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EXISTENCE OF SOLUTIONS FOR THE TWO-DIMENSIONAL STATIONARY EULER SYSTEM FOR IDEAL FLUIDS WITH ARBITRARY FORCE

EXISTENCE DE SOLUTIONS STATIONNAIRES POUR LE SYSTÈME D'EULER BIDIMENSIONNEL DES FLUIDES PARFAITS INCOMPRESSIBLES AVEC UNE FORCE ARBITRAIRE

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ABSTRACT. – We generalize a theorem by J.-M. Coron (see [Sur la stabilisation des fluides parfaits incompressibles bidimensionnels, in: Séminaire Équations aux Dérivées Partielles, École Polytechnique, Centre de Mathématiques, 1998–1999, exposé VII]) and prove the existence of steady states of the Euler system for inviscid incompressible fluids with an arbitrary force term, in a plane bounded domain not necessarily simply connected, if one allows the fluid to pass through a prescribed region of the boundary, which satisfies the necessary condition that each connected component of the boundary is met by it.

RÉSUMÉ. – Nous généralisons un théorème de J.-M. Coron (voir [Sur la stabilisation des fluides parfaits incompressibles bidimensionnels, in: Séminaire Équations aux Dérivées Partielles, École Polytechnique, Centre de Mathématiques, 1998–1999, exposé VII]), en prouvant l'existence d'états stationnaires pour le système d'Euler pour les fluides parfaits incompressibles avec un terme de force arbitraire. Ce résultat se place dans un domaine borné du plan non nécessairement simplement connexe, où le fluide peut entrer à travers une partie prescrite du bord, qui satisfait la condition nécessaire, qu'elle en rencontre toutes les composantes connexes.

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1. Introduction

Let Ω be a nonempty connected bounded smooth open domain in $\mathbb{R}^2$. Consider Σ a nonempty open part of the boundary $\partial\Omega$ of Ω. Denote by ν the unit outward normal on $\partial\Omega$.

The problem that we study in this paper is the existence of solutions of the stationary Euler system for ideal (i.e. inviscid and incompressible) fluids, that is,

$$(y.\nabla)y + \nabla p = f \quad \text{in } \Omega, \tag{1}$$

$$\operatorname{div} y = 0 \quad \text{in } \Omega, \tag{2}$$

where $y : \Omega \to \mathbb{R}^2$ is the velocity field and $p : \Omega \to \mathbb{R}$ is the pressure, for any local force term $f : \Omega \to \mathbb{R}^2$. We consider the following constraint at the boundary:

$$y.\nu = 0 \quad \text{on } \partial\Omega \backslash \Sigma, \tag{3}$$

that is, the fluid is allowed to pass through the boundary only on Σ (and slips on the rest of the boundary).

We show the following result:

THEOREM 1. – *If Σ meets each connected component of $\partial\Omega$, then for any $f \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$, there exist $y \in C^\infty(\overline{\Omega}; \mathbb{R}^2)$ and $p \in C^\infty(\overline{\Omega}; \mathbb{R})$ such that* (1)–(2) *and* (3) *are satisfied.*

Remark 1. – For the closed system (i.e. when $\Sigma = \emptyset$), it is well known that (1)–(3) has no solution in general. For example, consider f with a non trivial circulation on a given connected component of the boundary. Then the Kelvin law for the stationary Euler system, which states that:

$$\int_\Gamma \big[(y.\nabla)y + \nabla p\big].\vec{d\tau} = 0,$$

for any Jordan curve Γ in the domain along which y is everywhere tangent, ensures that there is no solution for that f. By the way, this objection also points out that the condition on Σ that it must meet each connected component of the boundary is necessary.

Remark 2. – Even if we consider only f with vanishing circulations around each connected component of the boundary, there is no solution in general unless Σ meets each connected component of the boundary. Consider indeed f such that $\operatorname{curl} f > 0$ on a given "uncontrolled" connected component Γ of $\partial\Omega$. Let y be a solution of the system. Then, taking the curl of (1), one gets

$$(y.\nabla)(\operatorname{curl} y) = \operatorname{curl} f. \tag{4}$$

This involves in particular that $y \neq 0$ on Γ. As y must be tangent to Γ, it has a constant orientation on Γ. With $\operatorname{curl} f > 0$, this makes (4) impossible.

Theorem 1 was established in the particular case of a simply connected domain by J.M. Coron (see [3]). One of the motivations concerns asymptotic stabilization of the non-stationary Euler system. Indeed, R.W. Brockett established a necessary condition for a finite-dimensional control system to be stabilizable, see [2]. The equivalent of this necessary condition in the infinite-dimensional system considered here is precisely what is proven in Theorem 1. For more precisions concerning the stabilization of ideal fluids, see [3] and [4].

For the three-dimensional system, we do not know whether such a result could be stated. An important step in that direction is given by the work of H.D. Alber (see [1]), which deals with the existence of non trivial steady-states with vanishing force term, in a simply connected domain. But this result uses as an assumption the existence of a reference solution; the existence of such a solution is an open problem in the general case (up to our knowledge). Also, it would be an interesting question to generalize the present work to higher dimensions, in particular in the perspective of the stabilization of three-dimensional ideal fluids, which are known to be exactly controllable (see [6]).

As in [1] and [3], the idea is to find a solution of the problem close to a fixed reference solution. Here, this solution is a potential steady-state of the problem (for $f = 0$). One cannot in general make this solution fit all the requirements of the reference solution of [1]; in particular, in [1], the reference flow v_0 has to satisfy that v_0 does not vanish in $\overline{\Omega}$ and that, on the boundary of $\{x \in \Sigma \mid v_0.\nu < 0\}$, v_0 is pointing outside this set. In our case, when Ω is not simply connected, both conditions can no longer be required (for degree arguments). However, in the two-dimensional case, we can get rid of the latter assumption.

One of the major points in the proof of Theorem 1 is hence the statement of the following proposition, which proves the existence of an appropriate potential reference solution:

PROPOSITION 1. – *Consider Ω a nonempty bounded connected regular domain in $\mathbb{R}^2$. Let ν the unit outward normal on $\partial\Omega$. Consider Σ an open part of $\partial\Omega$, which meets each connected component $\Gamma_0, \ldots, \Gamma_g$ of $\partial\Omega$. Then there exists a function $\theta \in C^\infty(\overline{\Omega}; \mathbb{R})$ which satisfies the following conditions:*

$$\Delta\theta = 0 \quad \text{in } \Omega, \tag{5}$$

$$\partial_\nu\theta = 0 \quad \text{on } \partial\Omega \backslash \Sigma, \tag{6}$$

$$\left|\nabla\theta(x)\right| > 0 \quad \text{for any } x \text{ in } \overline{\Omega}, \tag{7}$$

$$\text{for } \gamma^+(\theta) := \{x \in \partial\Omega \mid \partial_\nu\theta > 0\} \text{ and } \gamma^-(\theta) := \{x \in \partial\Omega \mid \partial_\nu\theta < 0\},$$
$$\text{one has: } \overline{\gamma^+(\theta)} \cap \overline{\gamma^-(\theta)} = \emptyset, \tag{8}$$

$$\gamma^+(\theta) \text{ and } \gamma^-(\theta) \text{ are unions of a finite number of intervals of } \partial\Omega \text{ with disjoint closures,} \tag{9}$$

$$\text{there exist } g \text{ points } M_1, \ldots, M_g \text{ in } \gamma^-(\theta) \cap \Gamma_0, \text{ respectively sent on } \gamma^+(\theta) \cap \Gamma_1, \ldots, \gamma^+(\theta) \cap \Gamma_g \text{ by the flow of } \nabla\theta, \text{ the trajectories not touching } \partial\Omega \backslash \left[\gamma^+(\theta) \cup \gamma^-(\theta)\right]. \tag{10}$$

The proof of this proposition is postponed until Section 4. We consider it as established during Section 2, and aim at proving Theorem 1. In Section 3, we also discuss a generalization of Theorem 1.

2. Proof of Theorem 1

First, we introduce some notations.

2.1. Notations

We shall consider the open ball B_R in $\mathbb{R}^2$, centered in 0, with radius R large enough so that $\overline{\Omega} \subset B_R$. We will also use a regular linear operator π, which extends functions in $C^1(\overline{\Omega};\mathbb{R})$ to functions in $C^1_0(B_R;\mathbb{R})$ (i.e. C^1 functions with compact support), and which sends any C^k-regular function to a C^k-regular function.

Given a vector field $V \in C^1_0(B_R;\mathbb{R}^2)$, we will denote by ϕ^V the corresponding flow, that is the function in $C^1(\mathbb{R}\times\mathbb{R}\times B_R; B_R)$, defined by the following differential system:

$$\begin{cases} \phi^V(t_1,t_1,x) = x \quad \text{for any } (t_1,x)\in\mathbb{R}\times B_R, \\ \partial_{t_2}\phi^V(t_2,t_1,x) = V(\phi^V(t_2,t_1,x)) \quad \text{for any } (t_1,t_2,x)\in\mathbb{R}\times\mathbb{R}\times B_R. \end{cases}$$

When $y\in\phi^V(\mathbb{R}^+,0,x)$, we will write $\phi^V : x\to y$ for the path leading from x to y given by the flow of V.

Given a Jordan curve J, and given two points a and b in J, we denote by $[a,b]_J$ the interval which joins a and b in the direction given on the curve naturally by the orientation in the plane. Given a point $x_0\in J$ and given a positive real number ε, we will denote (when there is no ambiguity) by $x_0+\varepsilon$ the point in J situated at distance ε from x_0, considering the arc length, when following the orientation on the curve, and by $x_0-\varepsilon$ the point obtained when following the opposite way.

We shall introduce, given a point x_0 in a Jordan curve J in the plane, and given a positive (small) real number ε, an extension operator $\mathcal{P}^+_{\varepsilon,x_0}$ which associates to any function function g in $C^1([x_0-\varepsilon,x_0]_J;\mathbb{R})$, a function $\mathcal{P}^+_{\varepsilon,x_0}(g)$ in $C^1([x_0-\varepsilon,x_0+\varepsilon]_J;\mathbb{R})$ such that

$$\begin{cases} \mathcal{P}f = f \quad \text{in } [x_0-\varepsilon,x_0], \\ \operatorname{Supp}(\mathcal{P}f)\cap[x_0,x_0+\varepsilon]\subset[x_0,x_0+\varepsilon), \\ \mathcal{P}f\in C^k([x_0-\varepsilon,x_0+\varepsilon]_J;\mathbb{R}), \quad \forall f\in C^k([x_0-\varepsilon,x_0]_J;\mathbb{R}),\ \forall k\in\mathbb{N}, \\ \|\mathcal{P}f\|_{C^0([x_0-\varepsilon,x_0+\varepsilon]_J;\mathbb{R})}\leqslant\|f\|_{C^0([x_0-\varepsilon,x_0]_J;\mathbb{R})}, \\ \|\mathcal{P}f\|_{C^1([x_0-\varepsilon,x_0+\varepsilon]_J;\mathbb{R})}\leqslant\kappa(\varepsilon)\|f\|_{C^1([x_0-\varepsilon,x_0]_J;\mathbb{R})}. \end{cases}$$

We shall also introduce the operator $\mathcal{P}^-_{\varepsilon,x_0}$ directed in the other way on the curve.

