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ASYMPTOTIC STABILIZABILITY BY STATIONARY FEEDBACK OF THE TWO-DIMENSIONAL EULER EQUATION: THE MULTICONNECTED CASE*

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Abstract. We construct a feedback law which allows us to asymptotically stabilize the Euler system for incompressible inviscid fluids in two dimensions, in the case of a multiconnected bounded domain, by means of a control localized on a part of the boundary that meets every connected component of the boundary. This generalizes a result of Coron [*SIAM J. Control Optim.*, 37 (1999), pp. 1874–1896] concerning simply connected domains.

Key words. incompressible inviscid fluids, stabilization by feedback control

AMS subject classifications. 93D15, 35B37, 76B75, 93C20

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1. Introduction.

1.1. Statement of the problem. In this paper, we are concerned with the null asymptotic stabilization by means of a stationary feedback of the Euler system for inviscid incompressible fluids in two space dimensions, namely, the following system:

$$\left\{ \begin{array}{l} \partial_t v(t,x) + (v(t,x).\nabla)v(t,x) + \nabla p(t,x) = 0 \text{ for } (t,x) \text{ in } [0,T^*) \times \Omega, \\ \operatorname{div} v(t,x) = 0 \text{ for } (t,x) \text{ in } [0,T^*) \times \Omega. \end{array} \right. \tag{1.1}$$

In the above equation, t is the time (the problem under consideration is formulated for $T^* = +\infty$), and x is the position in the domain Ω. The function $v : [0,T^*) \times \Omega \to \mathbb{R}^2$ is the velocity field and $p : [0,T^*) \times \Omega \to \mathbb{R}$ is the pressure. The domain Ω is two-dimensional (2-D), bounded, regular and nonsimply connected (let us agree that the boundary $\partial\Omega$ is decomposed into $\partial\Omega = \Gamma_0 \cup \cdots \cup \Gamma_g$, where the components Γ_i are nonempty, connected, and disjoint).

The initial-boundary problem for equation (1.1) has been studied by Yudovich (see [10]). Given initial data

$$v_{|t=0} = v_0 \text{ in } \Omega, \tag{1.2}$$

where $v_0 : \overline{\Omega} \to \mathbb{R}^2$ is a divergence-free vector field, and appropriate boundary conditions, the system is well-posed. The boundary conditions can be taken as the following data:

- the normal component of the velocity $v(t,x) \cdot n(x)$ on the whole boundary $\partial\Omega$ for any time ($n(x)$ is the unit outward normal on $\partial\Omega$), which has to satisfy

$$\int_{\partial\Omega} v(t,x) \cdot n(x) dx = 0 \quad \forall t \in [0,T^*),$$

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- the vorticity $\omega(t,x) := \operatorname{curl} v(t,x)$ at the points of $\partial\Omega$ which, moved by the velocity flow, enter inside Ω, namely, the points in

$$\Sigma_{T^*}^{-} := \{(t,x) \in [0,T^*) \times \partial\Omega \ / \ v(t,x)\cdot n(x) < 0\}\,.$$

Let us emphasize that certain compatibility conditions between the solution and the boundary data must be taken into account in order to obtain a solution with suitable regularity.

In this paper, the boundary conditions (or a part of them) will be considered as a control (that is, a parameter to be determined, that we choose to influence the system). More precisely, we fix Σ an open part of the boundary; the part Σ of the boundary is the zone where we can choose the boundary conditions, whereas $\partial\Omega \setminus \Sigma$ represents a wall that cannot be crossed. In other words, we will consider the Euler system with

- the constraint

$$v(t,x)\cdot n(x) = 0 \quad \text{on } [0,T^*) \times (\partial\Omega\backslash\Sigma); \tag{1.3}$$

  that is, the fluid cannot enter or quit the domain through $\partial\Omega\backslash\Sigma$ (it must slip on it),
- the boundary condition on $[0,T^*)\times\Sigma$, which is the control to be chosen.

In this setting, the problem of controllability (i.e., to steer a prescribed initial state v_0 to a prescribed final state v_1 in an arbitrary time by choosing a relevant control) was answered affirmatively by Coron in [2], under the necessary condition that Σ meets each connected component of the boundary.

Here we are interested in the problem of asymptotic stabilizability of the equilibrium $v \equiv 0$ by means of a stationary feedback. In other words, we want to find a continuous function f of the state $\mathcal{S}(t)$ of the system at time t such that if the control $\mathcal{C}(t)$ is given at each time by $\mathcal{C}(t) = f(\mathcal{S}(t))$, then the resulting closed system makes 0 globally asymptotically stable in the sense that

- any solution defined on $[0,T^*)\times\overline{\Omega}$ with $T^* < +\infty$ can be extended for $t \geq T^*$;
- for any neighborhood $\mathcal{U}$ of 0, one can find another neighborhood $\mathcal{V}$ of 0 such that any solution of the closed system beginning in $\mathcal{V}$ is in $\mathcal{U}$ for any $t \geq 0$;
- any solution tends to 0 as $t \to +\infty$.

The above-mentioned problem was solved by Coron in the case of a simply connected domain; see [4].

Remark 1. As was already the case in the controllability problem, the condition that Σ meets any connected component of the boundary is a necessary condition to solve the problem. For instance, the vorticity around any "uncontrolled" connected component just slips on it and cannot be "modified." Another obstruction is the Kelvin law which states that the velocity circulation around any uncontrolled connected component of the boundary is constant. Throughout this paper, we will suppose that Σ meets every connected component of the boundary.

1.2. Mathematical setting. We have to specify which data will be the state of the system, and also the precise structure of the control. A natural state to consider would be the whole velocity field $v(t,\cdot)$ in $\overline{\Omega}$, but if we chose $\mathcal{S}$ this way, then (as we will consider solutions that are continuous up to the boundary) it would completely determine the choice of the control (since the boundary conditions described above would be given by the normal component of the trace of v on $[0,T^*)\times\Sigma$ and by the trace of $\operatorname{curl} v$ on $\Sigma_{T^*}^{-}$).

To avoid such a problem, as suggested in [3], we shall consider the following data as the state of the system:

$$\mathcal{S}(t) = (\omega(t,\cdot), \lambda_1(t), \dots, \lambda_g(t)), \tag{1.4}$$

where $\omega(t,\cdot) : \overline{\Omega} \to \mathbb{R}$ is the vorticity field (which is scalar in two dimensions), that is,

$$\omega(t,x) := \operatorname{curl} v(t,x),$$

and where λ_i, for $i = 1,\dots,g$, is the velocity circulation around the component Γ_i of $\partial\Omega$, that is,

$$\lambda_i(t) := \int_{\Gamma_i} v(t,x) \cdot \vec{\tau}(x) dx.$$

Here $\vec{\tau}(x)$ is the unit tangent vector field on $\partial\Omega$, chosen so that $(\vec{\tau}, n)$ should be direct. (Let us remark that consequently $\vec{\tau}$ endows the curve Γ_i with an orientation that is positive if the curve is an inner component of $\partial\Omega$ and negative in the case of the outer component.)

Remark 2. Of course, only g circulations of v around Γ_i are needed among $(g+1)$ available, since the sum of all these circulations is related to ω by Green's formula.

Once given the state $\mathcal{S}(t)$ and $v(t,x)\cdot n(x)$ on Σ (which is a part of the control, say $\mathcal{C}_1$), one can reconstruct $v(t,\cdot)$ in $\overline{\Omega}$, for each $t \in [0,T^*)$, by means of the following system:

$$\begin{cases} \operatorname{curl} v(t,\cdot) = \omega(t,\cdot) \quad \text{in } \Omega, \\ \operatorname{div} v(t,\cdot) = 0 \quad \text{in } \Omega, \\ v(t,\cdot)\cdot n(\cdot) = \mathcal{C}_1(t) \quad \text{on } \Sigma, \\ v(t,\cdot)\cdot n(\cdot) = 0 \quad \text{on } \partial\Omega \setminus \Sigma, \\ \displaystyle\int_{\Gamma_i} v(t,x).\vec{\tau}(x)dx = \lambda_i(t) \ \text{ for } i = 1,\dots,g. \end{cases} \tag{1.5}$$

The Euler equation can be written in terms of the state $\mathcal{S}(t)$: as is well known, the vorticity in two dimensions satisfies

$$\partial_t\omega + (v\cdot\nabla)\omega = 0 \quad \text{in } (0,T^*)\times\Omega, \tag{1.6}$$

or, equivalently,

$$\partial_t\omega + \operatorname{div}(\omega v) = 0 \quad \text{in } (0,T^*)\times\Omega, \tag{1.7}$$

and the velocity circulations satisfy

$$\lambda_i(t) - \lambda_i(0) = \int_0^t \int_{\Gamma_i} v(s,x)\cdot n(x)\omega(s,x)dxds. \tag{1.8}$$

One easily sees that the group composed of (1.5), (1.7), and (1.8) is equivalent to (1.1).

We still need to specify the exact structure of the control that we use. The first part of the control is the normal component of the velocity on Σ, which we call $\mathcal{C}_1$. We must stipulate the other part of the control, which concerns the entering vorticity.

Since $\omega(t,\cdot)$ is now a part of the state, it seems inappropriate to consider ω on $\Sigma^-_{T^*}$ as the second part of the control (as will be specified below, $\omega(t,\cdot)$ is continuous up to the boundary in our problem). Thus a natural control to consider would be the following:

$$\mathcal{C}(t) = \begin{cases} v(t,x)\cdot n(x) & \text{on } \Sigma, \\ \partial_t \omega(t,x) & \text{on } \Sigma^-, \end{cases} \tag{1.9}$$

with Σ^- given by

$$\Sigma^- := \{x \in \partial\Omega \ / \ v(t,x)\cdot n(x) < 0\}. \tag{1.10}$$

(To simplify the notation, we omit the dependence of Σ^- on t and $\mathcal{C}_1$—besides, essentially, Σ^- will be constant in what follows.)

Let us point out that the control described in (1.9) was the one used in [4]. However, for technical reasons that will be explained in section 2.1, we will have to consider the following control of *mixed type*:

$$\mathcal{C}(t) = (\mathcal{C}_1(t), \mathcal{C}_2(t), \mathcal{C}_3(t)), \tag{1.11}$$

with

$$\mathcal{C}_1 = v(t,x)\cdot n(x) \quad \text{on } \Sigma, \tag{1.12}$$

$$\mathcal{C}_2 = \partial_t \omega_1(t,x) \quad \text{on } \Sigma^-, \tag{1.13}$$

$$\mathcal{C}_3 = \omega_2(t,x) \quad \text{on } \Sigma^-, \tag{1.14}$$

and the boundary condition for the entering vorticity obtained as

$$\omega(t,x) = \omega_1(t,x) + \omega_2(t,x) \quad \text{on } \Sigma^-. \tag{1.15}$$

In other words, the entering vorticity ω is the sum of two terms: ω_1, whose time derivative we control, and ω_2, which we control directly.

Remark 3. Let us remark that this choice of the form of the control is important. Indeed, even the stabilizability by means of a simpler feedback law of the form $\partial_t \mathcal{C} = f(\mathcal{S})$ does not necessarily imply the stabilizability by means of a feedback law of the form $\mathcal{C} = f(\mathcal{S})$. See, for instance, [5] in the context of finite-dimensional systems.

We can now be more specific about the problem under study.

DEFINITION 1.1. *Given a feedback law*

$$(\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3) = (\mathcal{C}_1(\mathcal{S}), \mathcal{C}_2(\mathcal{S}), \mathcal{C}_3(\mathcal{S})),$$

1. *we shall call "the closed-loop system," with $\mathcal{S}$ as the unknown, the system* (1.5), (1.7), (1.8), *with boundary conditions given by* (1.12)–(1.15);
2. *we shall call $\mathcal{S} = (\omega, \lambda_1, \dots, \lambda_g)$ a solution of the closed-loop system if*
   - $\mathcal{S} \in C^0([0,T^*)\times\overline{\Omega};\mathbb{R}) \times C^0([0,T^*),\mathbb{R})^g$ *(for some $T^* > 0$);*
   - *$v(t,\cdot)$ being for each $t \in [0,T^*)$ the unique solution (in the sense of distributions in Ω) of* (1.5), *the functions $(\lambda_i)_{i=1,\dots,g}$ satisfy* (1.8) *for all $t \in [0,T^*)$, and ω satisfies* (1.7) *in the sense of distributions in $(0,T^*)\times\Omega$ and* (1.13)–(1.15) *in the sense of distributions on the open manifold*

$$\Big\{(t,x) \in (0,T^*)\times\Sigma,\ \mathcal{C}_1[\mathcal{S}(t)](x) < 0\Big\};$$

3. *we call "maximal" any solution that cannot be extended over its maximal time T^*.*

The purpose of this paper is to establish the following result.

