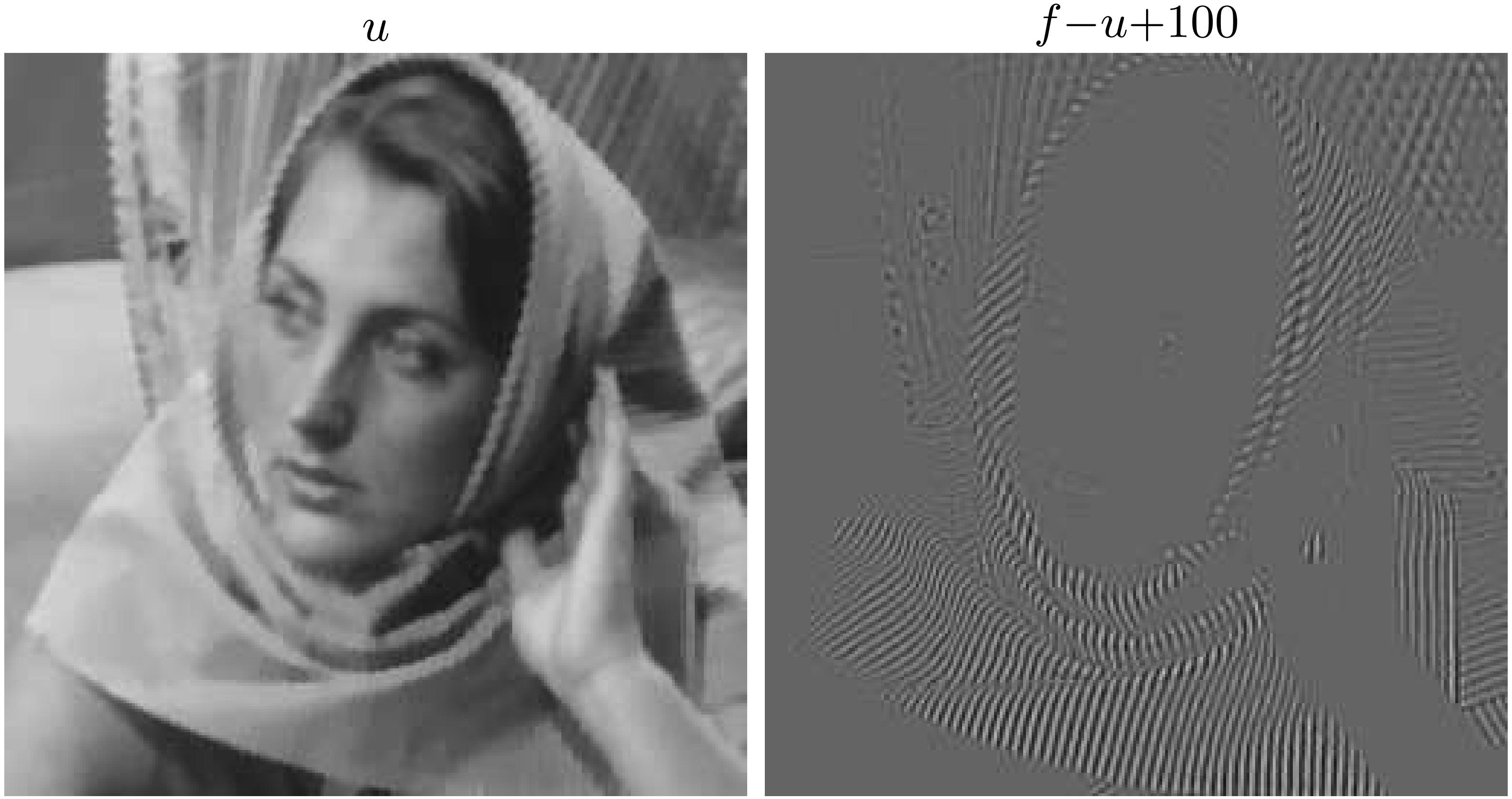
u

$f - u + 100$



A decomposition using $\mathcal{J}_a$ with $\alpha = 0.5$, $p = 1$, $\mu = 1$, and $\lambda = 0.5$.

Numerical results (cont.)