We will consider a function U_{ε,x_0} defined in $C^\infty_0([x_0-\varepsilon,x_0+\varepsilon]_J;\mathbb{R})$ and satisfying:

$$\begin{cases} U_{\varepsilon,x_0}\geqslant 0, \\ \int_{[x_0-\varepsilon,x_0+\varepsilon]_J} U_{\varepsilon,x_0} = 1. \end{cases} \tag{11}$$

Finally, we introduce the family $(\tau_i)_{i=0,\dots,g}$ of functions in $C^\infty(\overline{\Omega};\mathbb{R})$ defined for each i by

$$\begin{cases} \Delta\tau_i = 0 & \text{in } \Omega,\\ \tau_i = 0 & \text{on } \partial\Omega\backslash\Gamma_i,\\ \tau_i = 1 & \text{on } \Gamma_i. \end{cases} \tag{12}$$

It is well-known that $(\nabla^\perp\tau_i)_{i=1,\dots,g}$ is a basis for the first de Rham cohomology space of the domain Ω (and that $\sum_{i=0}^{g}\nabla^\perp\tau_i = 0$), where $\nabla^\perp := (-\partial_2, \partial_1)$.

2.2. Remarks concerning $\nabla\theta$

From this section, we consider a fixed function θ as in Proposition 1. We first give a property of it, and then describe some objects related to it.

PROPOSITION 2. – *There exists $\varepsilon_0 > 0$ such that for all $W \in C^1(\overline{\Omega};\mathbb{R}^2)$ satisfying*

$$\|W - \nabla\theta\|_{C^0(\overline{\Omega};\mathbb{R}^2)} < \varepsilon_0, \tag{13}$$

$$(W - \nabla\theta).\nu = 0 \quad \text{on } \partial\Omega, \tag{14}$$

and for all x in $\overline{\Omega}$, there exists $t > 0$ such that

$$\begin{aligned} &\text{either } \phi^W(0,t,x) \in \gamma^-(\theta),\\ &\text{or } \phi^W(0,t,x) \in \partial\gamma^-(\theta) \text{ with } W \text{ pointing outside } \gamma^-(\theta) \text{ at this point.} \end{aligned} \tag{15}$$

Proof of Proposition 2

First, we establish (15) in the particular case $W = \nabla\theta$. Starting from x, we let the time t become large. Then necessarily the point leaves the domain. Indeed, define $\Theta : t \mapsto \theta(\phi^{\pi(\nabla\theta)}(0,t,x))$. As long as the point $\phi^{\pi(\nabla\theta)}(0,t,x)$ has not left the domain, the derivative of Θ is $|\nabla\theta(\phi^{\pi(\nabla\theta)}(0,t,x))|^2$, and hence is bounded from below by a positive constant. We conclude by using the compactness of the domain. So one deduces (15).

Then, for W close enough to $\nabla\theta$ for the C^0 norm, the flow of W is close to the one of $\nabla\theta$, as shown by the following Gronwall inequality:

$$\begin{aligned} &\bigl|\phi^{\pi(W)}(0,t,x) - \phi^{\pi(\nabla\theta)}(0,t,x)\bigr| \leqslant \bigl\|\pi(W) - \pi(\nabla\theta)\bigr\|_{C^0(\overline{B_R})} \mathrm{e}^{t\|\pi(\nabla\theta)\|_{C^1(\overline{B_R})}},\\ &\forall x \in \overline{B_R},\ \forall t \in \mathbb{R}^+. \end{aligned} \tag{16}$$

Now, for $t > 0$ small, when going back in time a little bit more, the point $\phi^{\pi(\nabla\theta)}(0,t,x)$ is sent outside $\overline{\Omega}$. Thus if W is close enough to $\nabla\theta$, the point $\phi^{\pi(W)}(0,t,x)$ must go out $\overline{\Omega}$ too, which, with (14), involves (15).

Concerning $\nabla\theta$, we will consider the following constants computed from it:

$$\underline{m} := \min_{x\in\overline{\Omega}} \bigl|\nabla\theta(x)\bigr|,$$

$$\overline{T} := \max_{x\in\overline{\Omega}} \inf\bigl\{t \in \mathbb{R}^+ \mid d\bigl(\phi^{\pi(\nabla\theta)}(t,0,x), \overline{\Omega}\bigr) \geqslant d^\star\bigr\},$$

the distance $d^\star$ being chosen sufficiently small, in order that $\overline{T}$ is finite (by the same argument as in Proposition 2, each point in $\overline{\Omega}$ which follows the flow of $\nabla\theta$, must leave the domain in finite time; then the compactness of the domain allows us to define $\overline{T}$ properly).

We then introduce a second extension operator $\tilde{\pi}$ with the same properties as π, except that it satisfies

$$\tilde{\pi}(f)(x) = 0,\ \forall f \in C^1(\overline{\Omega}; \mathbb{R}),\ \forall x \in B_R \text{ such that } \operatorname{dist}(x, \overline{\Omega}) \geqslant d^\star/4. \tag{17}$$

2.3. Remarkable points in the domain

In this section, we distinguish some "special points" in the domain, depending on the function θ previously introduced.

Points A_i, B_i, C_i and D_i

We class the different points in $\partial[\gamma^-(\theta)] \cup \partial[\gamma^+(\theta)]$ in four different categories:

- *The* "*A points*": we will call A the points in $\partial[\gamma^-(\theta)]$ such that at these points, $\nabla\theta$ is pointing *inside* $\gamma^-(\theta)$.
- *The* "*B points*": we will call B the points in $\partial[\gamma^-(\theta)]$ such that at these points, $\nabla\theta$ is pointing *outside* $\gamma^-(\theta)$.
- *The* "*C points*": we will call C the points in $\partial[\gamma^+(\theta)]$ such that at these points, $\nabla\theta$ is pointing *inside* $\gamma^+(\theta)$.
- *The* "*D points*": we will call D the points in $\partial[\gamma^+(\theta)]$ such that at these points, $\nabla\theta$ is pointing *outside* $\gamma^+(\theta)$.

Then, we are interested in the different trajectories of these special points inside $\overline{\Omega}$ by the flow of $\nabla\theta$ and $-\nabla\theta$ (we stop as soon as the point goes out of $\overline{\Omega}$).

We observe that the trajectories of these remarkable points are described by the diagram in Fig. 1. In this diagram, the arrows symbolize the movement of the point under the flow of $\nabla\theta$ in $\overline{\Omega}$. That is, a certain point A comes from a point D (or from a point B) and is then sent to a point in $\gamma^+(\theta)$ or (maybe) to a new point D which itself is sent to a new point A or to a point C, etc. Points in $\gamma^+(\theta) \cup \{C\}$ are end points, whereas points in $\gamma^-(\theta) \cup \{B\}$ are starting points.

Points $\underline{A}_i^W$, $\underline{C}_i^W$ and $\underline{D}_i^W$

For any $W \in C^1(\overline{\Omega}; \mathbb{R}^2)$ sufficiently close to $\nabla\theta$ in $C^0(\overline{\Omega}; \mathbb{R}^2)$, and with the same normal trace on $\partial\Omega$ as $\nabla\theta$, we introduce, for $x \in \overline{\Omega}$,

$$\sigma_W(x) := \min\{t \in [0, +\infty) \mid \phi^{\pi(W)}(0, t, x) \in \gamma^-(\theta) \cup \{B_i,\ i = 1, \ldots\}\},$$

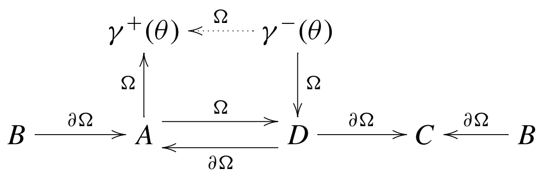

Fig. 1. Description of the trajectories of the points A, B, C and D by the flow of $\nabla\theta$.

which is well defined thanks to Proposition 2.

Now for a point A_i (respectively C_i and D_i) and W such as above, we introduce the point $\underline{A}_i^W$ (respectively $\underline{C}_i^W$ and $\underline{D}_i^W$) as the "point in $\gamma^-(\theta) \cup \{B\}$ where the point A_i (respectively C_i and D_i) is coming from", i.e. more precisely

$$\underline{A}_i^W := \phi^{\pi(W)}\big(0, \sigma_W(A_i), A_i\big), \tag{18}$$

and the equivalent for $\underline{C}_i^W$ and $\underline{D}_i^W$.

For each of the three categories A, C and D, we distinguish two types of points. We denote A° (respectively C°, D°) the set of the A_i points (respectively C_i, D_i) for which $\underline{A}_i^{\nabla\theta}$ (respectively $\underline{C}_i^{\nabla\theta}$, $\underline{D}_i^{\nabla\theta}$) belongs to $\gamma^-(\theta)$. The others, for which $\underline{A}_i^{\nabla\theta}$ (respectively $\underline{C}_i^{\nabla\theta}$, $\underline{D}_i^{\nabla\theta}$) is a B point, will be denoted by A^B (respectively C^B, D^B).

Points $\overline{M}_i^W$

We will consider in the same way, the "points in $\gamma^+(\theta)$ where the points M_i are going to", i.e.

$$\begin{aligned}&\overline{M}_i^W := \phi^{\pi(W)}\big(\sigma'_W(M_i), 0, M_i\big),\\ &\quad \text{where } \sigma'_W(M_i) := \min\big\{t \in (0, +\infty) \mid \phi^{\pi(W)}(t, 0, M_i) \in \gamma^+(\theta)\big\}.\end{aligned}$$

The existence of the points $\overline{M}_i^W$ for W close to $\nabla\theta$ is clear thanks to (10) and (16) – see the remark below.

Points b_i

To any B_i, we associate a point b_i in B_R obtained as

$$\begin{gathered} b_i = \phi^{\pi(\nabla\theta)}(0, t, B_i) \quad \text{for a } t > 0,\\ \frac{3}{4}d^\star \leqslant \operatorname{dist}(b_i, \overline{\Omega}) \leqslant d^\star. \end{gathered}$$

Two constants

Then we consider the following finite set of $\partial\Omega$:

$$\mathcal{M} := \big\{M, \overline{M}^{\nabla\theta}, \underline{(D^\circ)}^{\nabla\theta}\big\}.$$

For the rest of this paper, the positive real number ε_1 (depending only on $\nabla\theta$ – consequently on the domain and Σ) will be defined by

$$\begin{aligned}\varepsilon_1 := \min\big[&\big\{\operatorname{dist}(x, y), x, y \in \mathcal{M}, x \neq y\big\}\\ &\cup \big\{\operatorname{dist}(x, \mathcal{M}), x \in \partial\Omega\backslash\big(\gamma^-(\theta) \cup \gamma^+(\theta)\big)\big\}\\ &\cup \big\{\operatorname{dist}(x, y),\ x \in \overline{\gamma^-(\theta)},\ y \in \overline{\gamma^+(\theta)}\big\}\big].\end{aligned}$$

Also, we fix ε_2 such that for any $W \in C^1(\overline{\Omega}; \mathbb{R}^2)$ with the same normal trace as $\nabla\theta$ on $\partial\Omega$ and for which $\|W - \nabla\theta\|_{C^0(\overline{\Omega})} < \varepsilon_2$, the previous points $\underline{A}_i^W$, $\underline{C}_i^W$, $\underline{D}_i^W$ and $\overline{M}_i^W$ are well defined. We also require, using (16), that the trajectories $\phi^W(t, 0, M_i)$ do not meet $\partial\Omega\backslash\Sigma$.