THEOREM 1.2. *If Σ meets every connected component of the boundary, one can find three continuous functions $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$ defined on $C^0(\overline{\Omega};\mathbb{R}) \times \mathbb{R}^g$ and with values in $C^0(\Sigma;\mathbb{R})$, $C^0(\Sigma^-;\mathbb{R})$, and $C^0(\Sigma^-;\mathbb{R})$, respectively, such that the following properties are fulfilled:*

P1. *For any $(\omega_0, \lambda_1^0, \dots, \lambda_g^0) \in C^0(\overline{\Omega};\mathbb{R}) \times \mathbb{R}^g$, the closed-loop system with initial condition*

$$\mathcal{S}(0) = (\omega_0, \lambda_1^0, \dots \lambda_g^0) \tag{1.16}$$

has a global in time solution, and any local in time solution can be extended to $T^* = +\infty$ (in other words, any maximal solution is global).

P2. *For any $\varepsilon > 0$, there exists $\eta > 0$ such that if*

$$\max(\|\omega_0\|_{L^\infty(\Omega)}, |\lambda_1^0|, \dots, |\lambda_g^0|) < \eta,$$

then one has

$$\max(\|\omega(t,\cdot)\|_{L^\infty(\Omega)}, |\lambda_1(t)|, \dots, |\lambda_g(t)|) \leq \varepsilon,$$

for all $t \geq 0$ and any global in time solution of the closed-loop system satisfying (1.16).

P3. *Any global in time solution of the closed-loop system satisfies*

$$\max(\|\omega(t,\cdot)\|_{L^\infty(\Omega)}, |\lambda_1(t)|, \dots, |\lambda_g(t)|) \to 0 \textit{ as } t \to +\infty.$$

We will describe in section 2 the feedback law $(\mathcal{C}_1(\mathcal{S}), \mathcal{C}_2(\mathcal{S}), \mathcal{C}_3(\mathcal{S}))$ which involves Theorem 1.2. The precise result expressed in terms of this feedback law is given in section 2.5.

1.3. A related problem concerning the stationary equation. As pointed out in [3], the problem of asymptotic stabilizability by stationary feedback is connected with a problem concerning the stationary equation. Indeed, Brockett established a necessary condition for a finite-dimensional control system to be stabilizable; see [1].

THEOREM 1.3 (see Brockett [1]). *A necessary condition for the control system $\dot{x} = f(x,u)$ to be locally asymptotically stabilizable at the equilibrium point x_0 (satisfying $f(x_0, 0) = 0$) by a stationary feedback is that the image by f of any neighborhood of $(x_0, 0)$ is a neighborhood of 0.*

The corresponding statement of this necessary condition in the infinite-dimensional system considered here is precisely what is proved in [6], that is, the existence of solutions for the stationary problem with a small force term (and by scaling arguments, with any force term). Hence, the study in [6] can be viewed as a preliminary step before this one. As we will see in section 2, some tools developed in [6] are essential in the construction here. For more details, see [3], [4], and [6].

1.4. Structure of the paper. In the next section, we begin by giving the main ideas concerning the construction of the feedback law that yields Theorem 1.2, then we detail this construction (which is rather involved), and finally we state our precise result (Theorem 2.4), which takes the stated form of the feedback law into account and clearly involves Theorem 1.2.

In section 3, we fix the notation and give preliminary elementary statements, which are classical for the construction of global in time solutions to the Euler system in two dimensions. At the end of this section, a proposition that is central in the proof is stated.

Sections 4 and 5 prove the existence of local in time solutions of the closed-loop system defined with the control law introduced in section 2. This is done in two steps. In section 4, we construct a certain operator $\mathcal{F}$. Then section 5 proves by Schauder's fixed point theorem that the operator $\mathcal{F}$ has fixed points, which actually give local in time solutions to the closed-loop system.

Section 6 finishes the proof of Theorem 2.4, by proving that maximal solutions of the closed-loop system are global and satisfy the asymptotic stability properties P2 and P3 described in Theorem 1.2.

Finally, we put in the appendix the proofs of the most technical lemmas.

2. Description of the feedback.

2.1. Basic ideas. The most important feature of the 2-D Euler equation is a straightforward consequence of (1.6) (or (1.7)), precisely the following:

$$\text{the vorticity } \omega \text{ follows the flow of the velocity } v. \tag{2.1}$$

A direct consequence of this fact is that, to perform P3 in Theorem 1.2, a global solution of the closed-loop has to satisfy

$$\text{the flow of the velocity } v \text{ makes any point in } \overline{\Omega} \text{ go out of the domain} \tag{2.2}$$

(except perhaps for configurations with important zones of null vorticity from the beginning, but this situation is essentially nongeneric).

To get (2.2), we examine the Hodge decomposition of the velocity, which in non-simply connected domains takes the following form (as usual, $\nabla^\perp := (-\partial_{x^2}, \partial_{x^1})$):

$$v(t,x) = \nabla\phi(t,x) + \nabla^\perp\psi(t,x) + \sum_{k=1}^{g} \mu_i(t)\nabla^\perp\tau_i, \tag{2.3}$$

where $\tau_i \in C^\infty(\overline{\Omega};\mathbb{R})$, $i \in \{1,\dots,g\}$, is the solution of the system

$$\begin{cases} \Delta\tau_i = 0 & \text{in } \Omega, \\ \tau_i = 1 & \text{on } \Gamma_i, \\ \tau_i = 0 & \text{on } \partial\Omega\backslash\Gamma_i, \end{cases} \tag{2.4}$$

and where the different terms satisfy, for each t,

$$\begin{cases} \Delta\phi(t,x) = 0 & \text{in } \Omega, \\ \partial_n\phi(t,x) = v(t,x)\cdot n(x) & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

$$\begin{cases} \Delta\psi(t,x) = \operatorname{curl} v(t,x) & \text{in } \Omega, \\ \psi(t,x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.6}$$

$$\lambda_i(t) = -\sum_{j=1}^{g} \mu_j(t)\left(\int_\Omega \nabla\tau_i\cdot\nabla\tau_j\right) - \int_\Omega \omega(t,x)\tau_i(x)dx \quad \forall\, i \in \{1,\dots,g\}. \tag{2.7}$$

The family $\nabla^\perp\tau_i$ being clearly linearly independent, the matrix $(\int_\Omega \nabla\tau_i.\nabla\tau_j)_{i,j=1,\dots,n}$ is invertible.

Now to obtain (2.2), it seems rather arduous to rely on the last two terms in (2.3), because μ_j and $\nabla^\perp\psi$ are fixed at the beginning by the state and then slowly evolve according to the flow of v itself. On the contrary, the $\nabla\phi$ part is directly obtained from the control. Hence a natural idea, which is also present in [2], [4], and [6], is to fix the $v \cdot n$ part of the control so that the $\nabla\phi$ part in (2.3) should prevail over the other two in such a way that (2.2) is satisfied.

As in [4], this program is fulfilled by finding a function $\theta : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} \Delta\theta = 0 \quad \text{in } \Omega, \\ \partial_n\theta = 0 \quad \text{on } \partial\Omega \setminus \Sigma, \\ |\nabla\theta(x)| > 0 \quad \text{in } \overline{\Omega}. \end{cases} \tag{2.8}$$

Indeed, given such a function θ, one can hope that the control

$$\mathcal{C}_1 = f(\omega, \lambda_1, \dots, \lambda_g)\partial_n\theta \tag{2.9}$$

with $f(\omega, \lambda_1, \dots, \lambda_g)$ an adequate nonnegative function, which should be large when $(\omega, \lambda_1, \dots, \lambda_g)$ is large, will satisfy the requirements.

Once this part of the control is imposed, the idea is to choose the vorticity part of the control in the form

$$\partial_t\omega = -K(\omega, \lambda_1, \dots, \lambda_g)\omega \quad \text{in } \Sigma^-,$$

where $K(\omega, \lambda_1, \dots, \lambda_g)$ is an appropriate positive function. In this way, one can hope that the vorticity inside the domain will gradually be replaced by a smaller one.

However, there remain two issues:

- This might not be sufficient to get rid of the velocity circulations. The natural idea to diminish these circulations is to inject additional vorticity through $\Sigma^- \cap \Gamma_i$, as motivated by (1.8). This can raise a problem, because this injected vorticity could influence the other λ_j. In order to avoid this, we make this vorticity leave the domain through Γ_0.
- Because of (2.1), at a point of $\partial\Sigma^-$ where v is *pointing inside* Σ^-, there must be compatibility conditions on the control in vorticity so that the solution will have proper regularity. The reason for this is that, on one side of this point in $\partial\Omega$, the vorticity is determined by the control, whereas on the other side, it is determined by the incoming flow along the uncontrolled part of the boundary (see the points A in figures below). This is the main reason we must consider a control in the form (1.11)–(1.15): the continuity of the entering vorticity at this point is ensured by the $\mathcal{C}_3$-part of the control.
  Moreover, it will be technically simpler if these points in $\partial\Sigma^-$ where v is pointing inside Σ^- do not depend on the state. In fact, it can be expected that by choosing f in (2.9) properly, these points will be exactly those for which $\nabla\theta$ is pointing inside Σ^-.

Remark 4. It would seem natural to require the function θ to satisfy, besides (2.8),

$$\text{at any point of } \partial\gamma^-, \ \nabla\theta \text{ is pointing outside } \gamma^-, \text{ where } \gamma^- = \{x \in \Sigma \ / \ \partial_n\theta(x) < 0\}. \tag{2.10}$$

In the case of a simply connected domain, this is possible; see [4]. But this is no longer possible in the case of a nonsimply connected domain, since this would result in a null index of the vector field $\nabla\theta$ around the outer component of the boundary and in positive indices of $\nabla\theta$ around inner components, which would be inconsistent with (2.8).

Now that we have sketched the main ideas, we can construct a function θ satisfying (2.8); other conditions are required, either for technical reasons or to address the above-mentioned issues. The feedback law, which relies on θ, is constructed subsequently.

2.2. The function $\boldsymbol{\theta}$. The function θ that we introduce here has been used to prove the existence of solutions for the stationary problem; see section 1.3 and reference [6]. Precisely, we have the following result.

PROPOSITION 2.1 (see [6, Prop. 1]). *Consider Ω a nonempty, bounded, connected and regular domain in $\mathbb{R}^2$, assumed to be not simply connected. Denote $\Gamma_0, \dots, \Gamma_g$ the connected components of its boundary. Let n be the unit outward normal on $\partial\Omega$. Consider Σ an open part of $\partial\Omega$, which meets each connected component of $\partial\Omega$. Then there exists a function $\tilde\theta \in C^\infty(\overline{\Omega};\mathbb{R})$ that satisfies the following conditions:*

$$\Delta\tilde\theta = 0 \quad \text{in } \Omega, \tag{2.11}$$

$$\partial_n\tilde\theta = 0 \quad \text{on } \partial\Omega\backslash\Sigma, \tag{2.12}$$

$$|\nabla\tilde\theta(x)| > 0 \quad \text{for any } x \text{ in } \overline{\Omega}, \tag{2.13}$$

$$\text{for } \gamma^+(\tilde\theta) := \{x \in \partial\Omega \ / \ \partial_n\tilde\theta > 0\} \quad \text{and} \quad \gamma^-(\tilde\theta) := \{x \in \partial\Omega \ / \ \partial_n\tilde\theta < 0\}, \quad \text{one has: } \overline{\gamma^+(\tilde\theta)} \cap \overline{\gamma^-(\tilde\theta)} = \emptyset, \tag{2.14}$$

$$\gamma^+(\tilde\theta) \text{ and } \gamma^-(\tilde\theta) \text{ are unions of a finite number of intervals of } \partial\Omega \text{ with disjoint closures,} \tag{2.15}$$

$$\text{there exist } g \text{ points } \tilde M_1, \dots, \tilde M_g \text{ in } \gamma^-(\tilde\theta) \cap \Gamma_0, \text{ sent respectively on } \gamma^+(\tilde\theta) \cap \Gamma_1, \dots, \gamma^+(\tilde\theta) \cap \Gamma_g \text{ by the flow of } \nabla\tilde\theta, \text{ with the trajectories not touching } \partial\Omega\backslash[\gamma^+(\tilde\theta) \cup \gamma^-(\tilde\theta)]. \tag{2.16}$$

To describe properties of the flow, it is more convenient to work in a domain that is invariant by the flow. To that end, we consider $R > 0$ such that $\overline{\Omega} \subset B_R$ and introduce an operator π that extends continuous (resp., C^1) vector fields defined on Ω to continuous (resp., C^1) and compactly supported vector fields on B_R; see a more precise definition of π in section 3.1.

