A complex-analytic approach to the problem of uniform controllability of a transport equation in the vanishing viscosity limit

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Abstract

We revisit a result by Coron and Guerrero stating that the one-dimensional transport-diffusion equation

$$u_t + Mu_x - \varepsilon u_{xx} = 0 \text{ in } (0,T)\times(0,L),$$

controlled by the left Dirichlet boundary value is zero-controllable at a bounded cost as $\varepsilon \to 0^+$, when $T > 4.3\,L/M$ if $M > 0$ and when $T > 57.2\,L/|M|$ if $M < 0$. By a completely different method, relying on complex analysis, we prove that this still holds when $T > 4.2\,L/M$ if $M > 0$ and when $T > 6.1\,L/|M|$ if $M < 0$.

1 Introduction

Let us fix $L > 0$ and $M \neq 0$. We consider the following transport-diffusion equation:

$$\begin{cases} u_t + Mu_x - \varepsilon u_{xx} = 0 \text{ in } (0,T)\times(0,L), \\ u_{|t=0} = u_0 \text{ in } (0,L), \\ u_{|x=0} = v(t) \text{ in } (0,T), \quad u_{|x=L} = 0 \text{ in } (0,T), \end{cases} \tag{1}$$

In the above equation v is a boundary control and ε is a small positive parameter, intended to tend to zero.

The problem which we consider for this parabolic equation is connected to the zero-controllability. We recall that the problem of zero-controllability is to determine whether it is possible given a time $T > 0$ and an initial data u_0 in $L^2(0,L)$, to find a control $v \in L^2(0,T)$ such that the corresponding solution of (1) satisfies

$$u(T,x) = 0 \text{ for all } x \in [0,L]. \tag{2}$$

The controllability of parabolic equations in dimension 1, such as the one considered here for fixed $\varepsilon > 0$, was established by Fattorini and Russell [6]. The controllability of parabolic equation in higher dimensions was established independently by Fursikov and Imanuvilov (see [7]) and Lebeau and Robbiano (see [13]) in slightly different frameworks, and with different methods (both using the so-called Carleman estimates, though).

In this paper, we investigate the cost of the control in the vanishing viscosity limit $\varepsilon \to 0^+$, and in particular to determine in which situation it is possible to obtain a control which remains bounded as $\varepsilon \to 0^+$. We will say that the system is *uniformly zero-controllable* if this property is satisfied.

A motivation for studying the controllability of a transport equation in the vanishing viscosity limit, comes from the topic of the control of systems of conservation laws, in the context of weak entropy solutions, see for instance [1, 2, 4, 8]. These solutions are discontinuous solutions (admitting shocks), which can be obtained via a vanishing viscosity limit. It is hence interesting in order to understand better the control properties of these equations, to know how the control behaves for small but not zero

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viscosity. Of course the linear model which we consider here is the simplest possible example of scalar conservation law. A first example of controllability result of a nonlinear conservation law in the vanishing viscosity limit was given in [9].

The problem under view was first introduced and studied by Coron and Guerrero [5]. Next Guerrero and Lebeau [10] extended some of the results of [5] in arbitrary dimension and with a variable vector field M. In these papers, it is proven that if the vector field M is such that the transport equation is not controllable (because there is a characteristic of M which stays in the domain without reaching the control zone ω) then the size of the control can grow as $e^{C/\varepsilon}$. On the other side, if all the characteristics stay sufficiently long in the control zone ω or outside $\overline{\Omega}$, then the system uniformly zero-controllable. These results require that T is large enough, and in particular in [5] it is proven that in the one-dimensional case that (1) is uniformly zero-controllable when $M > 0$ provided that $T > 4.3L/M$, and when $M < 0$ provided that $T > 57.2L/|M|$. Clearly the transport equation ($\varepsilon = 0$) is controllable for $T \geqslant L/|M|$ (this time being optimal), so one could expect that in both cases the uniform controllability to hold for any time $T > L/|M|$. A very surprising result of [5] is that when $M < 0$, the control can blow up exponentially for any $T < 2L/|M|$, while this is shown only for times $T < L/M$ when $M > 0$ (which is much more intuitive).

What we establish in this paper is that we can improve the times $4.3\,L/M$ and $57.2\,L/|M|$ of Coron and Guerrero's paper to $T > 4.2\,L/M$ and $T > 6.1\,L/|M|$ respectively. Also (and perhaps more importantly), our proof is of completely different nature. Coron and Guerrero used a Carleman estimate to prove the observability inequality of the adjoint problem, and showed that the explosive nature of the constant coming from this Carleman estimate as $\varepsilon \to 0^+$ can be compensated by the constant of a dissipation estimate (the solution of (1) or its adjoint equation naturally decreases for $T > 1/|M|$, exponentially in $-1/\varepsilon$ as $\varepsilon \to 0^+$), provided that T is large enough. Here, our method is closer to Russell's harmonic analysis approch to some controllability problems (see in particular Fattorini-Russell [6] and Russell [18]). The observation inequality for the adjoint system is connected to a question concerning sum of exponentials. This requires the construction of some bi-orthogonal family to the family of exponentials, which relies on the Paley-Wiener theorem. Some analogous methods can be found for instance in [20, 22, 21, 16, 23], but here the core of the proof is slightly different and relies on the construction of a complex "multiplier" due to Beurling and Malliavin [3].

Precisely, we show the following result.