Strips

We fix a ball around b_i, say $B(b_i, r^\star)$, with $r^\star < d^\star/8$. Under the flow of $\pi(\nabla\theta)$, this ball describes a strip around the parts of $\partial\Omega\backslash[\gamma^+(\theta) \cup \gamma^-(\theta)]$ that b_i "visits", viz. the trajectories $B \to A$ or C, and possibly $A \leftrightarrow D$ and $D \to C$. Let us call $\underline{\tau}$ a positive minimizer of the thickness (according to the normal of the trajectory) of all these strips for each b_i.

Now we fix an interval $[\underline{D}_i^{\nabla\theta} - \varepsilon_1/2, \underline{D}_i^{\nabla\theta} + \varepsilon_1/2]_{\gamma^-(\theta)}$. Under the flow of $\pi(\nabla\theta)$, it describes also a strip around the parts of $\partial\Omega$ visited by $\underline{D}_i^{\nabla\theta}$ (this strip contains in particular D_i). Note that, because $\partial_\nu\theta < 0$ on $\gamma^-(\theta)$, the interval $[\underline{D}_i^{\nabla\theta} - \varepsilon_1/2, \underline{D}_i^{\nabla\theta} + \varepsilon_1/2]_{\gamma^-(\theta)}$ (which does not touch $\partial\gamma^-(\theta)$) is non characteritic for $\nabla\theta$. We reduce $\underline{\tau} > 0$ in order that it is also inferior to the thickness of these strips.

A remark

We remark that the function $W \mapsto \underline{A}_i^W$ (respectively $\underline{C}_i^W$, $\underline{D}_i^W$ and $\overline{M}_i^W$) is continuous for the C^0 topology, if W is close enough to $\nabla\theta$. (This would not have been necessarily true if one had considered, instead of (18),

$$\phi^{\pi(W)}\big(0, s_W(A_i), A_i\big), \quad \text{with } s_W(x) := \min\big\{t \in [0, +\infty) \mid \phi^{\pi(W)}(0, t, x) \in \overline{\gamma^-(\theta)}\big\}.)$$

Indeed, by the same argument as for the proof of Proposition 2, if one considers $\phi^{\pi(\nabla\theta)}(0, t, A_i)$ for $t \in (\sigma_{\nabla\theta}(A_i), \sigma_{\nabla\theta}(A_i) + \varepsilon)$, we get a point in $B_R\backslash\overline{\Omega}$. For W close enough to $\nabla\theta$, the corresponding point $\phi^{\pi(W)}(0, t, A_i)$ also lies outside $\overline{\Omega}$, which gives the continuity of $\sigma_W(\cdot)$.

2.4. Defining the operators F and $\mathcal{G}$

In this section, we introduce two continuous operators F and $\mathcal{G}$. The solution of the problem will be found as a fixed point of the latter.

Domain of F

The operator F will be defined for (f, W) in

$$C^2\big(\overline{\Omega}; \mathbb{R}^2\big) \times \mathcal{T}_\varepsilon \text{ into } C^1(\overline{\Omega}; \mathbb{R}),$$

for ε sufficiently small, where $\mathcal{T}_\varepsilon$ is defined in the following way

$$\mathcal{T}_\varepsilon := \big\{W \in C^1\big(\overline{\Omega}; \mathbb{R}^2\big) \mid W.\nu = \partial_\nu\theta \text{ on } \partial\Omega,\ \operatorname{div} W = 0 \text{ in } \Omega,\\ \text{and } \|W - \nabla\theta\|_{C^1(\overline{\Omega};\mathbb{R}^2)} < \varepsilon\big\}. \tag{19}$$

In order for the operator to be well defined, we reduce ε_2 in order that any $W \in \mathcal{T}_\varepsilon$ with $\varepsilon < \varepsilon_2$, satisfies the five following conditions:

$$\big|W(x)\big| \geqslant \underline{m}/2 \quad \text{in } \overline{\Omega}, \tag{20}$$

$$\text{for any } x \in \overline{\Omega},\ \exists t \in [0, 2\overline{T}],\ \text{s.t.} \quad \phi^{\pi(W)}(t, 0, x) \notin \overline{\Omega}, \tag{21}$$

$$\big\|\overline{M}_i^W - \overline{M}_i^{\nabla\theta}\big\| < \varepsilon_1/10, \qquad \big\|\underline{D}_i^W - \underline{D}_i^{\nabla\theta}\big\| < \varepsilon_1/10, \tag{22}$$

$$W \text{ and } \nabla\theta \text{ are pointing in the same direction}$$
$$\text{at the points of } \partial\gamma^-(\theta) \cup \partial\gamma^+(\theta), \tag{23}$$

$$\text{when following the flow of } W, \text{ the ball } B(b_i, r^\star) \text{ describes a strip}$$
$$\text{around the trajectory of } b_i \text{ in the flow of } \nabla\theta, \text{ of thickness at least } \underline{\tau}/2,$$
$$\text{and the equivalent for the strips corresponding to } \underline{D}_i^{\nabla\theta}. \tag{24}$$

(We remark that in fact (23) is a consequence of the validity of (20) for any $W \in \mathcal{T}_\varepsilon$. Conditions (21), (22) and (24) can be obtained thanks to (16).)

From now, reducing if needed $\underline{\tau}$ or ε_1, we consider that $\underline{\tau} = \varepsilon_1/4$.

Expression of F

We fix f. To any W in $\mathcal{T}_\varepsilon$, we are going to associate in a first time a real-valued function $\omega_W^\flat$ in $C^1(\overline{\gamma^-(\theta)}; \mathbb{R})$. In that order, we construct the two families of functions $\mathcal{A}_i$ and $\mathcal{B}_i$, defined respectively at the neighborhood of A_i and B_i in $\overline{\gamma^-(\theta)}$.

We consider the function $T_{B_i}^W$ defined in a nonempty open set in B_R by

$$\begin{cases} T_{B_i}^W = 0 \quad \text{on } B(b_i, r^\star), \\ (\pi(W).\nabla) T_{B_i}^W = \tilde{\pi}(\operatorname{curl} f) \quad \text{in } B_R. \end{cases} \tag{25}$$

This function is well (regularly) defined – note that the first equation is satisfied in $B(b_i, r^\star)$, thanks to (17) – at least on the strip that we mentioned in (24). In particular, it is defined in a neighborhood of B_i in $\overline{\gamma^-(\theta)}$ of length $\underline{\tau}/2$. We call $\mathcal{B}_i$ the restriction of $T_{B_i}^W$ on this interval of $\partial\Omega$. The same way, $T_{B_i}^W$ is defined in a neighborhood of A_i in $\overline{\gamma^-(\theta)}$ of length $\underline{\tau}/2$, for each $A_i \in A^B$. We denote the corresponding restriction $\mathcal{A}_i$.

It remains to define $\mathcal{A}_i$ for $A_i \in A^\circ$. For this, we consider the function $T_{D_i}^W$ defined in a nonempty open set in B_R by

$$\begin{cases} T_{D_i}^W = 0 \quad \text{on } [\underline{D}_i^{\nabla\theta} - \varepsilon_1/2, \underline{D}_i^{\nabla\theta} + \varepsilon_1/2]_{\gamma^-(\theta)}, \\ (\pi(W).\nabla) T_{D_i}^W = \operatorname{curl} f \quad \text{in } B_R. \end{cases} \tag{26}$$

All the same, thanks to the transversality of W at the beginning of the strip and to (24), $T_{D_i}^W$ is well (regularly) defined in a strip containing a neighborhood of A_i in $\gamma^-(\theta)$ of length $\underline{\tau}/2$, for each $A_i \in A^\circ$. We again denote the corresponding restriction $\mathcal{A}_i$.

Then, we introduce $\omega_W^\flat$ by the following formula (we consider in this expression that the direction in $\partial\Omega$ at points A_i or B_i pointing inside $\gamma^-(\theta)$ is the positive one; replace "+" by "−" if needed):

$$\omega_W^\flat = \begin{cases} \mathcal{P}^+_{B_i+\tau/2,\varepsilon_1/4}(\mathcal{B}_i) & \text{in } [B_i, B_i + 3\varepsilon_1/8]_{\partial\Omega}, \\ \mathcal{P}^+_{A_i+\tau/2,\varepsilon_1/4}(\mathcal{A}_i) & \text{in } [A_i, A_i + 3\varepsilon_1/8]_{\partial\Omega}, \\ 0 & \text{anywhere else in } \gamma^-(\theta). \end{cases} \tag{27}$$

Thanks to (19) and (20), we may now define $\tilde{\omega}_W \in C^0(\overline{\Omega}; \mathbb{R})$ to be the unique function satisfying the following relations:

$$\begin{cases} (W.\nabla)\tilde{\omega}_W = \operatorname{curl} f & \text{in } \overline{\Omega}, \\ \tilde{\omega}_W = \omega^\flat & \text{on } \overline{\gamma^-(\theta)}. \end{cases} \tag{28}$$

We now define the function $\omega_W^\#$ in $C_0^1(\gamma^-(\theta); \mathbb{R})$ as

$$\omega_W^\# = \sum_{i=1}^{g} \mu_i U_{\varepsilon_1/4, M_i} \quad \text{on } \gamma^-(\theta), \tag{29}$$

where the coefficients μ_i are computed with the help of the following relation

$$\mu_i \int_{\Gamma_i} (\partial_\nu \theta) U_{\varepsilon_1/4, M_i} = \int_{\Gamma_i} f.\vec{dx} - \int_{\Gamma_i} (\partial_\nu \theta)\tilde{\omega}_W. \tag{30}$$

Let us remark that, thanks to (10) and (11), relation (30) uniquely determines the μ_i for any W in $\mathcal{T}_\varepsilon$, with $\varepsilon < \varepsilon_2$.

As previously, we introduce a function $\overline{\omega}_W \in C^1(\overline{\Omega}; \mathbb{R})$ as the solution of the following system:

$$\begin{cases} (W.\nabla)\overline{\omega}_W = 0 & \text{in } \overline{\Omega}, \\ \overline{\omega}_W = \omega_W^\# & \text{on } \overline{\gamma^-(\theta)}. \end{cases} \tag{31}$$

We finally give the definition of F by

$$F(f, W) := \check{\omega}_W := \tilde{\omega}_W + \overline{\omega}_W \quad \text{in } \overline{\Omega}. \tag{32}$$

Of course, one deduces from (28) and (31) that $\check{\omega}_W$ satisfies

$$(W.\nabla)\check{\omega}_W = \operatorname{curl} f \quad \text{in } \overline{\Omega}. \tag{33}$$

Regularity of $F(f, W)$

Let us check that the image of F is actually included in C^1, and, more generally, let us study the regularity of $F(f, W)$ depending on the one of f and of W.