We have the following technical refinement of Proposition 2.1.

COROLLARY 2.2. *One can add the following requirement on $\tilde\theta$ (call $\tilde\Phi$ the flow of $\pi(\nabla\tilde\theta)$):*

$$\text{given any point } E \text{ in } \partial\gamma^+(\tilde\theta) \text{ such that } \nabla\tilde\theta(E) \text{ is pointing outside } \gamma^+(\tilde\theta), \text{ then for } t > 0,\ \tilde\Phi(t,0,E) \text{ does not meet another point in } \partial\gamma^+(\tilde\theta) \text{ pointing outside } \gamma^+(\tilde\theta) \text{ before leaving } \overline{\Omega}. \tag{2.17}$$

The proof of this corollary is postponed to the appendix.

In fact, the function θ used in this paper is given by $-\tilde\theta$; hence θ satisfies (2.11), (2.12), (2.13), (2.14), and (2.15). However, (2.16) must be replaced by

$$\text{there exist } g \text{ points } M_1, \dots, M_g \text{ in } \gamma^-(\theta) \cap \Gamma_1, \dots, \gamma^-(\theta) \cap \Gamma_g, \text{ sent respectively on } \gamma^+(\theta) \cap \Gamma_0 \text{ by } \Phi, \text{ with the trajectories not touching } \partial\Omega\backslash[\gamma^+(\theta) \cup \gamma^-(\theta)] \tag{2.18}$$

(here Φ is the flow of $\pi(\nabla\theta)$, and we define $\gamma^+(\theta) := \gamma^-(\tilde{\theta})$ and $\gamma^-(\theta) := \gamma^+(\tilde{\theta})$), and (2.17) must be replaced by

(2.19) given any point A in $\partial\gamma^-(\theta)$ such that $\nabla\theta(A)$ is pointing inside $\gamma^-(\theta)$, $\Phi(t,0,A)$ has not met another point in $\partial\gamma^-(\theta)$ at which $\nabla\theta$ is pointing inside $\gamma^-(\theta)$ for $t<0$ such that $\Phi([t,0],0,E)\subset\overline{\Omega}$.

A representation of what $\nabla\theta$ may look like is given in Figure 2.1 below. The dotted lines represent some flow lines of $\nabla\theta$.

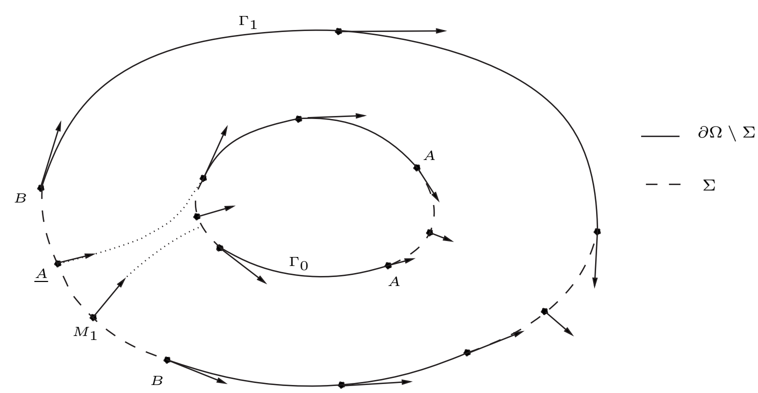

FIG. 2.1. *A representation of* $\nabla\theta$.

2.3. Some constructions relying on θ. We denote

$$\gamma^+ = \gamma^+(\theta) = \left\{x \in \partial\Omega \;\Big/\; \partial_n\theta(x) > 0\right\} \text{ and } \gamma^- = \gamma^-(\theta) = \left\{x \in \partial\Omega \;\Big/\; \partial_n\theta(x) < 0\right\}.$$

We also introduce

$$V(\theta) = \max_{\overline{\Omega}}\theta - \min_{\overline{\Omega}}\theta. \tag{2.20}$$

As in [6] we call A the points in $\partial\gamma^-$ on which $\nabla\theta$ is pointing *inside* γ^- and B the points in $\partial\gamma^-$ on which $\nabla\theta$ is pointing *outside* γ^-. In what follows, we denote by $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{M}$ the sets of A, B, and M_i points, respectively.

For each $A\in\mathcal{A}$, we introduce γ_A as the component of $\partial\Omega\backslash(\overline{\gamma^-}\cup\overline{\gamma^+})$ whose closure contains A (and the same for B). We also consider the points $\underline{A}$ defined as

$$\underline{A} := \Phi(t'_A,0,A), \text{ where } t'_A := \min\left\{t\le 0 \;/\; \Phi([t,0],0,A)\subset\overline{\Omega}\right\}. \tag{2.21}$$

Using (2.19), one sees that $\underline{A}\in\gamma^-(\theta)\cup\mathcal{B}$.

Given θ, we shall introduce some functions on $\overline{\gamma^-}$, called Γ_A and Λ_i, defined for each $A\in\underline{\mathcal{A}}$ and each $M_i\in\mathcal{M}$, respectively, and supported in a neighborhood of this point in $\overline{\gamma^-}$. Precisely, given $A\in\mathcal{A}$, call $\mathcal{V}_A$ a closed neighborhood of A in $\overline{\gamma^-}$, small enough that it contains neither any M_i point, nor any other point of $\mathcal{A}$, nor any $\underline{A}$ or

B point. Then define a function $\Gamma_A \in C^\infty(\overline{\gamma^-};\mathbb{R})$ satisfying

$$\begin{cases} -1 \leq \Gamma_A \leq 1, \\ \mathrm{Supp}(\Gamma_A) \subset \mathcal{V}_A, \\ \Gamma_A \equiv 1 \quad \text{in a neighborhood of } A, \\ \int_{\mathcal{V}_A} \Gamma_A(x) \nabla\theta(x) \cdot n(x) dx = 0. \end{cases} \tag{2.22}$$

Now, given an $M_i \in \mathcal{M}$, call $\mathcal{V}_{M_i}$ a closed neighborhood of M_i in $\gamma^- \cap \Gamma_i$, small enough that it contains neither any $\underline{A}$ point nor points of $\partial\gamma^-$, and such that all $\mathcal{V}_{M_i}$ are sent by Φ to $\gamma^+(\theta) \cap \Gamma_0$, with the trajectories not touching $\partial\Omega \backslash (\overline{\gamma^+} \cup \overline{\gamma^-})$ (as made possible by (2.18) and Gronwall's lemma; see Lemma 3.4 in section 3.3 below). Then define $\Lambda_i \in C^\infty(\overline{\gamma^-};\mathbb{R})$ satisfying

$$\begin{cases} \Lambda_i \leq 0, \\ \mathrm{Supp}(\Lambda_i) \subset \mathcal{V}_{M_i}, \\ \int_{\mathcal{V}_{M_i}} \Lambda_i(x) \nabla\theta(x) \cdot n(x) dx = 1. \end{cases} \tag{2.23}$$

Note that the last condition can be easily obtained since $\nabla\theta(x) \cdot n(x)$ is negative on $\mathcal{V}_{M_i}$.

We reduce if necessary the supports of Γ_A and Λ_i in order to obtain

$$\begin{array}{ll} \mathrm{Supp}(\Gamma_A) \cap \mathrm{Supp}(\Lambda_i) = \emptyset & \text{for any } A \in \mathcal{A} \text{ and any } i = 1, \dots, g, \\ \mathrm{Supp}(\Gamma_A) \cap \mathrm{Supp}(\Gamma_{A'}) = \emptyset & \text{for any } A, A' \in \mathcal{A} \text{ such that } A \neq A'. \end{array} \tag{2.24}$$

To make the notation lighter, we write

$$\mathrm{Supp}(\Lambda) := \bigcup_{i=1}^{g} \mathrm{Supp}(\Lambda_i),$$
$$\mathrm{Supp}(\Gamma) := \bigcup_{A \in \mathcal{A}} \mathrm{Supp}(\Gamma_A).$$

Also, we introduce

$$\begin{aligned} \|\Lambda\|_\infty &:= \max_{i=1}^{g} \|\Lambda_i\|_\infty, \\ \mathcal{T}(\Gamma) &= \sum_{A \in \mathcal{A}} \int_{\gamma^-} |\Gamma_A(x) \nabla\theta(x) \cdot n(x)| dx. \end{aligned} \tag{2.25}$$

We denote by ℓ a strict minimizer of the distance between the connected components of $\gamma^+ \cup \gamma^-$ and of the distances between the various $\mathrm{Supp}(\Gamma_A)$ with $A \in \mathcal{A}$, $\mathrm{Supp}(\Lambda_i)$ with $i \in \{1, \dots, g\}$, and points $B \in \mathcal{B}$.

The requirements on the supports are summarized in Figure 2.2 (where the arrows represent $\nabla\theta$).

2.4. The feedback law. Let us now describe the feedback law that we use. It is given by the following rule:

- If $(\omega(t), \lambda_1(t), \dots, \lambda_g(t)) = 0$, then fix

$$v \cdot n = \mathcal{C}_1 := 0 \text{ on } \Sigma. \tag{2.26}$$

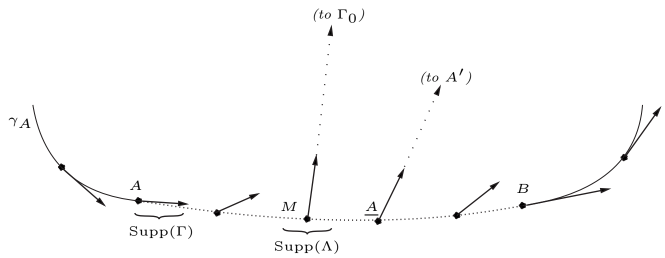

FIG. 2.2. *A representation of* $\Sigma^- \cap \Gamma_i$.

- If $(\omega(t), \lambda_1(t), \dots, \lambda_g(t)) \neq 0$, then fix

  (2.27)
$$\begin{cases} v \cdot n = \mathcal{C}_1 := K \max(\|\omega(t,\cdot)\|_{L^\infty(\Omega)}, |\lambda_1(t)|, \dots, |\lambda_g(t)|) \nabla\theta(x) \cdot n(x) \text{ on } \Sigma, \\ \omega = \omega_1 + \omega_2 \text{ on } \gamma^-, \end{cases}$$

  where ω_1 and ω_2 are given by

  (2.28)
$$\begin{cases} \partial_t \omega_1 = \mathcal{C}_2 := -M \max(\|\omega(t,\cdot)\|_{L^\infty(\Omega)}, |\lambda_1(t)|, \dots, |\lambda_g(t)|)\omega_1 \text{ on } \gamma^-, \\ \omega_2 = \mathcal{C}_3 := \sum_{A\in\mathcal{A}} \omega(t,A)\Gamma_A(x) - \sum_{i=1}^g \lambda_i(t)\Lambda_i(x) \text{ on } \gamma^-. \end{cases}$$

Consequently, we will have $\Sigma^- = \gamma^-$ except in the case $(\omega(t), \lambda_1(t), \dots, \lambda_g(t)) = 0$.

Remark that ω_1 is a function of the state since

$$\omega_1(t,\cdot) = \omega(t,\cdot) - \sum_{A\in\mathcal{A}} \omega(t,A)\Gamma_A(\cdot) + \sum_{i=1}^g \lambda_i(t)\Lambda_i(\cdot) \text{ on } \gamma^-.$$

The constants K and M are to be chosen large enough, as will be seen more precisely later.

Remark 5. Let us remark that, as the vorticity functions ω considered here are in the class $C^0([0,T) \times \overline{\Omega})$, the functions $t \mapsto \omega(t,A)$ are well-defined and continuous. Consequently, the feedback law is equivalent (in a distributional sense) to

$$\begin{aligned} \partial_t \omega(t,x) = &- M \max(\|\omega(t,\cdot)\|_{L^\infty(\Omega)}, |\lambda_1(t)|, \dots, |\lambda_g(t)|) \\ &\times \left[\omega(t,x) - \sum_{A\in\mathcal{A}} \omega(t,A)\Gamma_A(x) + \sum_{i=1}^g \lambda_i(t)\Lambda_i(x) \right] \\ &+ \sum_{A\in\mathcal{A}} \partial_t\omega(t,A)\Gamma_A(x) - \sum_{i=1}^g \left(\frac{d}{dt}\lambda_i(t)\right)\Lambda_i(x), \end{aligned} \tag{2.29}$$

where $\partial_t\omega(t,A)$ and $\frac{d}{dt}\lambda_i(t)$ can be recovered, in a formal sense for the first one, from the state thanks to (1.7)–(1.8).

