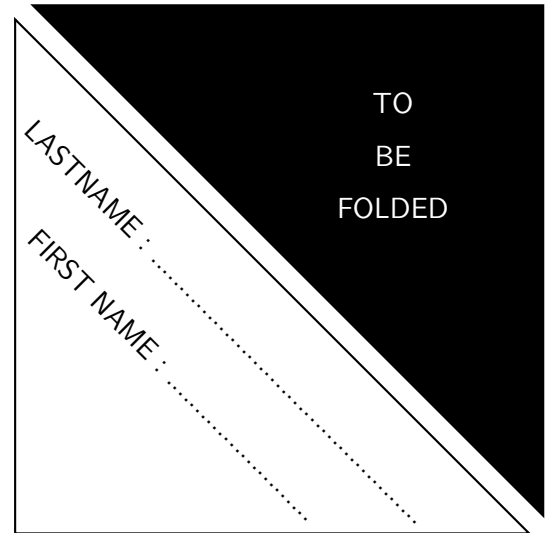


Monte Carlo Methods

Final Exam - 11/01/2025

2H00 – DOCUMENTS AND CALCULATOR PROHIBITED



Ex. 1	Ex. 2	Ex. 3	Total
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Problems are independent. Answers need be written on this document. There is sufficient space allotted to each question for accomodating a proper answer. If needed, additional space is available at the end of the booklet.

Notation: *i.i.d.*: independent and identically distributed

Problem 1

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Let f and g be densities with respect to the Lebesgue measure on \mathbb{R} such that

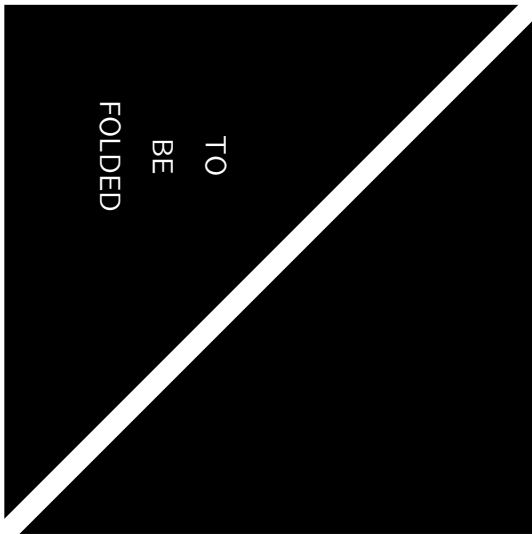
$$\forall x \in \mathbb{R}, \quad f(x) = c\tilde{f}(x) \quad \text{and} \quad g(x) = d\tilde{g}(x),$$

where both positive functions \tilde{f} and \tilde{g} are known and computable, and both constants c and d are unknown. **For questions 1. to 4.**, we consider the special case of an interval $]a, b[\subset \mathbb{R}$ such that

$$\forall x \notin]a, b[, \quad \tilde{f}(x) = 0 \quad \text{and} \quad \sup_{x \in \mathbb{R}} \tilde{f}(x) = M < \infty.$$

1. Considering $U = (U_1, U_2)$ a uniform random point on $\mathcal{R} =]a, b[\times]0, M[$, compute the probability $\mathbb{P}[U_2 \leq \tilde{f}(U_1)]$. Given an *i.i.d.* sequence $U^1, \dots, U^n, n \in \mathbb{N}^*$, of uniform random points on \mathcal{R} , deduce a converging estimator of c , \hat{c}_n , and justify the convergence of \hat{c}_n in n .

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2. Is this estimator \hat{c}_n unbiased?

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3. Let $df(x)$ be a vectorised R function that computes $\tilde{f}(x)$. Write an R code that computes \hat{c}_n .

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From now on, it is no longer assumed that f is bounded and has bounded support. Nevertheless, we assume from now on that $\{x \in \mathbb{R} \mid f(x) = 0\} \subseteq \{x \in \mathbb{R} \mid g(x) = 0\}$.

4. Let X be a random variable with density f . Show that $\mathbb{E}[\tilde{g}(X)/\tilde{f}(X)] = c/d$.

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5. Given an *i.i.d.* sequence X_1, \dots, X_n of random variables with density f , deduce from question 4. a converging estimator of c/d .

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6. Let $\alpha(\cdot)$ be a positive function on \mathbb{R} such that

$$\int_{\mathbb{R}} \alpha(x) \tilde{f}(x) \tilde{g}(x) dx < +\infty.$$

Show that if X is a random variable with density f and Y a random variable with density g , then

$$\mathbb{E}[\alpha(X) \tilde{g}(X)] / \mathbb{E}[\alpha(Y) \tilde{f}(Y)] = c/d.$$

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7. Deduce from the previous question a converging estimator of c/d based on two sequences X_1, \dots, X_n and Y_1, \dots, Y_n of *i.i.d.* random variables with density f and g , respectively. Justify the convergence of this estimator and provide a corresponding R function ratio of (n) .