The "$\overline{\omega}_W$-part" of $F(f, W)$ is clearly C^m-regular when f is C^{m+1}-regular and when W is C^m-regular. This is a consequence of the fact that we chose the support of U in a region of $\gamma^-(\theta)$ transverse to any $W \in \mathcal{T}_\varepsilon$, with $\varepsilon < \varepsilon_2$.

Now we concentrate on the regularity of $\tilde{\omega}_W$. Let us distinguish three cases:

- in the "$\underline{D}_i^W$-strip", we have this regularity, because $\tilde{\omega}_W$ coincides there with $T_{D_i}^W$ which is regular thanks to the transversality of the interval at the beginning,
- in the "b_i-strip", we get all the same that $\tilde{\omega}_W$ is C^m-regular when f is C^{m+1}-regular and when W is C^m-regular, for we can all the same find a suitable transversal interval in $B(b_i, r^\star)$,
- for the other points in Ω, we get again the same regularity, because they come from points in $\gamma^-(\theta)$ at a distance of at least $\underline{\tau}/2$ from $\partial\gamma^-(\theta)$ and stay away from $\partial\gamma^+(\theta) \cup \partial\gamma^-(\theta)$.

Finally, we get the following regularity result

$$f \in C^{m+1}(\overline{\Omega}; \mathbb{R}^2) \quad \text{and} \quad W \in C^m(\overline{\Omega}; \mathbb{R}^2) \cap \mathcal{T}_\varepsilon \quad \Rightarrow \quad F(f, W) \in C^m(\overline{\Omega}). \tag{34}$$

Note that, in particular, the operator F actually sends $C^2(\overline{\Omega}; \mathbb{R}^2) \times \mathcal{T}_\varepsilon$ in $C^1(\overline{\Omega}; \mathbb{R})$.

Defining the operator $\mathcal{G}$

To any $\omega \in C^1(\overline{\Omega}; \mathbb{R})$, one can continuously associate the unique vector field $y_\omega \in C^1(\overline{\Omega}; \mathbb{R}^2)$ as the solution of the following system

$$\begin{cases} \operatorname{div} y_\omega = 0 & \text{in } \Omega, \\ \operatorname{curl} y_\omega = \omega & \text{in } \Omega, \\ y_\omega.\nu = \partial_\nu \theta & \text{on } \partial\Omega, \end{cases} \tag{35}$$

and

$$\int_\Omega y_\omega.\nabla^\perp \tau_i = 0, \quad \text{for all } i \text{ in } \{1, \dots, g\}. \tag{36}$$

(In fact, y_ω is more regular than C^1, e.g. $y_\omega \in C^{1+\alpha}(\overline{\Omega})$ for any $\alpha \in (0, 1)$.)

We now define

$$\mathcal{G} : (\omega, f) \mapsto F(f, y_\omega) \tag{37}$$

for ω in

$$\mathcal{X} := \{\omega \in C^1(\overline{\Omega}; \mathbb{R}) \text{ s.t. } \|\omega\|_{C^1(\overline{\Omega})} < \varepsilon_3\}, \tag{38}$$

and f in $C^2(\overline{\Omega}; \mathbb{R}^2)$, with ε_3 small enough, computed from ε_2, so that $\mathcal{G}$ is well defined, i.e. for instance $y_\omega \in \mathcal{T}_{\varepsilon_2/2}$.

2.5. Back to the proof of Theorem 1

The main idea is to prove that, at least for f small in the C^2 norm, the operator $\mathcal{G}$ satisfies the assumptions of the Leray–Schauder fixed point Theorem. Then it is to prove that this fixed point solves the problem. Finally, we get rid of the assumption of smallness on f.

Step 1. We remark that, fixed f,

$$\mathcal{X} \text{ is a convex compact subset of the Banach space } C^0(\overline{\Omega}).$$

This is a clear consequence of Ascoli's theorem.

Step 2. We show that if we restrict ourselves to small f for the C^2 norm, then one gets

$$\mathcal{G}(\mathcal{X}) \subset \mathcal{X}. \tag{39}$$

It follows from the construction that, on $\overline{\gamma^-(\theta)}$, one gets

$$\|\mathcal{G}(\omega, f)\|_{C^1(\mathcal{V})} \leqslant C\|f\|_{C^2(\overline{\Omega})}, \tag{40}$$

where $\mathcal{V}$ is a neighborhood of the points "B" in $\gamma^-(\theta)$. This is also valid close to the points $\underline{D}_i^W$ in $\gamma^-(\theta)$, since a straightforward computation shows that at these points, one has $\nabla\tilde{\omega}.\nu = \operatorname{curl} f/\partial_\nu(\theta)$. Moreover, for $\omega \in \mathcal{X}$, the norm $\|y_\omega\|_{C^1(\overline{\Omega})}$ is bounded (by $(1+\varepsilon_2)\|\nabla\theta\|_{C^1(\overline{\Omega};\mathbb{R}^2)}$). Then the flow $\phi^{y_\omega}(t,0,\cdot)$ is bounded in the $C^1(\overline{\Omega})$ norm, uniformly in $t \in [-2\overline{T}, 2\overline{T}]$.

Then the estimate (40) (with a perhaps greater constant) propagates inside $\overline{\Omega}$ – remember (21) – and at the neighborhood of the "A" points, exactly as for (34). Hence, one gets (39), at least for f small.

Step 3. We show that, fixed f,

$$\mathcal{G} \text{ is continuous.}$$

When considering the construction, one can see that it is sufficient to prove the continuity of the functions $W \mapsto T_{b_i}^W(\cdot)$ and $W \mapsto T_{D_i}^W(\cdot)$. This continuity is again a consequence of (16).

Step 4. We hence find, by the Leray–Schauder theorem, a fixed point, say ζ of $\mathcal{G}(\cdot, f)$ (for f small enough). Let us show that ζ is a solution of the system.

From (30) and the relation

$$\int_\Omega \big[(y_\omega.\nabla)y_\omega\big].\nabla^\perp\tau_i = \int_{\Gamma_i} (y_\omega.\nu)\omega \quad \text{for all } i \text{ in } \{1,\dots,g\},$$

one gets

$$\int_\Omega \big[(y_\zeta.\nabla)y_\zeta\big].\nabla^\perp\tau_i = \int_\Omega f.\nabla^\perp\tau_i, \quad \text{for all } i \text{ in } \{1,\dots,g\}. \tag{41}$$

Together with (33), (35)–(36) and (37), this leads to (1), (2) and (3). The $C^\infty(\overline{\Omega};\mathbb{R}^2)$-regularity of ζ is a consequence of (34).

Conclusion. We have shown that the problem has a solution when $\|f\|_{C^2(\overline{\Omega};\mathbb{R}^2)}$ is small enough. The general case naturally follows from the previous one: it suffices to consider the homogeneity of the equation. So Theorem 1 is established.

3. A generalization of Theorem 1

3.1. Setting of the result

The solution y of (1)–(3) is of course highly non-unique. Even, one could ask for supplementary properties of the solution.

The natural question is the possibility to prescribe the entering vorticity. Indeed, for the non-stationary system, the choice of the normal velocity and of the entering vorticity (and of the initial range) uniquely determines the system (see e.g. [8]). In our method, it is essential that the normal velocity is fixed as the same as the one of the reference solution. But we can wonder if the entering vorticity could be demanded.

Theorem 1 can be generalized the following way:

THEOREM 2. – *Consider an open region $\mathcal{I}$ in $\gamma^-(\theta)$ such that*

- $\overline{\mathcal{I}}$ *does not touch* $\partial\gamma^-(\theta)$,
- $\overline{\mathcal{I}}$ *does not contain the points* M_i.

Then for any $\kappa \in C_0^\infty(\mathcal{I};\mathbb{R})$ *and for any* $(\lambda_1,\dots,\lambda_g) \in \mathbb{R}^g$, *there exists a solution* y *of* (1)–(3) *which moreover satisfies*

$$\operatorname{curl} y = \kappa \quad \text{on } \mathcal{I}, \tag{42}$$

$$\int_\Omega y.\nabla^\perp \tau_i = \lambda_i, \quad \forall i \in \{1,\dots,g\}, \tag{43}$$

where $(\tau_i)_{i=1,\dots,g}$ *is defined by* (12).

Remark 3. – In fact, many points could play the same role as M_i. So one should read the second assumption in Theorem 2 as "there are points $\widetilde{M_i}$ in $\gamma^-(\theta)\backslash\overline{\mathcal{I}}$ that could replace the points M_i in Proposition 1".

3.2. Sketch of the proof of Theorem 2

Let us briefly establish Theorem 2.

In the definition (32) of F, we add the following function $\omega^\star$ defined by:

$$\begin{cases} (W.\nabla)\omega^\star_W = 0 & \text{in } \overline{\Omega}, \\ \omega^\star_W = \kappa & \text{on } \mathcal{I}, \\ \quad = 0 & \text{elsewhere on } \overline{\gamma^-(\theta)}. \end{cases} \tag{44}$$

We may have to reduce $\varepsilon_1 = \underline{\tau}/4$ in order that the supports of the functions $\mathcal{P}^+_{A_i+\tau/2,\varepsilon_1/4}(\mathcal{A}_i)$ and of $\mathcal{P}^+_{B_i+\tau/2,\varepsilon_1/4}(\mathcal{B}_i)$ do not meet $\overline{\mathcal{I}}$. We have also, in the definition of (29), to choose ε_1 small enough so that $\operatorname{Supp} U_{\varepsilon_1/4,M_i}$ does not meet $\overline{\mathcal{I}}$, and to replace (30) by

$$\mu_i \int_{\Gamma_i} (\partial_\nu\theta) U_{\varepsilon_1/4,M_i} = \int_{\Gamma_i} f.\vec{dx} - \int_{\Gamma_i} (\partial_\nu\theta)[\tilde{\omega}_W + \omega^\star_W]. \tag{45}$$

Finally, in the definition of y_ω, we replace (36) by

$$\int_\Omega y_\omega.\nabla^\perp \tau_i = \lambda_i, \quad \forall i \in \{1,\dots,g\}. \tag{46}$$

We can then define the operator $\mathcal{G}$ all the same way. Then the only delicate point in order to prove that $\mathcal{G}$ has a fixed point is (39). To obtain this, we in a first step restrict ourselves to the case where κ and $(\lambda_i)_{i=1,\dots,g}$ are small enough. Then estimating $\mathcal{G}$ the same way allows us to affirm (39) if they are all small. Then we get a fixed point of $\mathcal{G}$. This is again a solution of our problem, for the same reasons. (Relation (46) does not influence (41).)