2.5. The result. We rewrite the definition of the solutions of the system with the above described feedback.

DEFINITION 2.3. *A function* $(\omega, \lambda_1, \dots, \lambda_g)$ *in* $C^0([0,T^*) \times \overline{\Omega}; \mathbb{R}) \times C^0([0,T^*); \mathbb{R})^g$ *is a solution of the closed-loop system with the feedback law of section* 2.4 *if and only if it satisfies*

- *the relation* (1.8) *for all* $t \in [0, T^*)$ *and the equation* (1.7) *in the sense of distributions, where* v *is defined for each* t *by* (1.5), *with* $\mathcal{C}_1$ *fixed by* (2.27),
- *that on the domain* $\{t \in [0, T^*) \ / \ (\omega(t, \cdot), \lambda_1(t), \ldots, \lambda_g(t)) \not\equiv 0\} \times \gamma^-$ *(which is an open bidimensional manifold), the function*

$$\omega_1(t,x) = \omega(t,x) - \sum_{A \in \mathcal{A}} \omega(t,A)\Gamma_A(x) + \sum_{i=1}^{g} \lambda_i(t)\Lambda_i(x)$$

satisfies (2.28) *in a distributional sense.*

The following theorem is the main result of the paper; it clearly involves Theorem 1.2.

THEOREM 2.4. *If the constant* K *is large enough, and* M *is large enough depending on* K*, then for any initial condition* $(\omega_0, \lambda_1^0, \ldots, \lambda_g^0)$ *in* $C^0(\overline{\Omega}; \mathbb{R}) \times \mathbb{R}^g$*, there are solutions in* $C^0([0,T^*) \times \overline{\Omega}; \mathbb{R}) \times C^0([0,T^*); \mathbb{R})^g$ *of the closed-loop system (for some* $T^* > 0$*) satisfying*

$$(\omega, \lambda_1, \ldots, \lambda_g)_{|t=0} = (\omega_0, \lambda_1^0, \ldots, \lambda_g^0). \tag{2.30}$$

Moreover, any maximal solution is global and satisfies, for some $\mathcal{K} > 0$ *depending only on* Ω *and* Σ *(and on the functions* θ*,* Γ_A*, and* Λ_i *constructed on* (Ω, Σ)*),*

$$\max(\|\omega(t,\cdot)\|_{L^\infty(\Omega)}, |\lambda_1(t)|, \ldots, |\lambda_g(t)|) \leq \mathcal{K} \max(\|\omega_0\|_{L^\infty(\Omega)}, |\lambda_1^0|, \ldots, |\lambda_g^0|) \quad \forall\ t \geq 0, \tag{2.31}$$

$$\max(\|\omega(t,\cdot)\|_{L^\infty(\Omega)}, |\lambda_1(t)|, \ldots, |\lambda_g(t)|) \to 0 \quad \text{as } t \to +\infty. \tag{2.32}$$

3. Notation and prerequisites.

3.1. Notation. We essentially keep the notation of [4]. The velocity field will now be designated by y. We write $\Omega_T := [0,T] \times \overline{\Omega}$ and $\Sigma_T := [0,T] \times \partial\Omega$. In that context we write $pr_1(t,x) = t$ and $pr_2(t,x) = x$.

For X a nonempty compact subset of $\mathbb{R}^n$ and f a continuous function $X \to \mathbb{R}$, we introduce $\Xi_X[f]$ as the following function $\mathbb{R}^{+*} \to \mathbb{R}^{+*} \cup \{+\infty\}$:

$$\Xi_X[f](\varepsilon) := \sup\Big\{\eta > 0 \ / \ \forall x, x' \in X,\ |x - x'| \leq \eta \Rightarrow |f(x) - f(x')| \leq \varepsilon\Big\}; \tag{3.1}$$

for $x \in X$ we introduce $\Xi_X^x[f]$ as

$$\Xi_X^x[f](\varepsilon) := \sup\Big\{\eta > 0 \ / \ \forall x' \in X,\ |x - x'| \leq \eta \Rightarrow |f(x) - f(x')| \leq \varepsilon\Big\}. \tag{3.2}$$

These two functions are clearly related to the modulus of continuity of f.

Given K a compact set in $\mathbb{R}^2$ and f a continuous function $K \to \mathbb{R}^2$, we introduce the *log-Lipschitz* norm

$$q_K(f) := \|f\|_\infty + \sup\left\{\frac{|f(x) - f(x')|}{r(|x - x'|)},\ (x,x') \in K^2,\ x \neq x'\right\}, \tag{3.3}$$

where $r(s) = s - s\ln(s)$ in $(0,1)$ and $r(s) = s$ in $[1, +\infty)$.

We call log-Lipschitz the functions for which $q_K(f) < +\infty$, and denote by $\mathcal{LL}(K)$ their space, which we endow with the norm described in (3.3). We denote by $\mathcal{L}ip(K)$ the space of Lipschitz functions on K.

In the notation of a functional space, an index 0 refers to functions with compact support.

Let R be a positive real number, large enough so that $\overline{\Omega}$ is included in the open 0-centered ball with radius R, denoted by B_R.

We consider a linear continuous extension operator $\pi : C^0(\overline{\Omega}) \to C^0_0(B_R)$, which maps any $\mathcal{LL}(\overline{\Omega})$ function to an $\mathcal{LL}_0(B_R)$ function, and any $C^1(\overline{\Omega})$ function to a $C^1_0(B_R)$ function. We fix c_π a constant such that

$$\|\pi(f)\|_{C^0(\overline{B_R})} \leq c_\pi \|f\|_{C^0(\overline{\Omega})}, \qquad \|\pi(f)\|_{\mathcal{LL}(\overline{B_R})} \leq c_\pi \|f\|_{\mathcal{LL}(\overline{\Omega})}, \quad \text{and} \quad \|\pi(f)\|_{C^1(\overline{B_R})} \leq c_\pi \|f\|_{C^1(\overline{\Omega})}. \tag{3.4}$$

Frequently, we write (ω, λ_i) for $(\omega, \lambda_1, \ldots, \lambda_g)$. Also, using the canonical isomorphism $C^0([0,T^*) \times \overline{\Omega}; \mathbb{R}) \cong C^0([0,T^*); C^0(\overline{\Omega}; \mathbb{R}))$, we often write $\omega(t)$ for $\omega(t,\cdot)$ and $y(t)$ for $y(t,\cdot)$. In the same way, $\|\omega(\cdot)\|_{L^\infty(\Omega)}$ denotes the function

$$t \mapsto \operatorname*{ess\,sup}_{x\in\Omega} |\omega(t,x)|.$$

3.2. The Wolibner–Yudovich theorem. In this section, we introduce a classical tool to deal with flows of vector fields which do not satisfy the Lipschitz condition (in fact, the existence is the Peano theorem, and the uniqueness is the Osgood theorem). One has the following theorem.

THEOREM 3.1 (Wolibner–Yudovich theorem). *Consider $T > 0$ and a vector field $y \in L^\infty([0,T]; C^0_0(B_R; \mathbb{R}^2))$ such that for some constant C*

$$q_{\overline{B_R}}(y(t)) \leq C \ \textit{a.e. in } [0,T]. \tag{3.5}$$

Then there exists a unique map $\Phi^y \in C^0([0,T]\times[0,T]\times B_R; B_R)$, $(t,s,x) \mapsto \Phi^y(t,s,x)$, which is a flow of y, i.e., a function that satisfies

$$\Phi^y(t,s,x) = x + \int_s^t y(\tau, \Phi^y(\tau,s,x))d\tau \quad \forall (t,s,x) \in [0,T] \times [0,T] \times B_R. \tag{3.6}$$

Moreover, there are two positive constants C_{WY} and δ_{WY} depending only on (R,T,C) such that for any $(s,s',t,t',x,x') \in [0,T]^4 \times {B_R}^2$, one has

$$|\Phi^y(t,s,x) - \Phi^y(t',s',x')| \leq C_{WY}(|s-s'|^{\delta_{WY}} + |t-t'|^{\delta_{WY}} + |x-x'|^{\delta_{WY}}). \tag{3.7}$$

For a proof of this theorem, we refer to Wolibner [9], Yudovich [11, Lemma 6.3], or Kato [7].

Estimates such as (3.5) can easily be established by using the following theorem.

THEOREM 3.2 (Wolibner). *Consider $\omega \in C^0(\overline{\Omega}; \mathbb{R})$, $\lambda_1, \ldots, \lambda_g \in \mathbb{R}$. Then the function y defined in $C^0(\overline{\Omega}; \mathbb{R}^2)$ by*

$$\begin{cases} \operatorname{curl} y(x) = \omega(x) \quad \textit{in } \Omega, \\ \operatorname{div} y(x) = 0 \quad \textit{in } \Omega, \\ y(x)\cdot n(x) = 0 \quad \textit{on } \partial\Omega, \\ \displaystyle\int_{\Gamma_i} y(x)\cdot\vec{\tau}(x)dx = \lambda_i \quad \textit{for } i = 1,\ldots g, \end{cases} \tag{3.8}$$

satisfies the estimate

$$\|y\|_{\mathcal{LL}(\overline{\Omega})} \leq C_{\mathcal{LL}} \max\left(\|\omega\|_{C^0}, |\lambda_1|, \ldots, |\lambda_g|\right). \tag{3.9}$$

We refer, for instance, to [7, Lemma 1.4] or [9].

3.3. Elementary tools. Here we recall two elementary Gronwall inequalities.

LEMMA 3.3. *Consider two vector fields* $y_1 \in L^\infty([0,T]; \mathcal{L}ip_0(B_R;\mathbb{R}^2))$ *and* $y_2 \in L^\infty([0,T]; \mathcal{LL}_0(B_R;\mathbb{R}^2))$ *and their corresponding flows* Φ_1 *and* Φ_2 *(obtained with the Cauchy–Lipschitz theorem for the first and with the Wolibner–Yudovich theorem for the latter). Then one has, for all* $t \in [0,T]$,

(3.10)

$$\|\Phi_1(t,0,\cdot) - \Phi_2(t,0,\cdot)\|_{L^\infty(B_R)} \leq \exp\left(\int_0^t \|y_1(\tau)\|_{\mathcal{L}ip(B_R)} d\tau\right) \|y_1 - y_2\|_{L^1([0,t],L^\infty(B_R))}.$$

LEMMA 3.4. *Consider* $y \in L^\infty([0,T], \mathcal{L}ip_0(B_R;\mathbb{R}^2))$ *and its flow* Φ^y. *One has, for all* $(t,x,x') \in [0,T] \times B_R^2$,

$$|\Phi^y(t,0,x) - \Phi^y(t,0,x')| \leq \exp\left(\int_0^t \|y(\tau)\|_{\mathcal{L}ip(B_R)} d\tau\right) |x - x'|. \tag{3.11}$$

These are very classical and elementary statements.

3.4. A proposition concerning flows under the feedback law. We begin with a remark.

Remark 6. The flow Φ of $\pi(\nabla\theta)$ satisfies the following properties:

(i) for x in $\mathcal{B} \cup \gamma^-(\theta)$ and for any $t \in (0,T]$, $\exists \nu > 0$ s.t. $\Phi([t-\nu,t],t,x) \subset \mathbb{R}^2 \backslash \overline{\Omega}$;

(ii) for x in $\gamma^-(\theta)$ and for any $t \in [0,T)$, $\exists \nu > 0$ s.t. $\Phi([t,t+\nu],t,x) \subset \Omega$;

(iii) for any $B \in \mathcal{B}$ and for any $t \in [0,T)$, $\exists \nu > 0$ s.t. $\Phi([t,t+\nu],t,B) \subset \gamma_B$;

(iv) for any $A \in \mathcal{A}$, for any $t \in [0,T)$, and for $\tau < t$ s.t. $(t-\tau)\|\nabla\theta\|_\infty \leq \ell/2$, one has $\Phi(\tau,t,A) \in \gamma_A$;

(v) for any $A \in \mathcal{A}$, for any $t \in (0,T]$, and for $\tau > t$ s.t. $(\tau - t)\|\nabla\theta\|_\infty \leq \ell/2$, one has $\Phi(\tau,t,A) \in \Omega$;

(vi) for all $M_i \in \mathcal{M}$, one has $\Phi(t,0,M_i) \notin \partial\Omega \backslash [\overline{\gamma^+} \cup \overline{\gamma^-}]$ for $t > 0$ s.t. $\Phi([0,t],0,M_i) \subset \overline{\Omega}$; that is, the trajectories of M_i do not touch the set $\partial\Omega \backslash [\overline{\gamma^+} \cup \overline{\gamma^-}]$ before leaving the domain.