Theorem 1. *Given $M \neq 0$ and $T > 0$, the system (1) is uniformly zero-controllable in the sense that there exist constants $\kappa > 0$ and $K > 0$ such that for any $u_0 \in L^2(0, L)$, any $\varepsilon \in (0, 1)$, there exists $v \in L^2(0, T)$ such that the solution of (1) satisfies (2), and moreover*

$$\|v\|_{L^2(0,T)} \leqslant K \exp\Big(- \frac{\kappa}{\varepsilon}\Big) \|u_0\|_{L^2(0,L)}, \tag{3}$$

provided that:

$$T > 4.2\,\frac{L}{M} \text{ if } M > 0, \tag{4}$$

$$T > 6.1\,\frac{L}{|M|} \text{ if } M < 0. \tag{5}$$

Remark 1. *The conjecture that the optimal times should be $1/M$ and $2/|M|$ is hence still open. We believe that the complex analytic technique could be a good approach to solve the problem, probably by finding a more accurate complex multiplier.*

2 Notations and preliminaries

2.1 Observability inequality

It is a standard fact (see Lions [15] and Russell [18]) that proving Theorem 1 is equivalent to establish an observability inequality for the adjoint equation with a constant as in (3). Precisely the adjoint equation is the following

$$\begin{cases} \varphi_t + M\varphi_x + \varepsilon\varphi_{xx} = 0 \text{ in } (0,T)\times(0,L), \\ \varphi = 0 \text{ on } (0,T)\times\{0,L\}, \\ \varphi(T,\cdot) = \varphi_T \text{ in } (0,L). \end{cases} \tag{6}$$

It is then sufficient to show that for some $\kappa > 0$ and $K > 0$, one has for any $\varepsilon \in (0,1)$ and any $\varphi_T \in L^2(0,L)$, one has

$$\|\varphi(0,\cdot)\|_{L^2(0,L)} \leqslant K \exp\Big(-\frac{\kappa}{\varepsilon}\Big)\|\partial_x\varphi(\cdot,0)\|_{L^2(0,T)}. \tag{7}$$

2.2 The operator $-M\partial_x - \varepsilon\partial^2_{xx}$

To diagonalize the operator

$$P := -M\partial_x - \varepsilon\partial^2_{xx},$$

it suffices to remark that

$$\partial^2_{xx}(e^{\frac{Mx}{2\varepsilon}}u) = e^{\frac{Mx}{2\varepsilon}}\left(\partial^2_{xx}u + \frac{M}{\varepsilon}\partial_x u + \frac{M^2}{4\varepsilon^2}u\right),$$

that is to say with the obvious notation for the multiplication operator

$$P = -\varepsilon e^{-\frac{Mx}{2\varepsilon}} \circ \partial^2_{xx} \circ e^{\frac{Mx}{2\varepsilon}} + \frac{M^2}{4\varepsilon}\mathrm{Id}. \tag{8}$$

It follows that P is diagonalizable in $L^2(0,L)$, with eigenvectors

$$e_k(x) := \sqrt{2}\exp\Big(-\frac{Mx}{2\varepsilon}\Big)\sin\Big(\frac{k\pi x}{L}\Big). \tag{9}$$

for $k \in \mathbb{N}\setminus\{0\}$ and corresponding eigenvalues

$$\lambda_k := \varepsilon\frac{k^2\pi^2}{L^2} + \frac{M^2}{4\varepsilon}, \tag{10}$$

the family $\{e_k,\ k \in \mathbb{N}\setminus\{0\}\}$ being a Hilbert basis of $L^2(0,L)$ for the $L^2((0,L);\exp(\frac{Mx}{\varepsilon})\,dx)$ scalar product:

$$< u, v > := \int_0^L \exp\Big(\frac{Mx}{\varepsilon}\Big)u(x)v(x)\,dx. \tag{11}$$

3 Proof of Theorem 1

3.1 General strategy

The strategy to prove Theorem 1 is connected to the method of moments, see for instance [6, 16, 18, 20, 21, 22]. The idea is to construct a biorthogonal family in $L^2(0,T)$ to the family of exponentials

$$t \mapsto \exp(-\lambda_k(T-t)). \tag{12}$$

By the change of variables $t \mapsto T-t$, we can of course consider the family of exponentials

$$t \mapsto \exp(-\lambda_k t). \tag{13}$$

To that purpose, as in the complex-analytic proof of the Müntz-Szász theorem (see for instance [17, 19]) the idea is to construct a suitable family $J_k(z)$ of entire functions of exponential type (see e.g. [12]), satisfying

$$J_k(-i\lambda_j) = \delta_{jk}, \tag{14}$$

where δ_{jk} is the Kronecker symbol. Then using the Paley-Wiener theorem we deduce our biorthogonal family ψ_k as the inverse Fourier transform of $J_k(z)$ (up to a translation in time). The family $J_k(z)$ is constructed from a single entire function having simple poles at $(-i\lambda_k)_{k\in\mathbb{N}\setminus\{0\}}$. This function is naturally constructed as a Weierstrass product (which turns out to be explicit here), multiplied by a function (which we will designate as a “multiplier”) intended to make J_k of relevant exponential type and with suitable behaviour on the real axis. Such a method can be traced back to Paley and Wiener [17]. The construction of the multiplier which we employ here follows the work of Beurling and Malliavin [3].