Now, as for Theorem 1, we obtain the general case by homogeneity: if y is a solution for $[\varepsilon f, \varepsilon\kappa, (\varepsilon\lambda_i)]$, then y/ε is a solution for $[f, \kappa, (\lambda_i)]$.

3.3. Uniqueness and continuity of solutions with respect to "exterior conditions"

Now we wonder if two stationary solutions obtained by the previous process are close when the entering vorticity, the exterior force and the "λ_i" are close. We obtain

THEOREM 3. – *For any positive constant $\mathcal{M}$ and for $\mathcal{I}$ as above, there exists an amplification of $\nabla\theta$ (say $\lambda\nabla\theta_0$), such that the operator $\mathcal{Y}_{\mathcal{M}}$, which associates to any $[\kappa, f, (\lambda_i)_{i=1,\dots,g}] \in C_0^\infty(\mathcal{I};\mathbb{R}) \times C^\infty(\overline{\Omega};\mathbb{R}^2) \times \mathbb{R}^g$ satisfying*

$$\begin{aligned} \|\kappa\|_{C^1(\mathcal{I})} &\leqslant \mathcal{M}, \\ \|f\|_{C^2(\overline{\Omega})} &\leqslant \mathcal{M}, \\ |\lambda_i| \leqslant \mathcal{M}, &\quad \forall i = 1,\dots,g, \end{aligned} \tag{47}$$

a solution $\hat{y}$ satisfying (1)–(3) *and* (42)–(43) *constructed as above* for this $\nabla\theta$ *exists and is unique*.

Moreover, there exists $C(\mathcal{M},\mathcal{I}) > 0$ such that for $[\kappa_1, f_1, (\lambda_i^1)]$ and $[\kappa_2, f_2(\lambda_i^2)]$ satisfying (47) *one has*

$$\begin{aligned} &\big\|\mathcal{Y}_{\mathcal{M}}\big(\kappa_1, f_1, (\lambda_i^1)\big) - \mathcal{Y}_{\mathcal{M}}\big(\kappa_2, f_2, (\lambda_i^2)\big)\big\|_{H^1(\Omega)} \\ &\quad \leqslant C(\mathcal{M},\mathcal{I})\left[\|\kappa_1-\kappa_2\|_{L^2(\gamma^-(\theta))} + \|f_1-f_2\|_{C^1(\overline{\Omega})} + \sum_{i=1}^g \big|\lambda_i^1-\lambda_i^2\big|\right]. \end{aligned} \tag{48}$$

Remark 4. – For the existence, we required the $\nabla\theta$ to be sufficiently "amplified" too. Let us also underline that solutions of the 2-D stationary Euler system with prescribed normal velocity and entering vorticity are not in general unique (see e.g. [7]). The comparable geometry of the two stationary solutions (particularly the "domination" of the $\nabla\theta$ part in them) is essential here.

Remark 5. – This result gives in particular a 2-D equivalent of the work of H.D. Alber, but with other entering conditions (these depend on the dimension) and with a fixed and constructed reference solution.

3.4. Proof of Theorem 3

The existence was already proven. To establish uniqueness, it is sufficient to prove directly (48). The proof mainly follows [1]. Again using the homogeneity of the equation, we get that it is equivalent to fix $\nabla\theta$ and to show that the operator of Section 3.2 satisfies the required conditions *for $\mathcal{M}$ small enough*. We have for this $\nabla\theta$ an ε_2 such that the solutions are found in $\mathcal{T}_{\varepsilon_2/2}$. In particular, elements of $\mathcal{T}_{\varepsilon_2/2}$ do not have vanishing points in $\overline{\Omega}$, and their characteristics inside $\overline{\Omega}$ have a uniformly bounded length (say bounded by L).

We will need the following lemma ([1, Corollary A.2])

LEMMA 1 (Alber). – *Consider $W \in \mathcal{T}_{\varepsilon_2/2}$ and a function $q \in C^0(\overline{\Omega})$. For $x \in \gamma^-(\theta)$, denote by $l(x)$ the arc length of the characteristic of W starting from x. Consider finally the parameterization V of Ω: $(s,\zeta) \in \mathbb{R}^+ \times \gamma^-(\theta) \mapsto \phi^{W/|W|}(s,0,\zeta)$ for $s \leqslant l(\zeta)$. Then* *one gets*

$$\int_{\Omega} q(x)\,dx = \int_{\gamma^-(\theta)} \int_0^{l(\zeta)} q\big(V(s,\zeta)\big)\frac{|\partial_\nu \theta(\zeta)|}{|W(V(s,\zeta))|}\,ds\,d\zeta.$$

The proof is elementary (use the transform law and the incompressibility), and the original one in 3-D is also valid here.

Let us denote by $\hat{y}^1$ and $\hat{y}^2$ two solutions of the steady Euler equation obtained by the previous process for the same $\nabla\theta$ and for respective conditions $[\kappa_1, f_1, (\lambda_i^1)]$ and $[\kappa_2, f_2, (\lambda_i^2)]$, small enough in order to apply Theorem 2.

In the sequel, we will denote by c_i various constants depending on Ω, Σ, $\mathcal{I}$ and θ, but not on $\hat{y}^i$ (but nevertheless $\hat{y}^i$ is found in $\mathcal{T}_{\varepsilon_3}$, so that e.g. $\|\hat{y}^i\|_{C^1(\overline{\Omega})} \leqslant 2\|\nabla\theta\|_{C^1(\overline{\Omega})}$).

Thanks to (35) and (46), we get

$$\|\hat{y}^1 - \hat{y}^2\|_{H^1(\Omega)} \leqslant c_1\left(\|\operatorname{curl}\hat{y}^1 - \operatorname{curl}\hat{y}^2\|_{L^2(\Omega)} + \sum_{i=1}^{g}|\lambda_i^1 - \lambda_i^2|\right). \tag{49}$$

So it is sufficient to estimate the latter norm $\|\operatorname{curl}\hat{y}^1 - \operatorname{curl}\hat{y}^2\|_{L^2(\Omega)}$. We write:

$$\begin{aligned}\|\operatorname{curl}\hat{y}^1 - \operatorname{curl}\hat{y}^2\|_{L^2(\Omega)} &= \|F^{f_1}_{\kappa_1,\lambda_i^1}(\hat{y}^1) - F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^2)\|_{L^2(\Omega)}\\ &\leqslant \|F^{f_1}_{\kappa_1,\lambda_i^1}(\hat{y}^1) - F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^1)\|_{L^2(\Omega)}\\ &\quad + \|F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^1) - F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^2)\|_{L^2(\Omega)},\end{aligned}$$

where F^f_{κ,λ_i} is the operator of Section 3.2 corresponding to the fixed value of $\nabla\theta$, to the force f, to the entering condition κ and to condition (43) for λ_i. We now analyse separately the two terms of the right hand side of (50).

1st term. From the definition of $F(\cdot,\hat{y}^1)$, we get that

$$\begin{aligned}\|\mathcal{A}_i^{f_2} - \mathcal{A}_i^{f_1}\|_{L^2(\gamma^-(\theta))} &\leqslant c_2\|f_1 - f_2\|_{C^1(\overline{\Omega})},\\ \|\mathcal{B}_i^{f_2} - \mathcal{B}_i^{f_1}\|_{L^2(\gamma^-(\theta))} &\leqslant c_2\|f_1 - f_2\|_{C^1(\overline{\Omega})}.\end{aligned}$$

Together with (45), we deduce

$$\begin{aligned}&\|F^{f_1}_{\kappa_1,\lambda_i^1}(\hat{y}^1) - F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^1)\|_{L^2(\gamma^-(\theta))}\\ &\quad\leqslant c_3\big[\|\kappa_1 - \kappa_2\|_{L^2(\gamma^-(\theta))} + \|f_1 - f_2\|_{C^1(\Omega)}\big].\end{aligned} \tag{50}$$

Now, we write

$$\hat{y}^1.\nabla\big(F^{f_1}_{\kappa_1,\lambda_i^1}(\hat{y}^1) - F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^1)\big) = (\operatorname{curl} f_1 - \operatorname{curl} f_2).$$

Now we consider the parameterization V corresponding to $\hat{y}^1$. To simplify notations, we omit the V. We get

$$\frac{d}{ds}\big|F^{f_1}_{\kappa_1,\lambda_i^1}(\hat{y}^1)(s,\zeta) - F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^1)(s,\zeta)\big| \leqslant |\operatorname{curl} f_1 - \operatorname{curl} f_2|(s,\zeta).$$

We deduce

$$\big|F^{f_1}_{\kappa_1,\lambda_i^1}(\hat{y}^1)(s,\zeta) - F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^1)(s,\zeta)\big| \leqslant \big|F^{f_1}_{\kappa_1,\lambda_i^1}(\hat{y}^1)(0,\zeta) - F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^1)(0,\zeta)\big| + \int_0^{l(\zeta)} |\operatorname{curl} f_1 - \operatorname{curl} f_2|(s',\zeta)\,ds'.$$

Now we multiply by $|\partial_\nu\theta(\zeta)|/|\hat{y}^1(V(s,\zeta))|$ and integrate over $(s,\zeta)\in\mathbb{R}^+\times\gamma^-(\theta)$ with $s\leqslant l(\zeta)$. Using Lemma 1, the boundedness of the characteristics of $\hat{y}^1$ and the fact that $|\partial_\nu\theta(\zeta)|/|\hat{y}^1(V(s,\zeta))|$ is bounded for $(s,\zeta)\in\mathbb{R}^+\times\gamma^-(\theta)$ with $s\leqslant l(\zeta)$, we obtain

$$\big\|F^{f_1}_{\kappa_1,\lambda_i^1}(\hat{y}^1) - F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^1)\big\|_{L^2(\Omega)} \leqslant c_4\big[\|\kappa_1-\kappa_2\|_{L^2(\gamma^-(\theta))} + \|f_1-f_2\|_{C^1(\overline{\Omega})}\big].$$

2nd term. In this case, we have

$$F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^1) = F^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^2) \quad \text{on } \gamma^-(\theta)\backslash\bigcup \operatorname{Supp} U_{\varepsilon_1/4,M_i}.$$

(That is, the $\tilde{\omega}$ parts coincide on $\gamma^-(\theta)$.)

This needs not to be true on $\operatorname{Supp} U_{\varepsilon_1/4,M_i}$ because coefficients μ_i computed in each case are not necessarily equal.

We want to estimate the difference and, thanks to the definition of F, it is sufficient to check the difference of the integrals $\int_{\Gamma_i}(\partial\nu\theta)\tilde{\omega}$, or equivalently $\int_{\Gamma_i\cap\gamma^+(\theta)}(\partial\nu\theta)\tilde{\omega}$, computed in both cases.