These properties are easy to prove using the form of $\nabla\theta$, the uniqueness of the flow, and the definition of ℓ.

The idea of the following proposition is to prove that if one imposes a control of the form (2.27) with K large enough, some of the properties in Proposition 2.1 and Remark 6 are also true for the flow of the resulting velocity y.

PROPOSITION 3.5. *There exist* $\kappa > 0$ *and* $\overline{K} := \overline{K}(\theta) > 0$ *such that, for any* $K \geq \overline{K}$, *any* $T > 0$, *any* $(\omega, \lambda_i) \in C^0(\Omega_T;\mathbb{R}) \times C^0([0,T];\mathbb{R}^g)$, *and any* $\alpha \in C^0([0,T],\mathbb{R}^+)$ *positive satisfying*

$$\alpha(t) \geq \max\left(|\lambda_1(t)|, \ldots, |\lambda_g(t)|, \|\omega(t)\|_\infty\right), \tag{3.12}$$

the solution $y \in C^0(\Omega_T;\mathbb{R}^2)$ *of*

$$\begin{cases} \operatorname{curl} y(t,x) = \omega(t,x) \quad \textit{for } (t,x) \in \Omega_T, \\ \operatorname{div} y(t,x) = 0 \quad \textit{for } (t,x) \in \Omega_T, \\ y(t,x)\cdot n(x) = K\alpha(t)\nabla\theta(x)\cdot n(x) \quad \textit{for } (t,x) \in \Sigma_T, \\ \displaystyle\int_{\Gamma_i} y(t,x)\cdot \vec{\tau}(x)dx = \lambda_i(t) \quad \textit{for } t \in [0,T] \textit{ and } i \in \{1,\ldots,g\} \end{cases} \tag{3.13}$$

satisfies

$$y(t,x)\cdot\nabla\theta(x) \geq \kappa K \alpha(t) \textit{ in } \Omega_T, \tag{3.14}$$

and, Φ^y *being the flow of* $\pi(y)$ *in* B_R*,*

for any point x *in* $\mathcal{B} \cup \gamma^-(\theta)$ *and for any* $t \in (0,T)$*,*

$$\exists \nu > 0 \textit{ such that } \Phi^y([t-\nu, t), t, x) \subset \mathbb{R}^2 \backslash \overline{\Omega}, \tag{3.15}$$

$$\exists \nu > 0 \textit{ such that } \Phi^y((t, t+\nu], t, x) \subset \Omega \cup [\partial\Omega \backslash (\overline{\gamma^-} \cup \overline{\gamma^+})], \tag{3.16}$$

$$\textit{for any } x \in \operatorname{Supp}(\Lambda) \textit{ and for any } t \in [0,T], \textit{ one has } \Phi^y(\tau, t, x) \notin \bigcup_{i=1}^{g} (\Gamma_i \cap \overline{\gamma^+}) \\ \textit{for } \tau \in (t, T] \textit{ such that } \Phi^y([t,\tau], t, x) \subset \overline{\Omega}, \tag{3.17}$$

(3.18)
for any $A \in \mathcal{A}$ *and for any* $t \in (0,T]$*, one has* $\Phi^y(\tau, t, A) \in \gamma_A$ *for* $\tau \in [0,t)$ *such that*

$$c_\pi(K\|\nabla\theta\|_\infty + C_{\mathcal{L}\mathcal{L}}) \left(\int_\tau^t \alpha(s) ds \right) \leq \ell/2,$$

(3.19)
for any $A \in \mathcal{A}$ *and for any* $t \in [0,T)$*, one has* $\Phi^y(\tau, t, A) \in \Omega$ *for* $\tau \in (t,T]$ *such that*

$$c_\pi(K\|\nabla\theta\|_\infty + C_{\mathcal{L}\mathcal{L}}) \left(\int_t^\tau \alpha(s) ds \right) \leq \ell/2,$$

$$\textit{for any } A \in \mathcal{A} \textit{ and for any } t \in [0,T], \textit{ one has } \Phi^y(\tau, t, A) \notin \operatorname{Supp}(\Gamma) \cup \operatorname{Supp}(\Lambda) \\ \textit{for } \tau \in [0,t) \textit{ such that } \Phi^y([\tau,t], t, A) \subset \overline{\Omega}. \tag{3.20}$$

Of course, the previous flow has to be understood in the Wolibner–Yudovich sense.

The proof of Proposition 3.5 is delayed to the appendix.

Remark 7. Let us remark that, as a consequence of (3.14), the points A and B defined for $\nabla\theta$ are still valid for the velocity y described in (3.13); that is, for all t in $[0,T]$, $y(t,A)$ (resp., $y(t,B)$) is tangent to $\partial\Omega$ and pointing inside (resp., outside) γ^- (for $K \geq \overline{K}$, provided $\alpha(t) > 0$).

In what follows, we will systematically suppose $K \geq \overline{K}$.

4. Construction of the operator $\mathcal{F}$. In this section, we construct an operator $\mathcal{F} = (F, G_1, \ldots, G_g)$, whose fixed points give local in time solutions to the closed-loop system. Roughly speaking, $F[\omega, \lambda_i]$ is the solution of an initial-boundary problem, which is approximately the closed-loop system described above, where (1.7) is replaced by the following *linear* equation:

$$\partial_t F[\omega, \lambda_i] + \operatorname{div}(y_{\omega,\lambda_i} F[\omega, \lambda_i]) = 0 \quad \text{in } (0, T^*) \times \Omega,$$

where y_{ω,λ_i} is the solution of (1.5) corresponding to (ω, λ_i). Then G_i corresponds approximately to the solution of (1.8).

4.1. The domain X. First let us define the space X on which $\mathcal{F}$ is to be defined. The operator $\mathcal{F}$ is split into $\mathcal{F} = (F, G_i)$, with $F : X \to C^0([0,T] \times \overline{\Omega}; \mathbb{R})$

and $G_i : X \to C^0([0,T];\mathbb{R})$ for $i \in \{1,\dots,g\}$. We introduce

$$(4.1)\quad X := \Big\{(\omega,\lambda_i) \in C^0([0,T]\times\overline{\Omega};\mathbb{R}) \times \left[C^0([0,T];\mathbb{R})\right]^g \Big/$$
$$\begin{aligned}
&\text{(a)}\quad \omega(0,\cdot) = \omega_0,\\
&\text{(b)}\quad \|\omega(t,\cdot)\|_\infty \le \mathcal{N}_{\omega_0,\lambda_i^0} \quad \text{for } t\in[0,T],\\
&\text{(c)}\quad \lambda_i(0) = \lambda_i^0 \quad \text{for } i = 1\dots g,\\
&\text{(d)}\quad \left\|\frac{\partial\omega}{\partial t}\right\|_{L^\infty_t(H^{-1}_x)} \le \kappa_1 \max\left(\|\omega_0\|_\infty, |\lambda_1^0|,\dots,|\lambda_g^0|\right)^2,\\
&\text{(e)}\quad |\lambda_i(t)| \le \mathcal{M}_{\omega_0,\lambda_i^0} \quad \text{for } t\in[0,T],\\
&\text{(f)}\quad \left|\frac{d\lambda_i}{dt^-}\right|(t) \le \kappa_2 \max\left(\|\omega_0\|_\infty, |\lambda_1^0|,\dots,|\lambda_g^0|\right)^2,\\
&\text{(g)}\quad \forall A\in\mathcal{A},\ \forall\varepsilon\in(0,1),\ \Xi_{[0,T]}[\omega(A,\cdot)](\varepsilon) \ge \tfrac{1}{c}\left(\Xi_{\gamma_A}[\omega_0](\varepsilon)\right)^{\frac{1}{\delta}},\\
&\text{(h)}\quad \forall\varepsilon\in(0,1),\ \Xi_{[0,T]}[\|\omega(\cdot)\|_{L^\infty(\Omega)}](\varepsilon) \ge \tfrac{1}{c}\min\left[\left(\Xi_{\overline{\Omega}}[\omega_0](\varepsilon)\right)^{\frac{1}{\delta}},\varepsilon\right]\Big\},
\end{aligned}$$

where the constant c depends on Ω, θ, T, and (ω_0,λ_i^0) and will be chosen large enough later. The other constants are fixed as follows:

$$(4.2)\quad \mathcal{N}_{\omega_0,\lambda_i^0} := \left[3 + |\Omega| + \frac{V(\theta)}{\kappa}\mathcal{T}(\Gamma)\right](1+\|\Lambda\|_\infty)\max(|\lambda_1^0|,\dots,|\lambda_g^0|,\|\omega_0\|_\infty),$$

$$(4.3)\quad \mathcal{M}_{\omega_0,\lambda_i^0} := \left[2 + |\Omega| + \frac{V(\theta)}{\kappa}\mathcal{T}(\Gamma)\right]\max(|\lambda_1^0|,\dots,|\lambda_g^0|,\|\omega_0\|_\infty),$$

and δ is defined with reference to section 3.2 as

$$(4.4)\quad \delta := \delta_{WY}(R,T,c_\pi(C_{\mathcal{LL}} + K\|\nabla\theta\|_{\mathcal{LL}(\overline{\Omega})})\mathcal{N}_{\omega_0,\lambda_i^0}).$$

The constants κ_1 and κ_2 depend on the domain and on the choice of θ, Λ, and Γ but not on T:

$$(4.5)\quad \kappa_1 := 2|\Omega|^{\frac{1}{2}}(C_{\mathcal{LL}} + K\|\nabla\theta\|_{L^\infty(\Omega)})\left[3 + |\Omega| + \frac{V(\theta)}{\kappa}\mathcal{T}(\Gamma)\right]^2(1+\|\Lambda\|_\infty)^2,$$

$$(4.6)\quad \kappa_2 := |\Sigma|K\|\nabla\theta\|_{L^\infty(\Omega)}\left[3 + |\Omega| + \frac{V(\theta)}{\kappa}\mathcal{T}(\Gamma)\right]^2(1+\|\Lambda\|_\infty)^2.$$

In (4.2)–(4.6), $|\Omega|$ stands for the Lebesgue measure of Ω and $|\Sigma|$ for the length of Σ, κ is the constant in (3.14), $V(\theta)$ is defined in (2.20), $\mathcal{T}(\Gamma)$ and $\|\Lambda\|_\infty$ are defined in (2.25), c_π is defined in (3.4), and $C_{\mathcal{LL}}$ is introduced in (3.9).

The time T is chosen in the following way:

- if $\max(|\lambda_1^0|,\dots,|\lambda_g^0|,\|\omega_0\|_\infty) = 0$, then 0 is a clear solution of the system and we pass (throughout sections 4 and 5 we will suppose $(\omega_0,\lambda_i^0) \neq (0,0)$);
- if $\|\omega_0\|_\infty = 0$, but $|\lambda_k^0| > 0$ for some $k\in\{1,\dots,g\}$, we fix

$$(4.7)\quad \underline{T} := \frac{|\lambda_k^0|}{2\kappa_2\max\left(\|\omega_0\|_\infty,|\lambda_1^0|,\dots,|\lambda_g^0|\right)^2};$$

- if $\|\omega_0\|_\infty \neq 0$, then we fix

$$(4.8)\quad \underline{T} := \frac{\|\omega_0\|_\infty}{2\kappa_1\max\left(\|\omega_0\|_\infty,|\lambda_1^0|,\dots,|\lambda_g^0|\right)^2}.$$

Finally, we define T as

$$T := \min \left(\frac{\ell}{4c_\pi (C_{\mathcal{L}\mathcal{L}} + K\|\nabla\theta\|_\infty)\mathcal{N}_{\omega_0,\lambda_i^0}}, \underline{T} \right). \tag{4.9}$$

It is quite clear that X is convex, closed, and nonempty (since, for example, it contains the constant function $t \mapsto (\omega_0, \lambda_i^0)$).

Let us finish this paragraph with a remark concerning the choice of T.