Once the biorthogonal family is constructed with suitable estimates, obtaining the observability inequality (7) is rather straightforward.

We develop these main steps in the following subsections.

3.2 The Weierstrass product Φ

An entire function having the k^2, $k \in \mathbb{N} \setminus \{0\}$ as its simple zeros is the following one:

$$\prod_{k=1}^{\infty} \left(1 - \frac{z}{k^2}\right) = \frac{\sin(\pi\sqrt{z})}{\pi\sqrt{z}}, \tag{15}$$

which is an entire function (despite the square roots). Now one can construct a function having simple zeros exactly at $\{-i\lambda_k,\ k \in \mathbb{N} \setminus \{0\}\ \}$ by

$$\Phi(z) = \frac{\sin\left(\frac{L}{\sqrt{\varepsilon}}\sqrt{iz - \frac{M^2}{4\varepsilon}}\right)}{\frac{L}{\sqrt{\varepsilon}}\sqrt{iz - \frac{M^2}{4\varepsilon}}}. \tag{16}$$

It is elementary to see that Φ is of exponential type, and even satisfies

$$|\Phi(z)| \leqslant C(M,\varepsilon)\exp(\frac{L}{\sqrt{2\varepsilon}}\sqrt{|z|}) \text{ as } |z| \to +\infty. \tag{17}$$

A good candidate for $J_k(z)$ would be

$$\frac{\Phi(z)}{\Phi'(-i\lambda_k)(z + i\lambda_k)}, \tag{18}$$

but precisely because of (17), one could show by the Phragmen-Lindelöf method that such a function cannot be bounded on the real line, and hence it cannot be used directly to construct the family ψ_k by inverse Fourier transform. We must use a multiplier to “mollify” the function on the real line without perturbing too much the behavior at the above zeros.

3.3 Beurling and Malliavin’s multiplier

We follow Beurling and Malliavin’s construction [3] (see also Koosis [12, Chapter X]). We fix

$$a := \frac{T}{2\pi}, \tag{19}$$

and

$$\tilde{L} := L + \alpha\varepsilon^{1/4} \text{ and } \hat{L} := L + 2\alpha\varepsilon^{1/4}, \tag{20}$$

with α a positive real number independent of ε to be chosen later.

Let us introduce

$$s(t) = at - \frac{\tilde{L}}{\pi\sqrt{2\varepsilon}}\sqrt{t}. \tag{21}$$

Using that ([3, p. 294])

$$\int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| dt^{\gamma} = |x|^{\gamma}\pi\cot\frac{\pi\gamma}{2} \text{ for } 0 < \gamma < 2, \tag{22}$$

we see that

$$\int_0^{\infty} \log\left|1 - \frac{x^2}{t^2}\right| ds(t) = -\frac{\tilde{L}}{\sqrt{2\varepsilon}}\sqrt{|x|}. \tag{23}$$

We notice that s is increasing for t larger than

$$A := \frac{1}{2\varepsilon}\left(\frac{\tilde{L}}{T}\right)^2. \tag{24}$$

We also introduce

$$B := 4A = \frac{2}{\varepsilon}\left(\frac{\tilde{L}}{T}\right)^2, \tag{25}$$

which satisfies $s(B) = 0$. Now one defines ν as the restriction of the measure $ds(t)$ to the interval $[B, +\infty)$. Let us underline that this measure is positive.

Next we introduce for $z \in \mathbb{C}$:

$$U(z) := \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| d\nu(t) = \int_B^\infty \log\left|1 - \frac{z^2}{t^2}\right| ds(t), \tag{26}$$

and for $z \in \mathbb{C} \setminus \mathbb{R}$

$$g(z) := \int_0^\infty \log\left(1 - \frac{z^2}{t^2}\right) d\nu(t) = \int_B^\infty \log\left(1 - \frac{z^2}{t^2}\right) ds(t). \tag{27}$$

By "atomizing" the measure $d\nu$ in the above integral, we can define

$$\tilde{U}(z) := \int_0^\infty \log\left|1 - \frac{z^2}{t^2}\right| d[\nu(t)], \tag{28}$$

where $[\,\cdot\,]$ denotes the integer part and where

$$\nu(t) = \int_0^t d\nu. \tag{29}$$

In the same way as previously we introduce

$$h(z) := \int_0^\infty \log\left(1 - \frac{z^2}{t^2}\right) d[\nu](t). \tag{30}$$

Of course,

$$U(z) = \mathrm{Re}(g(z)) \text{ and } \tilde{U}(z) = \mathrm{Re}(h(z)).$$

The main advantage of $\tilde{U}$ (and h) over U is that now $\exp(h(z))$ is an entire function. Indeed, calling $\{\mu_k,\ k \in \mathbb{N}\}$ the discrete set in $\mathbb{R}$ consisting of the discontinuities of the function $t \mapsto [\nu(t)]$, we have

$$\exp(h(z)) = \prod_{k\in\mathbb{N}}\left(1 - \frac{z^2}{\mu_k^2}\right). \tag{31}$$

The convergence of this product is quite straightforward.

Finally, the multiplier which we will use is the following:

$$f(z) := \exp(h(z - i)). \tag{32}$$

3.4 Estimates on the multiplier

Before constructing the functions J_k themselves, let us prove some lemmas which will be useful to obtain properties on f.