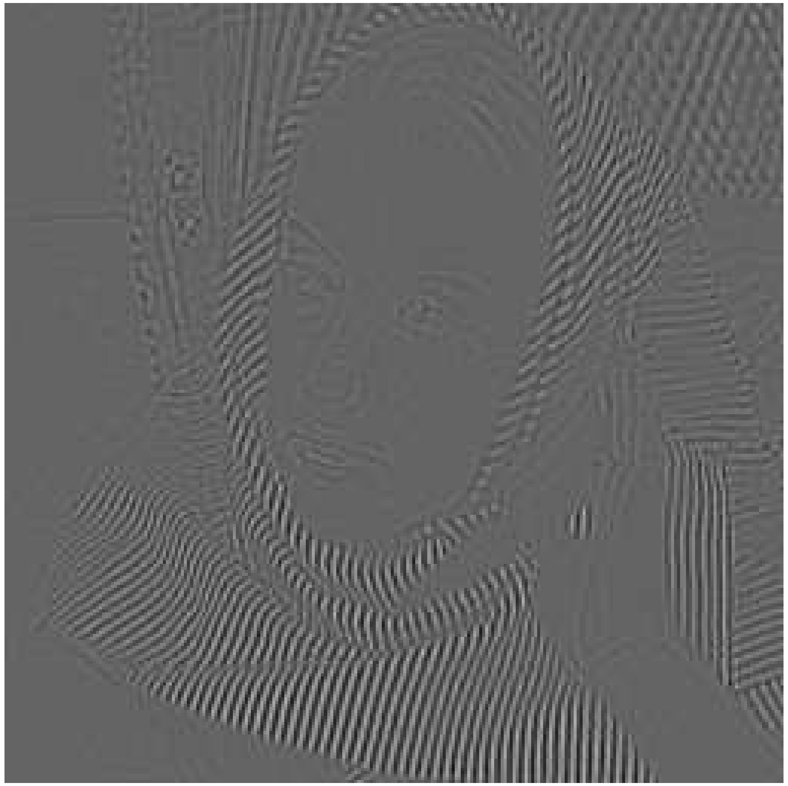


A decomposition using $\mathcal{J}_a$ with $\alpha = 0.1$, $p = 1$, $\mu = 1$, and $\lambda = 0.5$.

Numerical results (cont.)

u

$f - u + 100$

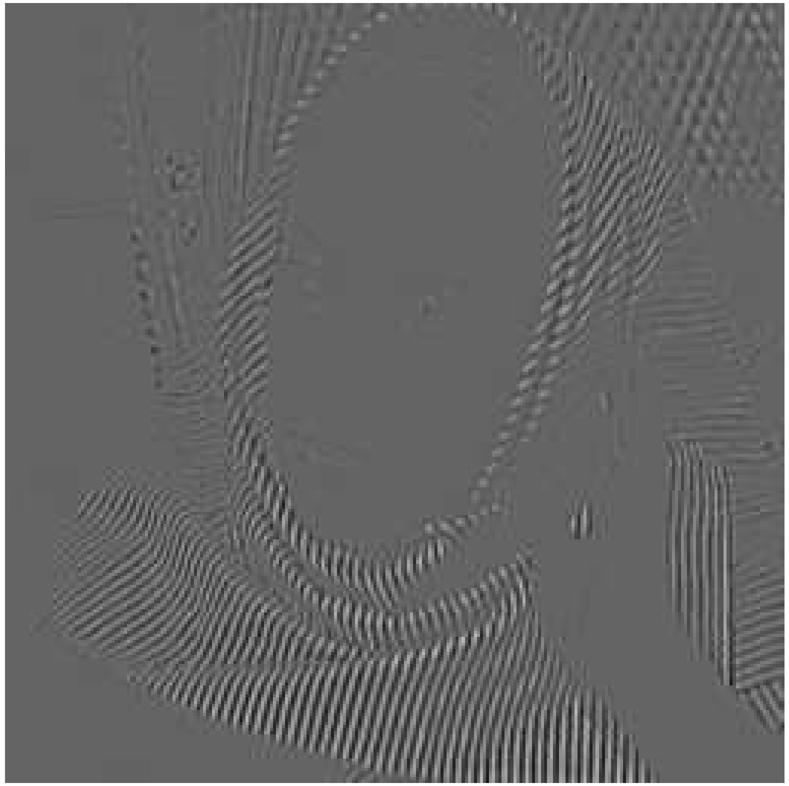


A decomposition using $\mathcal{J}_a$ with $\alpha = 1$, $p = \infty$, $\mu = 10$, and $\lambda = 1$.

Numerical results (cont.)

u

$f-u+100$



A decomposition using $\mathcal{J}_e$ with $\alpha = 1$, $p = 1$, $\lambda = 1500$.

Thank You!