Remark 8. Let us remark that T allows us to have the following properties:

- for any $A \in \mathcal{A}$ and any $(\omega, \lambda_i) \in X$, if one puts y as in (3.13) (with α satisfying (3.12)), then for any $t \in [0, T]$ one has $\Phi^y([0, t], t, A) \subset \overline{\gamma_A}$ (this is a consequence of the first part in the minimum in (4.9), using (3.18));
- for any $(\omega, \lambda_i) \in X$, one has as a consequence of the definition of $\underline{T}$ and of points (d) and (f) in (4.1), that
  - if $\underline{T}$ is defined by (4.7), then for all $t \in [0, T]$,

$$|\lambda_k(t)| \geq \frac{|\lambda_k^0|}{2} > 0, \tag{4.10}$$

  - if $\underline{T}$ is defined by (4.8), then for all $t \in [0, T]$,

$$\|\omega(t,\cdot)\|_\infty \geq \kappa_3^{-1} \frac{\|\omega_0\|_{H^{-1}(\Omega)}}{2} > 0, \tag{4.11}$$

    where κ_3 is some constant such that $\|\cdot\|_{H^{-1}(\Omega)} \leq \kappa_3 \|\cdot\|_{L^\infty(\Omega)}$.

4.2. The operator *F*. Let us now describe the operator F. Consider $(\omega, \lambda_i) \in X$. First, we associate with (ω, λ_i) the function $\alpha_{\omega,\lambda_i} \in C^0([0, T], \mathbb{R}^{+*})$ by

$$\alpha_{\omega,\lambda_i}(t) := \max\left(|\lambda_1(t)|, \ldots, |\lambda_g(t)|, \|\omega(t)\|_\infty\right). \tag{4.12}$$

Then, we can associate the following vector field $y_{\omega,\lambda_i} \in C^0(\Omega_T, \mathbb{R}^2)$ as the solution of

$$\begin{cases} \operatorname{curl} y_{\omega,\lambda_i}(t,x) = \omega(t,x) \quad \text{for } (t,x) \in \Omega_T, \\ \operatorname{div} y_{\omega,\lambda_i}(t,x) = 0 \quad \text{for } (t,x) \in \Omega_T, \\ y_{\omega,\lambda_i}(t,x) \cdot n(x) = K\alpha_{\omega,\lambda_i}(t)\nabla\theta(x) \cdot n(x) \quad \text{for } (t,x) \in \Sigma_T, \\ \displaystyle\int_{\Gamma_i} y_{\omega,\lambda_i}(t,x) \cdot \vec{\tau}(x)dx = \lambda_i(t) \quad \text{for } t \in [0,T], \quad i = 1, \ldots, g, \end{cases} \tag{4.13}$$

where $K \geq \overline{K}(\theta)$. Then we extend this vector field to $[0, T] \times B_R$ by

$$\tilde{y}_{\omega,\lambda_i}(t,\cdot) = \pi[y_{\omega,\lambda_i}(t,\cdot)]. \tag{4.14}$$

By the Wolibner–Yudovich theorem (see section 3.2), this vector field yields a flow $\Phi^{\omega,\lambda_i} : [0, T] \times [0, T] \times B_R \to B_R$, i.e., a solution of (3.6). Now, given this flow, we can introduce the following two functions on $[0, T] \times \overline{\Omega}$ (which, roughly speaking, represent, respectively, the time and location of entrance in the domain of the point located at x at time t, when following the flow):

$$s_{\omega,\lambda_i}(t,x) := \max\left\{\tau \in [0,t],\ \Phi^{\omega,\lambda_i}(\tau,t,x) \in \overline{\gamma^-}\right\}, \tag{4.15}$$

$$a_{\omega,\lambda_i}(t,x) := \Phi^{\omega,\lambda_i}(s_{\omega,\lambda_i}(t,x),t,x), \tag{4.16}$$

with the convention that when $\{\tau \in [0,t],\ \Phi^{\omega,\lambda_i}(\tau,t,x) \in \overline{\gamma^-}\} = \emptyset$, then

$$s_{\omega,\lambda_i}(t,x) := 0,$$

and correspondingly

$$a_{\omega,\lambda_i}(t,x) := \Phi^{\omega,\lambda_i}(0,t,x).$$

Note that in all cases, one has $a_{\omega,\lambda_i}(t,x) \in \overline{\Omega}$. Note that the function s_{ω,λ_i} is not necessarily continuous (contrary to what happened in the simply connected case; see [4, equations (3.36)–(3.37)]).

Now we can define $F[\omega,\lambda_i](t,x)$ for $(t,x) \in \Omega_T$. In that order, we distinguish four cases, corresponding to different situations for $a_{\omega,\lambda_i}(t,x)$. In what follows, the constant M (which appears in (2.28)) is to be chosen large enough later.

Case α: $a_{\omega,\lambda_i}(t,x) \in \Omega \cup (\partial\Omega \backslash \overline{\gamma^-})$. This case is possible only if $s_{\omega,\lambda_i}(t,x) = 0$. In that case, we fix

$$F[\omega,\lambda_i](t,x) := \omega_0(a_{\omega,\lambda_i}(t,x)). \tag{4.17}$$

Case β: $a_{\omega,\lambda_i}(t,x) \in \overline{\gamma^-} \backslash [\mathrm{Supp}(\Gamma) \cup \mathrm{Supp}(\Lambda)]$. In that case, we fix

$$F[\omega,\lambda_i](t,x) := \omega_0(a_{\omega,\lambda_i}(t,x)) \exp\left(-M \int_0^{s_{\omega,\lambda_i}(t,x)} \alpha_{\omega,\lambda_i}(\tau)d\tau\right). \tag{4.18}$$

Case γ: $a_{\omega,\lambda_i}(t,x) \in \mathrm{Supp}(\Gamma_A)$ for some $A \in \mathcal{A}$. In that case, we fix

$$\begin{aligned} F[\omega,\lambda_i](t,x) &:= \\ &\Big[\omega_0(a_{\omega,\lambda_i}(t,x)) - \omega_0(A)\Gamma_A(a_{\omega,\lambda_i}(t,x))\Big] \exp\left(-M \int_0^{s_{\omega,\lambda_i}(t,x)} \alpha_{\omega,\lambda_i}(\tau)d\tau\right) \\ &\qquad + \omega_0(\Phi^{\omega,\lambda_i}(0, s_{\omega,\lambda_i}(t,x), A))\Gamma_A(a_{\omega,\lambda_i}(t,x)). \end{aligned} \tag{4.19}$$

Case δ: $a_{\omega,\lambda_i}(t,x) \in \mathrm{Supp}(\Lambda_k)$ for some $k \in \{1,\dots,g\}$. In that case, we fix

$$\begin{aligned} F[\omega,\lambda_i](t,x) &:= \\ &\Big[\omega_0(a_{\omega,\lambda_i}(t,x)) + \lambda_k^0 \Lambda_k(a_{\omega,\lambda_i}(t,x))\Big] \exp\left(-M \int_0^{s_{\omega,\lambda_i}(t,x)} \alpha_{\omega,\lambda_i}(\tau)d\tau\right) \\ &\qquad - \lambda_k(s_{\omega,\lambda_i}(t,x))\Lambda_k(a_{\omega,\lambda_i}(t,x)). \end{aligned} \tag{4.20}$$

Another way to express this is that $F[\omega,\lambda_i]$ is given by

$$\begin{aligned} F[\omega,\lambda_i](t,x) = &\left[\omega_0(a_{\omega,\lambda_i}(t,x)) - \sum_{A\in\mathcal{A}} \omega_0(A)\Gamma_A(a_{\omega,\lambda_i}(t,x)) + \sum_{k=1}^{g} \lambda_k^0 \Lambda_k(a_{\omega,\lambda_i}(t,x))\right] \\ &\times \exp\left(-M \int_0^{s_{\omega,\lambda_i}(t,x)} \alpha_{\omega,\lambda_i}(\tau)d\tau\right) \\ &+ \sum_{A\in\mathcal{A}} \omega_0(\Phi^{\omega,\lambda_i}(0, s_{\omega,\lambda_i}(t,x), A))\Gamma_A(a_{\omega,\lambda_i}(t,x)) \\ &- \sum_{k=1}^{g} \lambda_k(s_{\omega,\lambda_i}(t,x))\Lambda_k(a_{\omega,\lambda_i}(t,x)) \end{aligned} \tag{4.21}$$

with at most one nonnull term in each summation.

We also define on $[0,T]\times\overline{\gamma^-}$

(4.22)
$$\begin{cases} \omega^\sharp(t,x) := \left[\omega_0(x) - \sum_{A\in\mathcal{A}}\omega_0(A)\Gamma_A(x) + \sum_{k=1}^{g}\lambda_k^0\Lambda_k(x)\right]\exp\left(-M\int_0^t \alpha_{\omega,\lambda_i}(\tau)d\tau\right), \\ \omega^\flat(t,x) := \sum_{A\in\mathcal{A}}\omega_0(\Phi^{\omega,\lambda_i}(0,t,A))\Gamma_A(x), \\ \omega^\natural(t,x) := -\sum_{k=1}^{g}\lambda_k(t)\Lambda_k(x). \end{cases}$$

Finally, let us call $\tilde{F}$ the same operator as F, with Λ_k replaced by 0 for each $k\in\{1,\dots,g\}$; that is, $\tilde{F}$ is constant along the flow of Φ^{ω,λ_i}, and on $\overline{\gamma^-}$, one has

$$\text{(4.23)}\quad \tilde{F}[\omega,\lambda_i](t,x) := \left[\omega_0(x) - \sum_{A\in\mathcal{A}}\omega_0(A)\Gamma_A(x)\right]\exp\left(-M\int_0^t \alpha_{\omega,\lambda_i}(\tau)d\tau\right) + \sum_{A\in\mathcal{A}}\omega_0(\Phi^{\omega,\lambda_i}(0,t,A))\Gamma_A(x).$$

4.3. The operators G_i. Define the function $\mathcal{T}_{\omega_0,\lambda_i^0}:\mathbb{R}\to\mathbb{R}$ as

$$\text{(4.24)}\qquad \begin{cases} \mathcal{T}_{\omega_0,\lambda_i^0}(x) = x \quad \text{in } [-\mathcal{M}_{\omega_0,\lambda_i^0},\mathcal{M}_{\omega_0,\lambda_i^0}], \\ \mathcal{T}_{\omega_0,\lambda_i^0}(x) = \mathcal{M}_{\omega_0,\lambda_i^0} \quad \text{in } [\mathcal{M}_{\omega_0,\lambda_i^0},+\infty), \\ \mathcal{T}_{\omega_0,\lambda_i^0}(x) = -\mathcal{M}_{\omega_0,\lambda_i^0} \quad \text{in } (-\infty,-\mathcal{M}_{\omega_0,\lambda_i^0}]. \end{cases}$$

Let us now introduce the operators G_k, $k=1,\dots,g$. We define $G_k(\omega,\lambda_1,\dots,\lambda_g)\in C^0([0,T],\mathbb{R})$ by

(4.25)
$$G_k(\omega,\lambda_1,\dots,\lambda_g)(t) := \mathcal{T}_{\omega_0,\lambda_i^0}\left[\lambda_k^0 + \int_0^t\int_{\Gamma_k} y_{\omega,\lambda_i}(s,x)\cdot n(x)F[\omega,\lambda_i](s,x)dsdx\right].$$

5. Proof that $\mathcal{F}$ admits a fixed point. In this section, we prove that the operator $\mathcal{F} := (F,G_1,\dots,G_g)$ that we have just constructed admits a fixed point. This is done by using the Leray–Schauder fixed point theorem. Accordingly, we have to prove three properties:

- $\mathcal{F}(X)\subset X$;
- $\mathcal{F}(X)$ is compact in X for the C^0 topology;
- $\mathcal{F}$ is continuous for the C^0 topology.

We prove this in three distinct subsections.

5.1. $\mathcal{F}(X)\subset X$. The first point to prove is that, for $(\omega,\lambda_i)\in X$, $F[\omega,\lambda_i]$ is a continuous function of (t,x). Fixing $(t,x)\in[0,T]\times\overline{\Omega}$, let us prove that $F[\omega,\lambda_i]$ is continuous at the point (t,x). Again, we distinguish the four cases α, β, γ, and δ, corresponding respectively to the case when $a_{\omega,\lambda_i}(t,x)\in\Omega\cup(\partial\Omega\backslash\overline{\gamma^-})$, $a_{\omega,\lambda_i}(t,x)\in\overline{\gamma^-}\backslash[\mathrm{Supp}(\Gamma)\cup\mathrm{Supp}(\Lambda)]$, $a_{\omega,\lambda_i}(t,x)\in\mathrm{Supp}(\Gamma)$, and $a_{\omega,\lambda_i}(t,x)\in\mathrm{Supp}(\Lambda)$.