Lemma 1. *For $x \in \mathbb{R}$, one has*

$$U(x) \leqslant -\frac{\tilde{L}}{\sqrt{2\varepsilon}}\sqrt{|x|} + C_1 aB, \tag{33}$$

where C_1 is the following positive (and finite) constant

$$C_1 := -\min_{x\in\mathbb{R}} \int_0^1 \log\left|1 - \frac{x^2}{t^2}\right| d(t - \sqrt{t}) \ \simeq 2.34 < 2.35. \tag{34}$$

Proof. Following (23), we have

$$U(x) + \frac{\tilde{L}}{\sqrt{2\varepsilon}}\sqrt{|x|} = -\int_0^B \log\left|1 - \frac{x^2}{t^2}\right| ds,$$

which immediately gives (34) after the change of variable $t \mapsto t/B$. Now that the constant C_1 is finite follows from explicit integration:

$$\int_0^1 \log\left|1 - \frac{x^2}{t^2}\right| d(t - \sqrt{t}) = -\pi\sqrt{x} + x\ln\left|\frac{x+1}{x-1}\right| - \sqrt{x}\ln\left|\frac{\sqrt{x}+1}{\sqrt{x}-1}\right| + 2\sqrt{x}\arctan(\sqrt{x}). \tag{35}$$

□

Lemma 2. *For* $\mathrm{Im}(z) < 0$, *we have*

$$U(z) = -\pi a \mathrm{Im}(z) - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}(z)U(t)}{|z-t|^2}\, dt. \tag{36}$$

Proof. This is essentially [12, Vol. I, Theorem G.1, p. 47] (see also [12, Vol. II, p. 161]). We recall this result for the reader's convenience.

Theorem 2. *Let* $f(z)$ *be analytic in* $\mathrm{Im}(z) > 0$ *and at the points of the real axis. Suppose that*

$$\log|f(z)| \leqslant \mathcal{O}(|z|),$$

for $\mathrm{Im}(z) \geqslant 0$ *and* $|z|$ *large, and that*

$$\int_{-\infty}^{+\infty} \frac{\log^+|f(x)|}{1+x^2} dx < \infty.$$

Then if $f(z)$ *has no zeros in* $\mathrm{Im}(z) > 0$,

$$\log|f(z)| = \mathcal{A}\,\mathrm{Im}(z) + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\mathrm{Im}(z)\log|f(t)|}{|z-t|^2}\, dt,$$

there where

$$\mathcal{A} = \limsup_{y\to+\infty} \frac{\log|f(iy)|}{y}.$$

We notice that for any $y \in \mathbb{R}$ we have

$$U(iy) = \int_0^\infty \log\left|1 + \frac{y^2}{t^2}\right| d\nu,$$

so that using

$$\frac{\nu(t)}{t} \to a \text{ as } t \to +\infty,$$

and integrating by parts we deduce

$$\limsup_{y\to+\infty} \frac{U(\pm iy)}{\pm y} = \pi a, \tag{37}$$

Now applying Theorem 2 to $\exp(g(-z))$ would yield the result, except that U is not analytic at the points of the real axis. But this is just a matter of considering $\exp(g(-z - i\tau))$ for small $\tau > 0$ and passing to the limit by dominated convergence.

□

Lemma 3. *For* $x \in \mathbb{R}$, *one has*

$$U(x - i) \leqslant \pi a + C_1 a B - \frac{\tilde{L}}{\sqrt{2\varepsilon}}\sqrt{|x|}. \tag{38}$$

Proof. We apply (33) and (36); since

$$\int_{-\infty}^{\infty} \frac{1}{|x-i-t|^2}\,dt = \int_{-\infty}^{\infty} \frac{1}{1+|x-t|^2}\,dt = \pi,$$

there is left to compute

$$\int_{-\infty}^{\infty} \frac{\sqrt{|t|}}{1+|x-t|^2}\,dt.$$

This can be cut into two integrals which are computed in a standard way via the respective changes of variable $u=\sqrt{t}$ and $u=\sqrt{-t}$:

$$\int_{0}^{\infty} \frac{\sqrt{t}}{1+(x-t)^2}\,dt = \frac{\pi}{\sqrt{2\sqrt{1+x^2}-2x}},$$

and

$$\int_{-\infty}^{0} \frac{\sqrt{-t}}{1+(x-t)^2}\,dt = \frac{\pi}{\sqrt{2\sqrt{1+x^2}+2x}}.$$

By considering $x>0$ and $x<0$ we see that

$$\left(\sqrt{2\sqrt{1+x^2}+2x}+\sqrt{2\sqrt{1+x^2}-2x}\right) \geqslant 2\sqrt{|x|},$$

and the result follows. □

Lemma 4. *We have for $z=x+iy\in\mathbb{C}$:*

$$\int_0^{\infty} \log\left|1-\frac{z^2}{t^2}\right| d([\nu](t)-\nu(t)) \leqslant \log\Big(\frac{\max(|x|,|y|)}{2|y|}+\frac{|y|}{2\max(|x|,|y|)}\Big). \tag{39}$$

Proof. This is [12, Vol. II, Lemma, p. 162]. □

Lemma 5. *Denote*

$$G(y) := \int_0^{1} \log\left|1+\frac{y^2}{t^2}\right| d(t-\sqrt{t}). \tag{40}$$