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We now assume that d is known. For an arbitrary $\omega > 0$, we further consider the **special case** when the auxiliary target density $h(x) \propto \tilde{f}(x) + \omega g(x)$ can be simulated, **even though its normalising constant is unknown**, that is, there exists an R function `mixt(N)` that returns N i.i.d. realisations with density $h(\cdot)$.

8. Show that a sample from $f(\cdot)$ can be extracted as a random subsample of an existing N -sample from $h(\cdot)$ —for instance, produced as `mixt(N)`—. What is the expected size of this subsample as a function of N ?

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9. Construct a valid algorithm that partitions an N -sample from $h(\cdot)$ into (i) a sample from $g(\cdot)$ and (ii) a sample from $f(\cdot)$. Deduce a converging estimator of the constant c .

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Problem 2

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Let $0 < \alpha < 1$ be the shape parameter of the Gamma distribution $\text{Ga}(\alpha, 1)$, with density

$$f(x; \alpha) = \frac{1}{\Gamma(\alpha)} x^{\alpha-1} \exp\{-x\} \quad x > 0$$

The goal is simulate from this distribution using a Generalized Exponential distribution $\text{GE}(\alpha, \lambda)$, with density

$$g(x; \alpha, \lambda) = \frac{\alpha}{\lambda} \left(1 - e^{-x/\lambda}\right)^{\alpha-1} e^{-x/\lambda} \quad \lambda, x > 0$$

We aim at sampling from f using the accept-reject algorithm with $g(\cdot; \alpha, \lambda)$ as proposal density.

1. Provide the CDF attached to $g(\cdot; \alpha, \lambda)$ and deduce the normalizing constant of $g(\cdot; \alpha, \lambda)$ is correct.

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2. Deduce a practical way to generate a random variable with density $g(\cdot; \alpha, \lambda)$.

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3. Show that

$$f(x; \alpha) = \frac{1}{\Gamma(\alpha + 1)} R(x) g(x; \alpha, 1) \quad x > 0 \quad (1)$$

with

$$R(x) = \left(\frac{x}{1 - e^{-x}} \right)^{\alpha - 1} \quad x > 0$$

and establish that $0 < R(x) \leq 1$ for $x \geq 0$.

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4. Construct an accept-reject algorithm to simulate $f(\cdot; \alpha)$ using $g(\cdot; \alpha, 1)$ by providing the acceptance bound on the uniform variate. Indicate the expected number of proposals needed to accept one realisation.

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5. Write an executable R code of this algorithm as an R function `zeini(N,alpha)` with inputs `N`, the number of simulations, and `alpha`, the $\text{Ga}(\alpha, 1)$ shape parameter.

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6. Since $R(\cdot)$ satisfies (no proof required!)

$$\frac{4 - (1 - \alpha)x}{4 + (1 - \alpha)x} \leq R(x) \leq \frac{4 + \alpha x}{4 - \alpha x}$$

deduce a faster accept-reject algorithm and write a corresponding R function squeeze(N,alpha).

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7. Since the choice $\lambda = 1$ made above in the proposal is arbitrary, other values of λ could lead to a higher efficiency. Give a precise mathematical meaning to "higher efficiency" and describe how you would run a Monte Carlo experiment to compare the choices $\lambda = 1/2$ and $\lambda = 2$. (**Bonus:** Write the associated R code.)

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Problem 3

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We revisit the simulation of upper truncated Normal $N^+(a)$ random variables, $a > 0$, with density

$$f(x; a) \propto \exp\{-x^2/2\} \mathbb{1}_{(a, \infty)}(x) \quad x \in \mathbb{R}$$

where the proportionality symbol is applying to both sides as functions of x .

1. Recall here (i) the exact value of the normalizing constant of $f(\cdot; a)$ and (ii) a standard accept-reject algorithm based on an Exponential proposal translated by a , **as seen in class**.

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2. Derive the density of $Z = X^2$ when $X \sim N^+(a)$ and show that an acceptance-rejection simulation of Z is possible when using as proposal an Exponential $E(1/2)$ random variable translated by a^2 .

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3. Deduce an acceptance-rejection algorithm for the simulation of $X \sim N^+(a)$ and provide an associated R function `marsa(N,a)` with inputs `N`, the number of simulations, and `a`, the $N^+(a)$ truncation parameter. (**Bonus:** Write a version with no `for`, no `while` and no `repeat` loop.)

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4. What is the average acceptance probability $\rho(a)$ for this algorithm? Given the following asymptotic approximation (when a goes to ∞) of the Normal cdf

$$\Phi(-a) \approx e^{-a^2/2}(a^{-1} - a^{-3})$$

give an asymptotic approximation of $\rho(a)$.

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5. An R experiment on the respective performances of both marsa and truncnorm (the standard algorithm) returns the following execution times:

```
function user system total
truncnorm 0.329 0.011 0.341
function user system total
mars 0.150 0.024 0.174
```

What is the conclusion of this comparison?

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