Case α: $a_{\omega,\lambda_i}(t,x)\in\Omega\cup(\partial\Omega\backslash\overline{\gamma^-})$. Therefore, $s_{\omega,\lambda_i}(t,x)=0$. By the continuity of the flow Φ^{ω,λ_i}, there exists a neighborhood of (t,x) on which $\Phi^{\omega,\lambda_i}(0,t',x')\in$

$\Omega \cup (\partial\Omega \backslash \overline{\gamma^-})$. Then the continuity of $F[\omega, \lambda_i]$ at the point (t, x) comes directly from the continuities of the flow and of ω_0, and from (4.17).

Case β: $a_{\omega,\lambda_i}(t,x) \in \overline{\gamma^-} \backslash [\mathrm{Supp}(\Gamma) \cup \mathrm{Supp}(\Lambda)]$. In this case, using (3.15) and the continuity of the flow, one sees that s_{ω,λ_i} is continuous at the neighborhood of (t, x). To be more precise the following hold:

– s_{ω,λ_i} is always upper semicontinuous, as follows from the continuity of the flow. Indeed, consider $s > s_{\omega,\lambda_i}(t, x)$; the trajectory

$$\Phi^{\omega,\lambda_i}(\tau, t, x) \quad \text{for } \tau \in [s, t]$$

does not touch $\overline{\gamma^-}$. Consequently, for (t', x') close enough to (t, x), the corresponding trajectory

$$\Phi^{\omega,\lambda_i}(\tau, t', x') \quad \text{for } \tau \in [s, t']$$

does not touch $\overline{\gamma^-}$ either. This leads to

$$\overline{\lim_{(t',x') \rightarrow (t,x)}} s_{\omega,\lambda_i}(t', x') \leq s_{\omega,\lambda_i}(t, x). \tag{5.1}$$

– s_{ω,λ_i} is lower semicontinuous in this neighborhood, as a consequence of (3.15). Indeed, for $s \in [s_{\omega,\lambda_i}(t,x) - \nu, s_{\omega,\lambda_i}(t,x))$, one has $\Phi^{\omega,\lambda_i}(s, t, x) \in B_R \backslash \overline{\Omega}$. Now using the continuity of the flow, this gives

$$\underline{\lim}_{(t',x') \rightarrow (t,x)} s_{\omega,\lambda_i}(t', x') \geq s_{\omega,\lambda_i}(t, x). \tag{5.2}$$

Then again, once we have obtained the continuity of s_{ω,λ_i}, the continuity of $F[\omega, \lambda_i]$ at the point (t, x) comes from the continuities of the flow and of ω_0, and from (4.21).

(Cases α and β are the only ones that arise in the simply connected case; see [4, Lemma 3.3].)

Case γ: $a_{\omega,\lambda_i}(t,x) \in \mathrm{Supp}(\Gamma_A)$ for some $A \in \mathcal{A}$. In this case, $s_{\omega,\lambda_i}(t,x)$ can be discontinuous, but only in the case where $a_{\omega,\lambda_i}(t,x) = A$, for the same reason as in case β. Indeed, when $a_{\omega,\lambda_i}(t,x) \neq A$, (3.15) is still valid, so the same argument stands true. So we suppose from now on that $a_{\omega,\lambda_i}(t,x) = A$. Consider (t', x') in a neighborhood of (t, x). We distinguish some subcases according to the locus of $a_{\omega,\lambda_i}(t', x')$.

- *Cases* β' *and* δ'*:* $a_{\omega,\lambda_i}(t', x') \in \overline{\gamma^-} \backslash \mathrm{Supp}(\Gamma_A)$ (including $a_{\omega,\lambda_i}(t', x') \in \mathrm{Supp}(\Gamma_{A'})$ for some $A' \in \mathcal{A} \setminus \{A\}$). These cases cannot happen if the neighborhood around (t, x) is chosen small enough (this is a clear consequence of the continuity of the flow).
- *Case* γ'*:* $a_{\omega,\lambda_i}(t', x') \in \mathrm{Supp}(\Gamma_A)$ (including A). Let us prove that in this case $a_{\omega,\lambda_i}(t', x')$ is close to $a_{\omega,\lambda_i}(t,x) = A$ and that $s_{\omega,\lambda_i}(t', x')$ is close to $s_{\omega,\lambda_i}(t,x)$ in the following sense: take a sequence (t'_n, x'_n) in the region of points in Case γ', converging to (t, x); then one has the corresponding convergences

$$a_{\omega,\lambda_i}(t'_n, x'_n) \rightarrow a_{\omega,\lambda_i}(t,x) \quad \text{and} \quad s_{\omega,\lambda_i}(t'_n, x'_n) \rightarrow s_{\omega,\lambda_i}(t,x) \quad \text{as } n \rightarrow +\infty.$$

Indeed,

  * given $\varepsilon > 0$, one has for n large enough $s_{\omega,\lambda_i}(t'_n, x'_n) \geq s_{\omega,\lambda_i}(t,x) - \varepsilon$. If not, for some subsequence of (t'_n, x'_n) (that we still call (t'_n, x'_n)), one has $s_{\omega,\lambda_i}(t'_n, x'_n) \rightarrow \overline{s}$, with

$$\overline{s} \leq s_{\omega,\lambda_i}(t,x) - \varepsilon.$$

By continuity of the flow, $\Phi^{\omega,\lambda_i}(s_{\omega,\lambda_i}(t'_n, x'_n), t'_n, x'_n)$ converges to $\Phi^{\omega,\lambda_i}(\overline{s}, t, x)$ as $n \to +\infty$. But as $\overline{s} \leq s_{\omega,\lambda_i}(t,x) - \varepsilon$ and using (3.18), one must have $\Phi^{\omega,\lambda_i}(\overline{s}, t, x) \in \gamma_A$, which contradicts the fact that it is a limit point of a sequence in $\overline{\gamma^-}$.

* we argue in the same way to get $s_{\omega,\lambda_i}(t'_n, x'_n) \leq s_{\omega,\lambda_i}(t,x) + \varepsilon$. If this does not happen, one finds a subsequence of (t'_n, x'_n) for which $s_{\omega,\lambda_i}(t'_n, x'_n)$ converges to $\overline{s} \geq s_{\omega,\lambda_i}(t,x) + \varepsilon$. This yields a contradiction with the continuity of the flow and (3.19).

Now, using the continuity of the flow and the convergence of $s_{\omega,\lambda_i}(t'_n, x'_n)$, one gets the continuity of $a_{\omega,\lambda_i}(t'_n, x'_n)$.
It follows from the choice of T—see in particular Remark 8—that

$$F^{\omega,\lambda_i}(t, A) = \omega_0(\Phi^{\omega,\lambda_i}(0, t, A))$$

and hence that $t \mapsto F^{\omega,\lambda_i}(t, A)$ is continuous. Now, using the continuity of Γ_A, we get a neighborhood of (t,x) in which the points in the Case γ' satisfy

$$|F^{\omega,\lambda_i}(t', x') - F^{\omega,\lambda_i}(t,x)| < \epsilon.$$

- *Case* α': $a_{\omega,\lambda_i}(t', x') \in \Omega \cup (\partial\Omega \backslash \overline{\gamma^-})$. Then one has

$$F[\omega, \lambda_i](t', x') = \omega_0(\Phi^{\omega,\lambda_i}[0, t', x']).$$

But by (4.19) we also have

$$F[\omega, \lambda_i](t,x) = \omega_0\left(\Phi^{\omega,\lambda_i}[0, s_{\omega,\lambda_i}(t,x), a_{\omega,\lambda_i}(t,x)]\right) = \omega_0(\Phi^{\omega,\lambda_i}[0,t,x]).$$

(Remember $a_{\omega,\lambda_i}(t,x) = A$.) Then again, using only the continuity of the flow and the continuity of ω_0, we get that $F[\omega,\lambda_i](t,x')$ can be made arbitrarily close to $F[\omega,\lambda_i](t,x)$ if we restrict x' to a small neighborhood of x of points in Case α'.

Case δ: $a_{\omega,\lambda_i}(t,x) \in \text{Supp}(\Lambda_k)$ for some $k \in \{1, \ldots, g\}$. This is again, as in Case β, a situation where s_{ω,λ_i} is continuous at the neighborhood of (t,x). Then the continuity in this case is a consequence of the continuities of the flow, of ω_0 and λ_k, and of (4.21).
From the continuity of $F(\omega,\lambda_i)$ and (4.25), we get that the functions $G_k(\omega, \lambda_1, \ldots, \lambda_g)$ are time continuous.

Once this is proved, we have to check that the points (a) to (h) in the definition of X are satisfied by $\mathcal{F}(\omega,\lambda_i)$ for $(\omega,\lambda_i) \in X$.

(a) That $F(\omega,\lambda_i)(0,\cdot) = \omega_0$ is a clear consequence of the construction of F.

(c) We have also $G_i(0) = \lambda_i^0$ for $i = 1, \ldots, g$, as a direct consequence of (4.3), (4.24), and (4.25).

(b) Let us check that for all $(t,x) \in [0,T] \times \overline{\Omega}$ one has $|F[\omega,\lambda_i](t,x)| \leq \mathcal{N}_{\omega_0,\lambda_i^0}$ by separating the four cases. Let us therefore consider (t,x) which achieves the maximum of $|F[\omega,\lambda_i](t,x)|$.

Case α: Suppose $a_{\omega,\lambda_i}(t,x) \in \Omega \cup (\partial\Omega \backslash \overline{\gamma^-})$. Then one has

$$|F[\omega,\lambda_i](t,x)| = |\omega_0(a_{\omega,\lambda_i}(t,x))| \leq \|\omega_0\|_\infty.$$

Case β: Suppose $a_{\omega,\lambda_i}(t,x) \in \overline{\gamma^-} \backslash [\text{Supp}(\Gamma) \cup \text{Supp}(\Lambda)]$. Then one has

$$|F[\omega,\lambda_i](t,x)| = |\omega_0(a_{\omega,\lambda_i}(t,x))| \exp\left(-M \int_0^{s_{\omega,\lambda_i}(t,x)} \alpha_{\omega,\lambda_i}(\tau) d\tau\right) \\ \leq \|\omega_0\|_\infty.$$

Case γ: Suppose $a_{\omega,\lambda_i}(t,x) \in \operatorname{Supp}(\Gamma_A)$ for some $A \in \mathcal{A}$. It is a consequence of (3.20) (or Remark 8) that

$$|F[\omega,\lambda_i](t,A)| \leq \|\omega_0\|_\infty.$$

Then one has, using (2.22) and (4.19),

$$\begin{aligned}|F[\omega,\lambda_i](t,x)| &= |\omega^\sharp(s_{\omega,\lambda_i}(t,x),a_{\omega,\lambda_i}(t,x)) + \omega^\flat(s_{\omega,\lambda_i}(t,x),a_{\omega,\lambda_i}(t,x))| \\ &\leq 3\|\omega_0\|_\infty \leq \mathcal{N}_{\omega_0,\lambda_i^0}.\end{aligned}$$

Case δ: Suppose $a_{\omega,\lambda_i}(t,x) \in \operatorname{Supp}(\Lambda_k)$ for some $k \in \{1,\dots,g\}$. As $F[\omega,\lambda_i](t,x)$ is transported by the flow inside Ω, it suffices to prove that for $(t,x) \in [0,T] \times \operatorname{Supp}(\Lambda_k)$ one has

$$\begin{aligned}&|\omega^\sharp(t,x) + \omega^\natural(t,x)| \\ &\qquad \leq \left[3 + |\Omega| + \frac{V(\theta)}{\kappa}\mathcal{T}(\Gamma)\right](1 + \|\Lambda\|_\infty)\max(|\lambda_1^0|,\dots,|\lambda_g^0|,\|\omega_0\|_\infty).\end{aligned}$$

Of course, one has

$$|\omega^\sharp(t,x)| \leq \|\omega_0\|_\infty + \|\Lambda\|_\infty|\lambda_k^0| \quad \text{on } [0,T]\times\operatorname{Supp}(\Lambda_k).$$

Now, using point (e), one gets, for $(t,x) \in [0,T]\times\operatorname{Supp}(\Lambda_k)$,

$$\begin{aligned}|\omega^\natural(t,x)| &\leq |\lambda_k(t)|\|\Lambda\|_\infty \\ &\leq \left[2 + |\Omega| + \frac{V(\theta)}{\kappa}\mathcal{T}(\Gamma)\right]\|\Lambda\|_\infty \max(|\lambda_1^0|,\dots,|\lambda_g^0|,\|\omega_0\|_\infty).\end{aligned}$$

Hence, one still gets the estimate (b) for $F(\omega,\lambda_i)$.