For any $y\in\mathbb{R}$ one has

$$\int_0^{\infty} \log\left|1+\frac{y^2}{t^2}\right| dt = \pi y, \tag{41}$$

$$\int_0^{\infty} \log\left|1+\frac{y^2}{t^2}\right| d\sqrt{t} = \pi\sqrt{2|y|}, \tag{42}$$

$$\int_0^{B} \log\left|1+\frac{y^2}{t^2}\right| ds = aBG\left(\frac{y}{B}\right). \tag{43}$$

Proof. These are easily obtained by integration by parts and change of variable, and noting $s(B)=0$. □

Lemma 6. *For all $y\in\mathbb{R}$ one has*

$$\int_B^{\infty} \log\left(1+\frac{y^2}{t^2}\right) d[s] \geqslant \int_B^{\infty} \log\left(1+\frac{y^2}{t^2}\right) ds - \log\left(1+\frac{y^2}{B^2}\right). \tag{44}$$

Proof. By integrating by parts, recalling that $s(B)=0$ and using $0\leqslant s(t)-[s(t)]\leqslant 1$, we obtain

$$\begin{aligned}
\int_B^{\infty} \log\left(1+\frac{y^2}{t^2}\right) d([s]-s) &\geqslant \int_B^{\infty} \partial_t\left[\log\left(1+\frac{y^2}{t^2}\right)\right](s(t)-[s(t)])\,dt \\
&\geqslant \int_B^{\infty} \partial_t\left[\log\left(1+\frac{y^2}{t^2}\right)\right] dt \\
&= -\log\left(1+\frac{y^2}{B^2}\right).
\end{aligned}$$

□

The conclusion of this paragraph is the following

Proposition 1. *The function $\tilde{U}$ constructed above satisfies the following properties for some $C > 0$:*

$$\forall x \in \mathbb{R},\ \tilde{U}(x-i) \leqslant -\frac{\tilde{L}}{\sqrt{2\varepsilon}}\sqrt{|x|} + aBC_1 + \log^+(|x|) + \pi a, \tag{45}$$

$$\forall y \in \mathbb{R}^-,\ \tilde{U}(iy) \geqslant \pi a|y| - \frac{\tilde{L}}{\sqrt{\varepsilon}}\sqrt{|y|} - \log\left(1+\frac{y^2}{B^2}\right) - aBG\left(\frac{y}{B}\right). \tag{46}$$

Proof. Estimate (45) is a direct consequence of Lemmata 3 and 4, while estimate (46) follows from Lemmata 5 and 6 and the fact that $y \mapsto \tilde{U}(iy)$ is monotonous on $\mathbb{R}^-$. □

3.5 The biorthogonal family ψ_k

Now we introduce the function for any $k \in \mathbb{N} \setminus \{0\}$:

$$\tilde{J}_k(z) := \frac{\Phi(z)}{\Phi'(-i\lambda_k)(z+i\lambda_k)}\frac{f(z)}{f(-i\lambda_k)}. \tag{47}$$

The construction of Paragraph 3.3 was performed in order to get the following result.

Proposition 2. *For any $k \in \mathbb{N} \setminus \{0\}$, the function $\tilde{J}_k$ is an entire function of exponential type πa. Moreover for $\varepsilon > 0$ small enough independent of k, it satisfies on the real line*

$$|\tilde{J}_k(x)| \leqslant C\exp\left(\frac{L|M|}{2\varepsilon} + \frac{1}{\pi}(C_1 - C_2)\frac{\tilde{L}^2}{T\varepsilon} - \frac{T}{2}\lambda_k + \frac{\hat{L}}{\sqrt{\varepsilon}}\sqrt{\lambda_k}\right)(1+|x|)^{-3/2}. \tag{48}$$

where

$$C_2 := -G(2) \simeq 1.97 > 1.95. \tag{49}$$

Proof. That $\tilde{J}_k$ is an entire function follows from the fact that Φ is entire and has only simple zeros at $-i\lambda_k$ and that f is an entire function with $f(-i\lambda_k) \neq 0$. From (18), we see that in order to prove that $\tilde{J}_k$ is of exponential type $\pi a = T/2$, it is sufficient to prove that f is of exponential type πa. That h satisfies $|h(z)| \leqslant C\exp(\pi a|z|)$ is a consequence of Theorem 2 and (37) being valid for $\tilde{U}$. It follows that f is also of exponential type $T/2$.

Now let us turn to estimate (48). Using (16) and the fact that for $y \in \mathbb{R}^-$, $x \in \mathbb{R} \mapsto \mathrm{Im}(\sqrt{ix+y} - \sqrt{ix})$ is maximal at $x = 0$, we infer

$$|\Phi(x)| \leqslant \frac{\exp\left(\frac{L|M|}{2\varepsilon} + \frac{L}{\sqrt{2\varepsilon}}\sqrt{|x|}\right)}{L\varepsilon^{-1/2}\left|x^2 + \frac{M^4}{16\varepsilon^2}\right|^{1/4}}.$$

Using (45), we infer

$$\begin{aligned} |\Phi(x)\exp(\tilde{U}(x-i))| &\leqslant \frac{\exp\left(\frac{L|M|}{2\varepsilon} - \frac{\tilde{L}-L}{\sqrt{2\varepsilon}}\sqrt{|x|} + aBC_1 + \log^+(|x|) + \pi a\right)}{L\varepsilon^{-1/2}\left|x^2 + \frac{M^4}{16\varepsilon^2}\right|^{1/4}} \\ &\leqslant C\varepsilon^{1/2}\frac{\exp\left(\frac{L|M|}{2\varepsilon} + aBC_1\right)}{\left|x^2 + \frac{M^4}{16\varepsilon^2}\right|^{1/4}}, \end{aligned}$$

provided that $\alpha \geqslant \sqrt{2}$ and with C independent of ε.