Consider the operator $\overline{F}$ defined exactly as F, but with $\mu_i\equiv 0$ instead of (45). We denote $w_i := \overline{F}^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^i)$ and we have

$$\begin{cases} w_i = \overline{F}^{f_2}_{\kappa_2,\lambda_i^2}(\hat{y}^i) & \text{on } \gamma^-(\theta),\\ \hat{y}^i.\nabla w_i = \operatorname{curl} f_2 & \text{in } \overline{\Omega}. \end{cases} \tag{51}$$

Then

$$\int_{\Gamma_i} w_j(\partial\nu\theta) = \int_\Omega w_j\hat{y}^j.\nabla\tau_i + \int_\Omega \tau_i \operatorname{curl} f_2,$$

for all $i=1,\dots,g$ and $j=1,2$. So we easily obtain

$$\left|\int_{\Gamma_i}(w_1-w_2)\partial_\nu\theta\right| \leqslant c_5\big[\|w_1-w_2\|_{L^2(\Omega)} + \|w_2\|_{L^2(\Omega)}\|\hat{y}^1-\hat{y}^2\|_{L^2(\Omega)}\big]. \tag{52}$$

(Remember $\hat{y}^i$ in bounded in $C^1(\overline{\Omega})$ by $2\|\nabla\theta\|_{C^1(\overline{\Omega})}$.)

Now we estimate $\|w_1-w_2\|_{L^2(\Omega)}$ by Lemma 1. We get as previously

$$\frac{d}{ds}\big|w_1(s,\zeta) - w_2(s,\zeta)\big| \leqslant \big|(\hat{y}^1-\hat{y}^2).\nabla \operatorname{curl}\hat{y}^2\big|(s,\zeta).$$

With $w_1 = w_2$ on $\gamma^-(\theta)$, this leads all the same way to:

$$\|w_1 - w_2\|_{L^2(\Omega)} \leqslant c_6 \|(\hat{y}^1 - \hat{y}^2).\nabla \operatorname{curl} \hat{y}^2\|_{L^2(\Omega)}.$$

Hence we deduce

$$|\mu_{i,1} - \mu_{i,2}| \leqslant c_7 \big[\|(\hat{y}^1 - \hat{y}^2).\nabla \operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} + \|w_2\|_{L^2(\Omega)} \|\hat{y}^1 - \hat{y}^2\|_{L^2(\Omega)}\big].$$

Note moreover that $\|w_2\|_{L^2(\Omega)} \leqslant \|\operatorname{curl} \hat{y}^2\|_{L^2(\Omega)}$. Now we can go back to the "full" operator F; we have now

$$\begin{aligned}\|F^{f_2}_{\kappa_2,\lambda^2_i}(\hat{y}^1) - F^{f_2}_{\kappa_2,\lambda^2_i}(\hat{y}^2)\|_{L^2(\gamma^-(\theta))} \leqslant c_8 \big[&\|(\hat{y}^1 - \hat{y}^2).\nabla \operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} \\ &+ \|\operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} \|\hat{y}^1 - \hat{y}^2\|_{L^2(\Omega)}\big].\end{aligned}$$

We have the following equation

$$\hat{y}^1.\nabla\big(F^{f_2}_{\kappa_2,\lambda^2_i}(\hat{y}^1) - F^{f_2}_{\kappa_2,\lambda^2_i}(\hat{y}^2)\big) = -(\hat{y}^1 - \hat{y}^2).\nabla F^{f_2}_{\kappa_2,\lambda^2_i}(\hat{y}^2),$$

and using as previously Lemma 1, we get

$$\begin{aligned}&\|F^{f_2}_{\kappa_2,\lambda^2_i}(\hat{y}^1) - F^{f_2}_{\kappa_2,\lambda^2_i}(\hat{y}^2)\|_{L^2(\Omega)} \\ &\quad\leqslant c_9 \big[\|(\hat{y}^1 - \hat{y}^2).\nabla \operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} + \|\operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} \|\hat{y}^1 - \hat{y}^2\|_{L^2(\Omega)}\big].\end{aligned}$$

Conclusion. So finally we get:

$$\begin{aligned}&\|\operatorname{curl} \hat{y}^1 - \operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} \\ &\quad\leqslant c_{10} \big[\|\kappa_1 - \kappa_2\|_{L^2(\gamma^-(\theta))} + \|f_1 - f_2\|_{C^1(\overline{\Omega})} \\ &\qquad + \|(\hat{y}^1 - \hat{y}^2)\nabla \operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} + \|\operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} \|\hat{y}^1 - \hat{y}^2\|_{L^2(\Omega)}\big].\end{aligned}$$

Now we use Sobolev and Hölder inequalities in order to find

$$\begin{aligned}&\|\operatorname{curl} \hat{y}^1 - \operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} \\ &\quad\leqslant c_{11} \big[\|\kappa_1 - \kappa_2\|_{L^2(\gamma^-(\theta))} + \|f_1 - f_2\|_{C^1(\overline{\Omega})} \\ &\qquad + \|(\hat{y}^1 - \hat{y}^2)\|_{L^2(\Omega)} \|\nabla \operatorname{curl} \hat{y}^2\|_{L^\infty(\Omega)} + \|\operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} \|\hat{y}^1 - \hat{y}^2\|_{L^2(\Omega)}\big].\end{aligned}$$

Using (49), we get

$$\begin{aligned}\|\operatorname{curl} \hat{y}^1 - \operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} \leqslant c_{12} &\left[\|\kappa_1 - \kappa_2\|_{L^2(\gamma^-(\theta))} + \|f_1 - f_2\|_{C^1(\overline{\Omega})} + \sum_{i=1}^{g} |\lambda^1_i - \lambda^2_i|\right] \\ &+ \mathcal{K} \|\operatorname{curl} \hat{y}^1 - \operatorname{curl} \hat{y}^2\|_{L^2(\Omega)} \|\operatorname{curl} \hat{y}^2\|_{W^{1,\infty}(\Omega)}.\end{aligned}$$

So we get (48) if we can restrict $\mathcal{M}$ enough in order that

$$\|\operatorname{curl} \hat{y}^2\|_{W^{1,\infty}(\Omega)} \leqslant \frac{1}{2\mathcal{K}}. \tag{53}$$

This is obtained as for Section 2.5, step 2. We observe again that $\phi^{y_\omega}(t, 0, \cdot)$ is bounded in the $C^1(\overline{\Omega})$ norm, uniformly in $t \in [-2\overline{T}, 2\overline{T}]$. So again, the smallness of $\operatorname{curl} \hat{y}^2$ for the C^1 norm in $\gamma^-(\theta)$ propagates inside $\overline{\Omega}$. Hence for $\mathcal{M}$ small enough, we get (53).

Remark 6. – The previous proof also shows that two solutions y_1 and y_2 of (1)–(2) for the same force f, and moreover satisfying $\operatorname{curl} y_1 = \operatorname{curl} y_2$ on $\gamma^-(\theta)$, $y_1.\nu = y_2.\nu = \partial_\nu \theta$ on $\partial\Omega$, $\int_\Omega y_1.\nabla^\perp \tau_i = \int_\Omega y_2.\nabla^\perp \tau_i$, and for which f, $\operatorname{curl} y_{1|\gamma^-(\theta)}$ and $\int_\Omega y_1.\nabla^\perp \tau_i$ are small with respect to $\partial_\nu\theta$, are equal. (This formulation does not involve the operator of Section 2.)

4. Proof of Proposition 1

Before proving precisely Proposition 1, we establish some preliminary results that will be useful during the proof.

4.1. Approximation results

We will use the two following propositions:

PROPOSITION 3. – *Consider Ω a nonempty regular bounded connected open set in $\mathbb{C}$, and let us be given Σ a nonempty open part of its boundary $\partial\Omega$. We consider $v \in C^\infty(\partial\Omega\backslash\Sigma;\mathbb{C})$. Then for any $\varepsilon > 0$, for any $k \in \mathbb{N}$, there exists $\phi \in H(\Omega) \cap C^\infty(\overline{\Omega};\mathbb{C})$ satisfying*

$$\|\phi - v\|_{C^k(\partial\Omega\backslash\Sigma;\mathbb{C})} < \varepsilon. \tag{54}$$

For the proof of Proposition 3, we refer to [5].

This proposition leads to the following corollary:

COROLLARY 1. – *Under the same assumptions as for Proposition* 3*, one can moreover require from ϕ, besides* (54)*, that either its real part or its imaginary one, exactly coincides with the one of v on $\partial\Omega\backslash\Sigma$.*

Proof of Corollary 1

Let us show this result in the case where the real parts are to be equal, and where Σ meets only one connected component of the boundary, say Γ_0. (The general case trivially follows.)

We consider an operator $\mathcal{O}_{(\partial\Omega\backslash\Sigma)\to\partial\Omega} : C^\infty(\partial\Omega\backslash\Sigma;\mathbb{R}) \to C^\infty(\partial\Omega;\mathbb{R})$, which satisfies, for all f in $C^\infty(\partial\Omega\backslash\Sigma;\mathbb{R})$,

$$\begin{cases} \mathcal{O}_{(\partial\Omega\backslash\Sigma)\to\partial\Omega}(f) \equiv f \quad \text{on } \partial\Omega\backslash\Sigma, \\ \|\mathcal{O}_{(\partial\Omega\backslash\Sigma)\to\partial\Omega}(f)\|_{C^i(\partial\Omega)} \leqslant C_{\mathcal{O}} \|f\|_{C^i(\partial\Omega\backslash\Sigma)}, \quad \forall i \in \{0,\dots,k+1\}, \\ \operatorname{dist}(x, \partial\Omega\backslash\Sigma) \leqslant \underline{d}, \quad \forall x \in \operatorname{Supp}[\mathcal{O}_{(\partial\Omega\backslash\Sigma)\to\partial\Omega}(f)] \cap \Sigma, \end{cases} \tag{55}$$

with $\underline{d}$ to be small enough, and for a suitable constant $C_{\mathcal{O}}$.

Given ε small enough and given $k \in \mathbb{N}^*$, we consider, by Proposition 3, a holomorphic function f, C^∞-regular up to the boundary and satisfying (54) in the space C^{k+1}.

We then introduce the function $\psi_1 \in C^\infty(\overline{\Omega};\mathbb{R})$ as the solution of the following Dirichlet problem:

$$\begin{cases} \Delta\psi_1 = 0 \quad \text{in } \Omega, \\ \psi_1 = \mathcal{O}_{(\partial\Omega\backslash\Sigma)\to\partial\Omega}(\operatorname{Re}(f - v)) \quad \text{on } \partial\Omega. \end{cases}$$

Our problem at this point, is that there does not necessarily exist a real harmonic function ψ_2 defined in $\overline{\Omega}$, such that the complex-valued function $\psi_1 + i\psi_2$ is holomorphic in Ω. Indeed, the Cauchy–Riemann equations would imply in that case that such a function should satisfy

$$\int_{\Gamma} \partial_\nu \psi_1 = \int_{\Gamma} \partial_\tau \psi_2 = 0,$$

for any connected component Γ of $\partial\Omega$. Conversely, when this latter condition is satisfied, it is then sufficient (in addition to the harmonicity of ψ_1) to define the holomorphic function $\psi_1 + i\psi_2$.