Now a direct computation yields

$$\Phi'(-i\lambda_k) = \frac{(-1)^k}{2\varepsilon\lambda_k}.$$

Finally, by (46) we get

$$|f(-i\lambda_k)| \geqslant c\exp\left(\pi a\lambda_k - \frac{\tilde{L}}{\sqrt{\varepsilon}}\sqrt{\lambda_k} - \log\left(1+\frac{\lambda_k^2}{B^2}\right) - aBG\left(\frac{\lambda_k}{B}\right)\right).$$

Using for instance $\log(1+y^2/24) \geqslant \sqrt{|y|}$, we infer that for α large enough and independently of k and $\varepsilon \in (0,1)$ one has

$$|f(-i\lambda_k)| \geqslant c \exp\left(\pi a \lambda_k - \frac{\hat{L}}{\sqrt{\varepsilon}}\sqrt{\lambda_k} - aBG\left(\frac{\lambda_k}{B}\right)\right).$$

Putting all these estimates together yields

$$|\tilde{J}_k(x)| \leqslant C \frac{\exp\left(\frac{L|M|}{2\varepsilon} + aBC_1 - \pi a \lambda_k + \frac{\hat{L}}{\sqrt{\varepsilon}}\sqrt{\lambda_k} - aBG\left(\frac{\lambda_k}{B}\right)\right)}{\left|x^2 + \frac{M^2}{4\varepsilon}\right|^{1/4} |x^2 + \lambda_k^2|^{1/2}}. \tag{50}$$

Concerning the last term in the exponential, we use that in both cases $T > 4L/|M|$ so that

$$\frac{\lambda_k}{B} \geqslant \frac{M^2}{8}\frac{T^2}{\tilde{L}^2} \geqslant 2, \tag{51}$$

(at least for ε small so that $T|M|/\tilde{L} > 4$) and the fact that G is a negative decreasing function. For larger ε it suffices to enhance a little bit the constant C in (50). □

Remark 2. *The constant C_2 could be optimized a little bit further by making the optimization later (see Proposition 3).*

Now from Proposition 2 and the Paley-Wiener theorem, we deduce that $\tilde{J}_k$ is the Fourier-Laplace transform of some function $\tilde{\psi}_k \in L^2(\mathbb{R})$, supported in $[-T/2, T/2]$. Now we define

$$J_k(z) = \frac{\exp(-i\frac{T}{2}z)}{\exp(-\frac{T}{2}\lambda_k)} \tilde{J}_k(z). \tag{52}$$

We deduce that J_k is the Fourier-Laplace transform of the function $\psi_k := \mathcal{T}_{T/2}\tilde{\psi}_k$, supported in $[0,T]$, where $\mathcal{T}_{T/2}$ is the translation at the source by $T/2$.

From (48) and (52), we moreover deduce that for $x \in \mathbb{R}$

$$|J_k(x)| \leqslant C \exp\left(\frac{L|M|}{2\varepsilon} + \frac{1}{\pi}(C_1 - C_2)\frac{\tilde{L}^2}{T\varepsilon} + \frac{\hat{L}}{\sqrt{\varepsilon}}\sqrt{\lambda_k}\right) \frac{1}{(1+|x|)^{3/2}}. \tag{53}$$

Moreover, due to (47) and (52), we have

$$J_k(i\lambda_j) = \delta_{jk}. \tag{54}$$

Finally Parseval's identity yields

$$\|\psi_k\|_{L^2(\mathbb{R})} \leqslant C \exp\left(\frac{L|M|}{2\varepsilon} + \frac{1}{\pi}(C_1 - C_2)\frac{\tilde{L}^2}{T\varepsilon} + \frac{\hat{L}}{\sqrt{\varepsilon}}\sqrt{\lambda_k}\right), \tag{55}$$

and (54) translates into

$$\int_0^T \psi_k(t) \exp(-\lambda_j t)\, dt = \delta_{jk}. \tag{56}$$

As mentionned in Paragraph 3.1, we will in fact consider $t \mapsto \psi_k(T-t)$. We will still call the resulting function ψ_k. The new family (ψ_k) still satisfies (55), and now (56) is replaced by

$$\int_0^T \psi_k(t) \exp(-\lambda_j (T-t))\, dt = \delta_{jk}. \tag{57}$$

3.6 The constants

The constants of the main statement appear in the next result.

Proposition 3. *We have for some* $\kappa > 0$

$$\frac{L|M|}{2\varepsilon} + \frac{1}{\pi}(C_1 - C_2)\frac{L^2}{T\varepsilon} - T\lambda_k + \frac{L}{\sqrt{\varepsilon}}\sqrt{\lambda_k} \leqslant -\kappa\lambda_k \quad \text{for all } k, \tag{58}$$

provided that

$$T > \frac{L}{|M|}c_+ \text{ with } c_+ := 2 + \sqrt{4 + \frac{4}{\pi}(C_1 - C_2)} < 4.2, \tag{59}$$

and we have for some $\kappa > 0$

$$\frac{L|M|}{\varepsilon} + \frac{1}{\pi}(C_1 - C_2)\frac{L^2}{T\varepsilon} - T\lambda_k + \frac{L}{\sqrt{\varepsilon}}\sqrt{\lambda_k} \leqslant -\kappa\lambda_k \quad \text{for all } k, \tag{60}$$

provided that

$$T > \frac{L}{|M|}c_- \text{ with } c_- := 3 + \sqrt{9 + \frac{4}{\pi}(C_1 - C_2)} < 6.1. \tag{61}$$

Proof. First we notice that

$$x \mapsto -Tx + \frac{L}{\sqrt{\varepsilon}}\sqrt{x},$$

is decreasing for values larger than $\frac{1}{4\varepsilon}\frac{L^2}{T^2} \leqslant \frac{M^2}{4\varepsilon}$ (in both cases). Next we only use that for all k,

$$\lambda_k \geqslant \frac{M^2}{4\varepsilon}, \tag{62}$$

hence we are led to decide when T is larger than the larger root of the polynomial

$$\frac{1}{\pi}(C_1 - C_2)\frac{L^2}{\varepsilon} - \frac{M^2}{2\varepsilon}X^2 + X\frac{L|M|}{\varepsilon},$$

for (58), respectively

$$\frac{1}{\pi}(C_1 - C_2)\frac{L^2}{\varepsilon} - \frac{M^2}{2\varepsilon}X^2 + X\frac{3L|M|}{2\varepsilon},$$

for (60). Obvious computations give (59)-(61), and the estimates of c_- and c_+ come from (34) and (49). □

Remark 3. *We do not use the "*$\varepsilon k^2\pi^2/L^2$*" part of* λ_k*, that is in some sense, we do not benefit from the high frequencies. Another possible strategy would be to use this part to absorb the term* $\frac{2}{\pi}(C_1 - C_2)\frac{L^2}{\varepsilon}$*, and to treat the low frequencies in another way, for instance by using the "spectral inequality" of Lebeau-Robbiano [13], Lebeau-Zuazua [14], Jerison-Lebeau [11] together with a dissipation estimate. But the constant appearing in this inequality is not explicit, so the constants* c_- *and* c_+ *would not be either.*

3.7 Deducing the observability inequality

Consider a solution φ of (6), where

$$\varphi_T(x) = \sum_{k=1}^{N} c_k e_k(x). \tag{63}$$

It is not restrictive to consider φ_T as the combination of a finite number of modes, since the inequalities which follow are independent of N. We see that

$$\varphi(t,x) = \sum_{k=1}^{N} c_k \exp(-\lambda_k(T-t))e_k(x), \tag{64}$$

and consequently

$$\sqrt{2}k\frac{\pi}{L}c_k = \int_0^T (\partial_x\varphi)(t,0)\,\psi_k(t)\,dt.$$

Hence we deduce

$$|c_k| \leqslant \frac{L}{\sqrt{2}\pi k}\|\partial_x\varphi_{|x=0}\|_{L^2(0,T)}\|\psi_k\|_{L^2(0,T)}. \tag{65}$$

And of course,

$$\varphi(0,x) = \sum_{k=1}^{N} c_k \exp(-\lambda_k T)e_k(x). \tag{66}$$

From (65) and (66) we deduce

$$\|\varphi(0,x)\|_{L^2(0,L)} \leqslant C\|\partial_x\varphi\|_{L^2(0,T)}\sum_{k=1}^{N}\frac{1}{k}\exp(-\lambda_k T)\|e_k(x)\|_{L^2(0,L)}\|\psi_k\|_{L^2(0,T)}. \tag{67}$$

Now let us distinguish between the two cases $M > 0$ and $M < 0$.

Case 1. If $M > 0$, then

$$\|e_k(x)\|_{L^2(0,L)} \leqslant 1.$$

Hence using (55) and (67), we finally deduce

$$\|\varphi(0,x)\|_{L^2(0,L)} \leqslant C\sum_{k=1}^{N}\frac{1}{k}\exp\left(\frac{L|M|}{2\varepsilon} + \frac{1}{\pi}(C_1 - C_2)\frac{\tilde{L}^2}{T\varepsilon} - T\lambda_k + \frac{\hat{L}}{\sqrt{\varepsilon}}\sqrt{\lambda_k}\right)\|\partial_x\varphi\|_{L^2(0,T)}. \tag{68}$$

Using (58) we deduce

$$\|\varphi(0,x)\|_{L^2(0,L)} \leqslant C\|\partial_x\varphi\|_{L^2(0,T)}\sum_{k=1}^{N}\exp(-\frac{\kappa}{2}\lambda_k + \frac{\hat{L}-L}{\sqrt{\varepsilon}}\sqrt{\lambda_k})\frac{1}{k}\exp(-\frac{\kappa}{2}\lambda_k).$$

It is not difficult to see that for some constant $C > 0$ independent of ε one has

$$-\frac{\kappa}{2}\lambda_k + \frac{\hat{L}-L}{\sqrt{\varepsilon}} + \frac{1}{\pi}(C_1 - C_2)\frac{\tilde{L}^2 - L^2}{T\varepsilon} \leqslant C - \frac{\kappa}{3}\lambda_k \leqslant C - \frac{\kappa}{3}\frac{M^2}{4\varepsilon},$$

and that

$$\begin{aligned}\sum_{k=1}^{N}\frac{1}{k}\exp(-\frac{\kappa}{2}\lambda_k) &\leqslant \sum_{k=1}^{N}\frac{1}{k}\exp\left(-\frac{\varepsilon\kappa\pi^2}{2L^2}k^2\right)\\ &\leqslant \sum_{k=1}^{\infty}\exp\left(-\frac{\varepsilon\kappa\pi^2}{2L^2}k\right)\\ &\leqslant \frac{C(T,L,M)}{\varepsilon}.\end{aligned}$$

This gives the desired result.