These Cauchy–Riemann conditions will be satisfied, if not by ψ_1, by a "modified version" of ψ_1, say $\psi_1 + \psi_1'$, where $(\psi_1')_{|\partial\Omega}$ will be supported in

$$\widehat{\Sigma} := \Sigma \backslash \mathrm{Supp}\big[\mathcal{O}_{(\partial\Omega\backslash\Sigma)\to\partial\Omega}\big(\mathrm{Re}(f - v)\big)\big].$$

Let us denote by $\Gamma_1, \ldots, \Gamma_g$ the other different connected components of $\partial\Omega$. One gets, for all $i \in \{1, \ldots, g\}$,

$$\int_{\Gamma_i} \partial_\nu(\psi_1 + \psi_1') = \int_{\partial\Omega} (\psi_1 + \psi_1')\partial_\nu \tau_i, \tag{56}$$

where $(\tau_i)_{i=0,1,\ldots,g}$ is the family of functions in $C^\infty(\overline{\Omega}; \mathbb{R})$ given again by (12).

The integral in the right side of (56), computed only on the part of the boundary $\mathrm{Supp}[\mathcal{O}_{(\partial\Omega\backslash\Sigma)\to\partial\Omega}(\mathrm{Re}(f-v))]$ is completely determined by the original definition of ψ_1. We wish to equilibrate this integral computed on the support of $\mathcal{O}_{(\partial\Omega\backslash\Sigma)\to\partial\Omega}(\mathrm{Re}(f - v))$ by the one computed on the rest of $\partial\Omega$, that is, $\int_{\widehat{\Sigma}} \psi_1' \partial_\nu \tau_i \, dx$. It remains to prove that this is possible simultaneously for all $i \in \{1, \ldots, g\}$. (The flux along the Γ_0 component will automatically follow.) For this, it suffices to observe that the family of $(\partial_\nu \tau_{j|I})_{j=1,\ldots,g}$, for any interval I nonempty and open in Γ_0, is a free family. Indeed, all the functions in this family vanish on Γ_0. If there existed a non-trivial linear dependence relation between the functions $(\partial_\nu \tau_{j|I})_{j=1,\ldots,g}$, then by harmonicity of the functions τ_i, the corresponding linear combination of the τ_i would vanish on the whole $\overline{\Omega}$. But this is impossible: for instance, consider traces of this combination on the other Γ_j components.

Then, it follows from this linear independence that one can find g distinct points in I, say $X_1, \ldots, X_g$, such that the vectors:

$$\begin{pmatrix} \partial_\nu \tau_1(X_j) \\ \vdots \\ \partial_\nu \tau_g(X_j) \end{pmatrix} \quad \text{for } j \in \{1, \ldots, g\},$$

are linearly independent. (This is easily done by induction.)

Then it follows that one can find a regular function ψ_1' with support in $\Sigma\backslash\mathrm{Supp}\,[\mathcal{O}_{(\partial\Omega\backslash\Sigma)\to\partial\Omega}(\mathrm{Re}(f-v))]$ and such that

$$\int_{\widehat{\Sigma}} \psi_1' \partial_\nu \tau_j = -\int_{\mathrm{Supp}(\mathcal{O}_{(\partial\Omega\backslash\Sigma)\to\partial\Omega}(\mathrm{Re}(f-v)))} \psi_1 \partial_\nu \tau_j.$$

For example, one can take for ψ_1' a linear combination of "bell functions" around the X_j (that is, functions which are very concentrated around the points X_j).

Once found such a ψ_1', we can consider the holomorphic function ϕ_2 (C^∞-regular up to the boundary), associated to $\psi_1 + \psi_1'$, i.e. with the following shape:

$$\phi_2 := \psi_1 + \psi_1' + i\psi_2.$$

Then $f - \phi_2$ will fit the requirements of Corollary 1, if one can establish that

$$\|\phi_2\|_{C^k(\overline{\Omega};\mathbb{C})} \leqslant C\varepsilon. \tag{57}$$

But (57) is a consequence of (54): it follows from this inequality, from (55) and from Schauder estimates that ψ_1 is of order ε for the $C^{k,\alpha}(\overline{\Omega})$-norm, for any $\alpha \in (0,1)$. It follows that ψ_1' is also of order ε, since the coefficients of the so-called "bell-functions" are linearly computed from the following integrals:

$$\int_{\partial\Omega} \psi_1 . \partial_\nu \tau_i \quad \text{for } i \in \{1, \ldots, g\},$$

the coefficients of the combinations being independent of ε.

This ends the proof of Corollary 1.

4.2. Another preliminary result

We will also need the following lemma

LEMMA 2. – *Consider I a compact connected subset, with nonempty interior, of a regular not insertecting curve in the plane. Let $f \in C_0^\infty(\mathring{I};\mathbb{R})$. Then for any $\alpha \in (0,1)$, for any $\varepsilon > 0$, there exists $\tilde{f} \in C^\infty(I;\mathbb{R})$ which satisfies the following properties*

$$\mathrm{Supp}\,\tilde{f} \subset \mathring{I}, \tag{58}$$

$$\int_I \tilde{f} = 0, \tag{59}$$

the set of all zeros of $f - \tilde{f}$ is the union of a finite number of intervals which have nonempty interiors, (60)

$$\|\tilde{f}\|_{C^\alpha(I;\mathbb{R})} \leqslant K(f)\varepsilon. \tag{61}$$

Proof of Lemma 2

Given this f, one can construct a "plateau function" Λ in $C^\infty(I;\mathbb{R})$ satisfying the following requirements:

$$\begin{cases} 0 \leqslant \Lambda \leqslant 1 \quad \text{on } I, \\ \Lambda \equiv 1 \quad \text{on Supp } f, \\ \operatorname{Supp} \Lambda \subset \mathring{I}, \\ I \backslash (\operatorname{Supp} \Lambda) \text{ has exactly 2 connected components}. \end{cases}$$

We introduce also a function $Z \in C^\infty(\mathbb{R};\mathbb{R})$ satisfying:

$$\begin{cases} 0 \leqslant Z \leqslant 1 \quad \text{on } \mathbb{R}, \\ Z \equiv 0 \quad \text{on } [-1/2, 1/2], \\ Z \equiv 1 \quad \text{on } (-\infty, -1] \cup [1, +\infty). \end{cases} \tag{62}$$

Let $\varepsilon > 0$ (small). By Sard's theorem, one can choose $\lambda \in (0, \varepsilon)$ such that λ is not a critical value of f. We then consider the function $F := f - \lambda\Lambda$. The zeros of this function on Supp f are simple and (hence) isolated. Let us denote these zeros by $x_1, \dots, x_N$, by ordering them increasingly on I. Then we define the function:

$$F_2(x) = F(x) \prod_{i \in \mathcal{I}} Z\left(\frac{x - x_i}{\varepsilon/2}\right) \prod_{i \in \mathcal{J}} (1 - \chi_{[x_i, x_{i+1}]}), \tag{63}$$

where we fixed

$$\mathcal{I} := \{1, \dots, N\} \backslash \{i \in \{2, \dots, N-1\} \text{ such that } |x_{i+1} - x_{i-1}| < 2\varepsilon\},$$
$$\mathcal{J} := \{i \in \{1, \dots, N\} \text{ such that } |x_{i+1} - x_i| < 2\varepsilon\},$$

and where χ_J is the characteristic function of the interval J, and where we transported Z on I by the arc length.

It is easy to see that the function F_2 constructed this way is $C^\infty(I)$-regular. Essentially, we will define $\tilde{f} := F_2 - f$. We now want to show that

$$\|F - F_2\|_{C^\alpha(I)} \leqslant K(f)\varepsilon^{1-\alpha}. \tag{64}$$

First, we remark that

$$\|F - F_2\|_{C^0(I)} \leqslant K(f)\varepsilon.$$

Indeed, F_2 differs from F only for points situated at distance at most ε from a zero of F. It follows immediately, together with (62), that $\|F - F_2\|_{C^0(I)} \leqslant (1 + \|f'\|_{C^0(I)})\varepsilon$.

We have yet to study the ratio

$$R(x, y) := \frac{|F(x) - F_2(x) - F(y) + F_2(y)|}{|x - y|^\alpha}.$$

For x and y such that $|x - y| \geqslant \varepsilon$, it follows from the previous point that $R(x, y) \leqslant K(f)\varepsilon^{1-\alpha}$. Now for x and y such that $|x - y| < \varepsilon$, one gets:

- either F and F_2 have identical values at points x and y, then $R(x, y) = 0$,
- or both points x and y are at distance at most 2ε from a zero of F. We then have three possible cases:
  - either x and y are both in an interval of the type $[x_i, x_{i+1}]$ with $i \in \mathcal{J}$, in which case F_2 vanishes for both x and y. Therefore in this case $R(x, y) \leqslant \|f'\|_{C^0(I)}\varepsilon^{1-\alpha}$,
  - or neither x nor y are in an interval of the type $[x_i, x_{i+1}]$ with $i \in \mathcal{J}$; then in (63), the two products in the right side are reduced to at most one term $Z(x - x_i/\frac{\varepsilon}{2})$, then we easily obtain $R(x, y) \leqslant K(f)\varepsilon^{1-\alpha}$,
  - or x is not in an interval of the type $[x_i, x_{i+1}]$ with $i \in \mathcal{J}$, but on the contrary y is in one of them (if needed, inverse x and y). But since the function $\widetilde{Z}$ which coincides with Z at the left of 0 and with 0 at the right of zero is all the same of class C^∞, one gets $R(x, y) \leqslant K(f)\varepsilon^{1-\alpha}$ as in the previous point.

  In all cases, we therefore get (64).

In order to get (59), we add a function with support in $\mathring{I}$, at the exterior of Supp Λ. We obtain this way the function $\tilde{f}$. This modification also has a cost of order ε for the norm $C^\alpha(I)$.

Finally, we obtain the condition (61) (renormalize ε to get it precisely). Conditions (58) and (60) follow from the construction. This ends the proof of Lemma 2.