Case 2. If $M < 0$, then

$$\|e_k(x)\|_{L^2(0,L)} \leqslant \exp\left(\frac{L|M|}{2\varepsilon}\right).$$

Hence using (55) and (67), we finally deduce

$$\|\varphi(0,x)\|_{L^2(0,L)} \leqslant C\sum_{k=1}^{N}\frac{1}{k}\exp\left(\frac{L|M|}{\varepsilon} + \frac{1}{\pi}(C_1 - C_2)\frac{\tilde{L}^2}{T\varepsilon} - T\lambda_k + \frac{\hat{L}}{\sqrt{\varepsilon}}\sqrt{\lambda_k}\right)\|\partial_x\varphi\|_{L^2(0,T)}, \tag{69}$$

and we conclude as previously by using (60). This concludes the proof of Theorem 1.

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References

[1] Ancona F., Coclite G.M., On the attainable set for Temple class systems with boundary controls, SIAM J. Control Optim. 43 (2005), no. 6, pp. 2166–2190.

[2] Ancona F., Marson A., On the attainable set for scalar nonlinear conservation laws with boundary control, SIAM J. Control Optim. 36 (1998), no. 1, pp. 290–312.

[3] Beurling, A., Malliavin P., On Fourier transforms of measures with compact support, Acta Math. 107 (1962), pp. 291–309.

[4] Bressan A., Coclite G.M., On the boundary control of systems of conservation laws, SIAM J. Control Optim. 41 (2002), no. 2, pp. 607–622.

[5] Coron J.-M., Guerrero S., Singular optimal control: A linear 1-D parabolic-hyperbolic example, Asymp. Anal. 44 **3,4** (2005), pp. 237–257.

[6] Fattorini H. O., Russell D. L., Exact controllability theorems for linear parabolic equations in one space dimension, Arch. Rational Mech. Anal. 43 (1971), pp. 272–292.

[7] Fursikov A., Imanuvilov O. Yu., *Controllability of Evolution Equations*, Lecture Notes #34, Seoul National University, Korea, 1996.

[8] Glass O., On the controllability of the 1-D isentropic Euler equation, J. European Math. Soc. 9, no. 3 (2007) pp. 427–486.

[9] Glass O., Guerrero S., On the uniform controllability of the Burgers equation, SIAM J. Control Optim. 46 (2007), no. 4, pp. 1211–1238.

[10] Guerrero S., Lebeau G., Singular optimal control for a transport-diffusion equation, Comm. Partial Differential Equations 32 (2007), no. 10-12, pp. 1813–1836.

[11] Jerison D., Lebeau G., Nodal sets of sums of eigenfunctions, *Harmonic analysis and partial differential equations* (Chicago, IL, 1996), pp. 223–239, Chicago Lectures in Math., Univ. Chicago Press, Chicago, IL, 1999.

[12] Koosis P., *The logarithmic integral I & II*, Cambridge Studies in Advanced Mathematics 12 (1988) & Cambridge Studies in Advanced Mathematics 21 (1992), Cambridge University Press, Cambridge.

[13] Lebeau G., Robbiano L., Contrôle exact de l'équation de la chaleur, Comm. Partial Differential Equations 20 (1995), pp. 335–356.

[14] Lebeau G., E. Zuazua, Null-controllability of a system of linear thermoelasticity. Arch. Rational Mech. Anal. 141 (1998), no. 4, 297–329.

[15] Lions J.-L., *Contrôlabilité exacte, stabilisation et perturbations de systèmes distribués*, Tomes 1 & 2. Masson, RMA 8 & 9, Paris 1988.

[16] Miller L., How violent are fast controls for Schrödinger and plate vibrations?, Arch. Ration. Mech. Anal. 172 (2004), no. 3, pp. 429–456.

[17] Paley R. C., Wiener N., *Fourier transforms in the complex domain.* Reprint of the 1934 original. American Mathematical Society Colloquium Publications, 19. American Mathematical Society, Providence, RI, 1987.

[18] Russell D. L., Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions, SIAM Review 20 (1978), pp. 639–739.

[19] Schwartz L., *Étude des sommes d'exponentielles*, 2ème éd. Publications de l'Institut de Mathématique de l'Université de Strasbourg, V. Actualités Sci. Ind., Hermann, Paris, 1959.

[20] Seidman T. I., Two results on exact boundary control of parabolic equations, Appl. Math. Optim. 11 (1984), no. 2, pp. 145–152.

[21] Seidman T. I., Avdonin S. A., Ivanov S. A., The "window problem" for series of complex exponentials, J. Fourier Anal. Appl. 6 (2000), no. 3, pp. 233–254.

[22] Seidman T. I., Gowda M. S., Norm dependence of the coefficient map on the window size, Math. Scand. 73 (1994), pp. 177–189.

[23] Tenenbaum G., Tucsnak M., New blow-up rates for fast controls of Schrdinger and heat equations. J. Differential Equations 243 (2007), no. 1, pp. 70–100.