4.3. Back to the proof of Proposition 1

Let us denote by $\Gamma_0, \Gamma_1, \ldots, \Gamma_g$ the different connected components of $\partial\Omega$, Γ_0 being the exterior one. We recall that ν is the unit outward normal vector on $\partial\Omega$, and we denote by τ the unit tangent along $\partial\Omega$ chosen in order that (τ, ν) is a direct basis of the plane. Finally, we note Φ the following function:

$$\Phi : \begin{cases} \mathbb{R}^2 \to \mathbb{C}, \\ (x, y) \mapsto x - iy. \end{cases}$$

We reduce the component of Σ in Γ_0 in a strictly smaller open set Σ', still regular, and which still intersects Γ_0. (We keep this way a kind of "margin".) On the other connected components of $\partial\Omega$ (for $i \in \{1, \ldots, g\}$), we let $\Sigma' \cap \Gamma_i := \Sigma \cap \Gamma_i$ except if $\Sigma \cap \Gamma_i = \Gamma_i$, in which case we choose $\Sigma' \subset\subset \Sigma$, in order to obtain generally

$$\Gamma_i \backslash \Sigma' \neq \emptyset, \quad \forall i \in \{0, \ldots, g\}. \tag{65}$$

Let us now define a vector field v on $\partial\Omega\backslash(\Sigma' \cap \Gamma_0)$, regular (in the C^∞ class). For $i \in \{1, \ldots, g\}$, one chooses $v_i \in C^\infty(\Gamma_i; \mathbb{R}^2)$ which satisfies the seven following conditions:

$$v_i = \tau \quad \text{on } \Gamma_i \backslash \Sigma', \tag{66}$$

$$\gamma_i^+(v_i) := \{x \in \Sigma' \cap \Gamma_i \mid v_i.\nu > 0\} \text{ and } \gamma_i^-(v_i) := \{x \in \Sigma' \cap \Gamma_i \mid v_i.\nu < 0\}$$
$$\text{are nonempty, connected and have disjoint closures,} \tag{67}$$

$$|v_i| \geqslant 1 \quad \text{on } \Gamma_i, \tag{68}$$

$$\deg(v_i, \Gamma_i, 0) = 0, \tag{69}$$

$$\int_{\Gamma_i} v_i \,.\vec{dx} = 0, \tag{70}$$

$$\int_{\gamma_i^-(v_i)} |v_i.\nu|\, dx \geqslant g, \tag{71}$$

$$\int_{\gamma_i^+(v_i)} v_i.\nu\, dx \leqslant 1. \tag{72}$$

It is easy to construct such vector fields v_i and we remark that these vector fields always satisfy the property:

$$\begin{aligned} &v_i \text{ is "pointing outside" } \gamma_i^+ \text{ on } \partial\gamma_i^+ \\ &\text{and "pointing inside" } \gamma_i^- \text{ on } \partial\gamma_i^-, \text{ for } i \in \{1, \dots, g\}. \end{aligned}$$

We will denote by Σ_τ^i the part of the boundary included in $\Sigma' \cap \Gamma_i$ and situated between γ_i^+ and γ_i^- for $i \in \{1, \dots, g\}$ (uniquely defined thanks to (65)). Remark that

$$v_i.\tau < 0 \quad \text{on } \Sigma_\tau^i. \tag{73}$$

For what concerns the Γ_0 component, we define v_0 only on $\Gamma_0 \backslash \Sigma'$ by condition (66). Finally, we set

$$v = \begin{cases} v_i & \text{on } \Gamma_i, \ \forall i \in \{1, \dots, g\}, \\ v_0 & \text{on } \Gamma_0 \backslash \Sigma'. \end{cases} \tag{74}$$

Thanks to (68) and (69), one may define $W := \log \Phi(v)$ on $\partial\Omega \backslash (\Sigma' \cap \Gamma_0)$.

We then use Corollary 1 on W, with $\varepsilon \in (0,1)$ and $k = 0$, and with $\Sigma' \cap \Gamma_0$ as the "window" in the boundary. We furthermore require that the imaginary parts should exactly coincide. We therefore get a function $\phi_\varepsilon \in H(\Omega) \cap C^\infty(\overline{\Omega}; \mathbb{C})$ such that

$$\|\phi_\varepsilon - W\|_{C^0(\partial\Omega\backslash(\Sigma'\cap\Gamma_0);\mathbb{C})} < \varepsilon, \tag{75}$$

$$\operatorname{Im}(\phi_\varepsilon) = \operatorname{Im}(W) \text{ on } \partial\Omega \backslash (\Sigma' \cap \Gamma_0). \tag{76}$$

The problem is that we are no longer sure that the circulations

$$\int_{\Gamma_i} \Phi^{-1} \exp(\phi_\varepsilon) \,.\vec{dx}, \quad \text{for } i = 1, \dots, g, \tag{77}$$

are exactly null (but we nevertheless know that these integrals are of order ε; let us say they are all of modulus inferior to $K\varepsilon$). These conditions are of course necessary in order for the vector field $\Phi^{-1}\exp(\phi_\varepsilon)$ to be a gradient.

To fix this problem, we define g functions $w_1, \dots, w_g$ in $C^\infty(\partial\Omega\backslash(\Sigma' \cap \Gamma_0); \mathbb{C})$ satisfying the following conditions ($i \in \{1, \dots, g\}$):

$$\begin{cases} \mathrm{Im}(w_i) = 0 & \text{on } \partial\Omega\backslash(\Sigma' \cap \Gamma_0), \\ \mathrm{Re}(w_i) = 0 & \text{on } \partial\Omega\backslash[(\Sigma' \cap \Gamma_0) \cup \Sigma_\tau^i], \\ \mathrm{Re}(w_i) \leqslant 0 & \text{on } \Sigma_\tau^i, \\ \int_{\Sigma_\tau^i} |\mathrm{Re}(w_i)| = 1. \end{cases} \tag{78}$$

We define, given a positive real number ε', approximations in the C^0 norm on $\partial\Omega\backslash(\Sigma' \cap \Gamma_0)$ of the functions w_i by Corollary 1, requiring again that the imaginary parts should exactly coincide. Let $W_i^{\varepsilon'}$ be the g functions obtained by this process.

The idea here is to consider, instead of ϕ_ε, a function defined by the following formula:

$$\tilde{\phi}_{\varepsilon,\varepsilon'}^{\lambda_1,\lambda_2,\dots,\lambda_g} := \phi_\varepsilon + \left(\lambda_1 W_1^{\varepsilon'} + \lambda_2 W_2^{\varepsilon'} + \dots + \lambda_g W_g^{\varepsilon'}\right), \tag{79}$$

the λ_i being real numbers, and then to find $\lambda_1, \dots, \lambda_g$, small, in order that the circulations (77) computed for $\tilde{\phi}_{\varepsilon,\varepsilon'}$ instead of ϕ_ε are null for $i = 1, \dots, g$. We denote $\psi_\varepsilon := \exp(\phi_\varepsilon)$ and $\tilde{\psi}_{\varepsilon,\varepsilon'} := \exp(\tilde{\phi}_{\varepsilon,\varepsilon'})$ (we omit the λ_i in the writing of $\tilde{\phi}_{\varepsilon,\varepsilon'}$ and $\tilde{\psi}_{\varepsilon,\varepsilon'}$).

But for λ_j all in $[-1, 1]$, for $i \in \{1, \dots, g\}$, one gets, using (68), (73) and (78):

$$\int_{\Gamma_i} \Phi^{-1}(\psi_\varepsilon)\,.\vec{dx} - \int_{\Gamma_i} \Phi^{-1}(\tilde{\psi}_{\varepsilon,\varepsilon'})\,.\vec{dx} \geqslant \left((1-\varepsilon)\lambda_i - C\varepsilon'\right) \quad \text{for } \lambda_i \in [0, 1], \tag{80}$$

$$\int_{\Gamma_i} \Phi^{-1}(\tilde{\psi}_{\varepsilon,\varepsilon'})\,.\vec{dx} - \int_{\Gamma_i} \Phi^{-1}(\psi_\varepsilon)\,.\vec{dx} \leqslant -\left((1-\varepsilon)\frac{\lambda_i}{2} - C\varepsilon'\right) \quad \text{for } \lambda_i \in [-1, 0], \tag{81}$$

the constant C being independent of ε', whatever the values of the others $\lambda_j \in [-1, 1]$. Indeed, we cut these integral in two: on $\Gamma_i\backslash\Sigma_i^\tau$, the "error" between the two integrals is of order ε'; on Σ_i^τ, the "growth" of the circulation is at least of $(1-\varepsilon)\lambda_i$, with still an error of order ε'.

From now, we take $\varepsilon < 1/2$ and $\varepsilon' := \frac{\varepsilon}{10C}$. Then we consider the application:

$$\mathcal{H}: \begin{cases} \mathbb{R}^g \to \mathbb{R}^g, \\ (\lambda_1, \dots, \lambda_g) \mapsto \left(\int_{\Gamma_i} \Phi^{-1}(\tilde{\psi}_{\varepsilon,\varepsilon'})\,.\vec{dx}\right)_{i=1,\dots,g}. \end{cases}$$

We endow $\mathbb{R}^g$ with the norm $\|(x_1, \dots, x_g)\| := \max(|x_1|, \dots, |x_g|)$. If we restrict the application $\mathcal{H}$ to the sphere (in fact, the cube) with center 0 and radius $4(K+1)\varepsilon$, say $\mathcal{S}(0, 4(K+1)\varepsilon)$ (denote by $B(0, 4(K+1)\varepsilon)$ the corresponding ball), then from (80) and (81), we deduce that 0 is not reached. So we can define:

$$\mathcal{H}': \begin{cases} \mathcal{S}(0, 4(K+1)\varepsilon) \to \mathcal{S}(0, 4(K+1)\varepsilon), \\ \lambda := (\lambda_1, \dots, \lambda_g) \mapsto 4(K+1)\varepsilon \frac{\mathcal{H}(\lambda)}{\|\mathcal{H}(\lambda)\|}. \end{cases}$$

This application is continuous and has a non-null degree – in fact, $\deg(\mathcal{H}') = 1$ – (for instance, by (80) and (81), no point is sent to its antipodal point). Hence,

$$\exists \bar{\lambda} \in B\big(0, 4(K+1)\varepsilon\big) \text{ such that } \quad \mathcal{H}(\bar{\lambda}) = 0.$$

That is, one finds a solution of the system, with scalars λ_i of order ε.

Therefore, we get a function ψ', holomorphic in Ω, C^∞-regular up to the boundary, such that the integrals $\int_{\Gamma_i} \Phi^{-1}(\psi')\,.\vec{dx}$, are null for $i \in \{1, \dots, g\}$ (and hence are null also for $i = 0$). When considering $\Phi^{-1}(\psi')$, one therefore obtains the gradient of a harmonic function, say θ_1, which satisfies (6) and (7). We have left to slightly modify this function in order to get (8)–(9) (which are satisfied everywhere except perhaps on Γ_0).

This is where we use Lemma 2, with $I := \Gamma_0 \cap \Sigma$, $f := \partial_\nu \theta_1$ and with a given ε'' to be fixed (f is actually compactly supported, thanks to the margin we kept on Σ). We hence find a certain function $\tilde{f}$, and define for this function the following solution of the Neumann problem, using (59),

$$\begin{cases} \Delta\theta_2 = 0 & \text{in } \Omega, \\ \partial_\nu \theta_2 = \tilde{f} & \text{on } \Gamma_0 \cap \Sigma, \\ \partial_\nu \theta_2 = 0 & \text{on } \partial\Omega \backslash \Gamma_0 \cap \Sigma, \\ \int_\Omega \theta_2 = 0. \end{cases}$$

We know that $|\nabla\theta_1(x)| \geqslant \underline{\mu} > 0$ in $\overline{\Omega}$. For ε'' small enough, using Schauder estimates, we get

$$|\nabla\theta_2|(x) \leqslant \underline{\mu}/2 \quad \text{in } \overline{\Omega}.$$

Consequently, for such an ε'', the function $\theta_1 - \theta_2$ satisfies the required properties (5) to (9). Finally, (10) is a consequence of the incompressibility of $\nabla\theta$, of the fact that $\nabla\theta$ is close to v, and of (71)–(72).

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