(e) That the functions G_i satisfy the constraint (e) is a direct consequence of (4.3) and (4.24).

(f) Point (f) is obtained as a consequence of (4.25). Consider $k \in \{1,\dots,g\}$ and $t \in (0,T]$. Then either t is a left accumulation of points where

$$\left|\lambda_k^0 + \int_0^t \int_{\Gamma_k} y_{\omega,\lambda_i}(t,x)\cdot n(x)F[\omega,\lambda_i](t,x)dx\right| \geq \mathcal{M}_{\omega_0,\lambda_i^0},$$

and in that case

$$\frac{dG_k(\omega,\lambda_i)}{dt^-} = 0,$$

or it is not, and one can write

$$\begin{aligned}\frac{dG_k(\omega,\lambda_i)}{dt^-} &= \int_{\Gamma_k} y_{\omega,\lambda_i}(t,x)\cdot n(x)F[\omega,\lambda_i](t,x)dx \\ &= K\int_{\Gamma_k}\alpha_{\omega,\lambda_i}(t)\nabla\theta(x)\cdot n(x)F[\omega,\lambda_i](t,x)dx.\end{aligned}$$

Using the fact that $(\omega,\lambda_i) \in X$ and consequently satisfies points (b) and (e), we get that

$$\|\alpha_{\omega,\lambda_i}(t)\nabla\theta(x)\cdot n(x)\|_{C^0([0,T]\times\partial\Omega)} \leq \mathcal{N}_{\omega_0,\lambda_i^0}\|\nabla\theta\|_\infty.$$

Using the fact that $F[\omega, \lambda_i]$ satisfies the estimate (b), this leads to

$$\left|\frac{dG_k(\omega, \lambda_i)}{dt^-}\right|(t) \leq \kappa_2 \max\left(\|\omega_0\|_\infty, |\lambda_1^0|, \dots, |\lambda_g^0|\right)^2.$$

(d) Define

$$\check{y}_{\omega,\lambda_i} = y_{\omega,\lambda_i} - K\alpha_{\omega,\lambda_i}\nabla\theta.$$

Using (3.9) and (b) and (e) in (4.1), one gets

$$\|\check{y}_{\omega,\lambda_i}\|_{L^\infty([0,T];\mathcal{LL}(\Omega))} \leq C_{\mathcal{LL}} \max(\mathcal{M}_{\omega_0,\lambda_i^0}, \mathcal{N}_{\omega_0,\lambda_i^0}) = C_{\mathcal{LL}}\mathcal{N}_{\omega_0,\lambda_i^0}.$$

It follows that one has

$$(5.3) \qquad \begin{cases} \|y_{\omega,\lambda_i}\|_{L^\infty([0,T]\times\Omega)} \leq C_{\mathcal{LL}}\mathcal{N}_{\omega_0,\lambda_i^0} + K\mathcal{N}_{\omega_0,\lambda_i^0}\|\nabla\theta\|_\infty, \\ \|y_{\omega,\lambda_i}\|_{L^\infty([0,T];\mathcal{LL}(\overline{\Omega}))} \leq C_{\mathcal{LL}}\mathcal{N}_{\omega_0,\lambda_i^0} + K\mathcal{N}_{\omega_0,\lambda_i^0}\|\nabla\theta\|_{\mathcal{LL}}. \end{cases}$$

Moreover, we have from point (b) that

$$\max_{t\in[0,T]} \|F[\omega, \lambda_i](t,\cdot)\|_{L^\infty(\Omega)} \leq \mathcal{N}_{\omega_0,\lambda_i^0}.$$

Consequently one gets

$$\|y_{\omega,\lambda_i}F[\omega, \lambda_i]\|_{L^\infty([0,T],L^\infty(\Omega))} \leq (C_{\mathcal{LL}} + K\|\nabla\theta\|_\infty)\mathcal{N}^2_{\omega_0,\lambda_i^0}.$$

But it follows from the construction that $F[\omega, \lambda_i]$ satisfies

$$\partial_t F[\omega, \lambda_i] + \operatorname{div}(y_{\omega,\lambda_i}F[\omega, \lambda_i]) = 0 \text{ in } \mathcal{D}'((0,T)\times\Omega).$$

This leads to the fact that $F[\omega, \lambda_i]$ satisfies constraint (d).

(g) This point follows from (3.7), (3.18), and (4.19). Indeed, one has, for any $A \in \mathcal{A}$ and any $(t, t') \in [0,T]^2$,

$$F[\omega, \lambda_i](t, A) - F[\omega, \lambda_i](t', A) = \omega_0(\Phi^{\omega,\lambda_i}(0, t, A)) - \omega_0(\Phi^{\omega,\lambda_i}(0, t', A)),$$

with $\Phi^{\omega,\lambda_i}(0, t', A)$ and $\Phi^{\omega,\lambda_i}(0, t, A)$ in $\overline{\gamma_A}$. Hence, so that

$$|F[\omega, \lambda_i](t, A) - F[\omega, \lambda_i](t', A)| \leq \varepsilon,$$

it is sufficient that $|\Phi^{\omega,\lambda_i}(0, t, A) - \Phi^{\omega,\lambda_i}(0, t', A)| \leq \Xi_{\overline{\gamma_A}}[\omega_0](\varepsilon)$. Using (3.7) and (5.3), one sees that it is sufficient that

$$|t - t'| \leq \left[\frac{\Xi_{\overline{\gamma_A}}[\omega_0](\varepsilon)}{C_{WY}(R, T, c_\pi\mathcal{N}_{\omega_0,\lambda_i^0}[C_{\mathcal{LL}} + K\|\nabla\theta\|_{\mathcal{LL}}])}\right]^{\frac{1}{\delta}}.$$

(h) We write $\hat{\omega} := F(\omega, \lambda_i)$. We divide the proof that $\hat{\omega}$ satisfies point (h) into two steps. First we estimate $\hat{t} - t$ so that

$$(5.4) \qquad \|\hat{\omega}(t,\cdot)\|_\infty - \|\hat{\omega}(\hat{t},\cdot)\|_\infty \leq \varepsilon \quad \text{for } \hat{t} > t,$$

and then we estimate $\hat{t} - t$ so that

$$(5.5) \qquad \|\hat{\omega}(t,\cdot)\|_\infty - \|\hat{\omega}(\hat{t},\cdot)\|_\infty \geq -\varepsilon \quad \text{for } \hat{t} > t.$$

In what follows, we suppose $\hat{t} > t$.

- *Sufficient condition for* (5.4). First, we state a lemma.

LEMMA 5.1. *There exist $\rho > 0$ and $\overline{\eta} > 0$ such that for any $\eta \in (0, \overline{\eta})$ and any $\overline{x} \in \overline{\Omega}$, there is some $\tilde{x} \in \Omega$ such that*

$$d(\tilde{x}, \overline{x}) \leq \eta \quad \textit{and} \quad d(\tilde{x}, \partial\Omega) \geq \rho\eta.$$

Proof of Lemma 5.1. We introduce $\mathcal{V}$, a tubular neighborhood of $\partial\Omega$ in $\mathbb{R}^2$. It is easy to see that the following procedure, which associates a point $\tilde{x}$ to any $\overline{x}$, allows us to find relevant ρ and $\overline{\eta}$:

– for $\overline{x}$ in $\mathcal{V} \cap \overline{\Omega}$, we pick a point $\tilde{x}$ in the direction of the inner normal to $\partial\Omega$,

– for $\overline{x}$ in $\overline{\Omega} \backslash \mathcal{V}$, we pick $\tilde{x} = \overline{x}$.

The details are left to the reader.

We go back to the sufficient condition for (5.4). Let us consider $t \in [0, T)$. We introduce $\overline{x} \in \overline{\Omega}$ such that

$$|\hat{\omega}(t, \overline{x})| = \|\hat{\omega}(t, \cdot)\|_\infty.$$

We have two possible situations.

– *First situation:* $d(\overline{x}, \overline{\gamma^+}) \geq \ell/4$. Then, considering (4.9), (5.3), the fact that $\hat{\omega}$ is constant along the flow of y_{ω,λ_i}, and the fact that a point following the flow of y_{ω,λ_i} can leave the domain only through $\overline{\gamma^+}$, one deduces that for $\hat{t} \in (t, T]$

$$\|\hat{\omega}(\hat{t})\|_{C^0(\overline{\Omega})} \geq |\hat{\omega}(\hat{t}, \Phi^{\omega,\lambda_i}(\hat{t}, t, \overline{x}))| = |\hat{\omega}(t, \overline{x})| = \|\hat{\omega}(t)\|_{C^0(\overline{\Omega})},$$

which is stronger than (5.4).

– *Second situation:* $d(\overline{x}, \overline{\gamma^+}) \leq \ell/4$, and hence, considering the definition of ℓ, one has

$$d(\overline{x}, \overline{\gamma^-}) \geq 3\ell/4.$$

Considering (4.9) and (5.3), one deduces that

$$\Phi^{\omega,\lambda_i}(s, t, x) \in \overline{\Omega} \backslash \overline{\gamma^-} \quad \forall s \in [0, T] \quad \text{and any } x \in \overline{\Omega} \text{ such that } d(x, \overline{x}) < \ell/2. \tag{5.6}$$

Moreover, using the fact that $\hat{\omega}$ is constant along the flow of y_{ω,λ_i} and the fact that a point following the flow of y_{ω,λ_i} cannot leave the domain except through $\overline{\gamma^+}$, we see that in order to have (5.4), it is sufficient that

$$|t - \hat{t}| \leq d(\tilde{x}, \overline{\gamma^+}) / \|y_{\omega,\lambda_i}\|_{L^\infty(\Omega_T)},$$

where $\tilde{x} \in \Omega$ is some point satisfying

$$|\hat{\omega}(t, \tilde{x}) - \hat{\omega}(t, \overline{x})| \leq \varepsilon. \tag{5.7}$$

Using (5.6), one sees that to get (5.7), it is sufficient to have

$$d(x, \overline{x}) < \ell/2 \text{ and } |\Phi^{\omega,\lambda_i}(0, t, \tilde{x}) - \Phi^{\omega,\lambda_i}(0, t, \overline{x})| \leq \Xi_{\overline{\Omega}}[\omega_0](\varepsilon),$$

and hence, using (3.7), that

$$|\tilde{x} - \overline{x}| \leq \min \left\{ \left(\frac{\Xi_{\overline{\Omega}}[\omega_0](\varepsilon)}{C_{WY}(R, T, c_\pi(C_{\mathcal{LL}} + K\|\nabla\theta\|_{\mathcal{LL}})\mathcal{N}_{\omega_0,\lambda_i^0})} \right)^{\frac{1}{\delta}}, \ell/4 \right\}.$$

It follows from Lemma 5.1 that one can find such a point $\tilde{x}$ satisfying (5.7) such that

$$d(\tilde{x}, \overline{\gamma^+}) \geq m \min \left(\left(\Xi_{\overline{\Omega}}[\omega_0](\varepsilon) \right)^{\frac{1}{\delta}}, 1 \right)$$

for some $m > 0$ independent from ε. So, using (5.3), one deduces that in order to have $\|\hat{\omega}(t, \cdot)\|_\infty - \|\hat{\omega}(\hat{t}, \cdot)\|_\infty \geq -\varepsilon$ (given $\varepsilon \in (0, 1)$), it is sufficient that

$$|t - \hat{t}| \leq \frac{1}{c} \min \left\{ \left(\Xi_{\overline{\Omega}}[\omega_0](\varepsilon) \right)^{\frac{1}{\delta}}, \varepsilon \right\}$$

for some $c > 0$ large enough depending on (ω_0, λ_i^0), on the domain, and on K and on θ, but not on (ω, λ_i) or on ε.

- *Sufficient condition for* (5.5). Let us prove that in order for (5.5) to happen, it is sufficient that

$$\|\hat{\omega}_{|\gamma^-}(s)\|_\infty - \|\hat{\omega}_{|\gamma^-}(t)\|_\infty \leq \varepsilon \quad \forall\, s \in [t, \hat{t}]. \tag{5.8}$$

Indeed, let us consider $\hat{x} \in \overline{\Omega}$ such that

$$|\hat{\omega}(\hat{t}, \hat{x})| = \|\hat{\omega}(\hat{t}, \cdot)\|_\infty.$$

- If $\Phi^{\omega,\lambda_i}([t, \hat{t}], \hat{t}, \hat{x})$ meets $\overline{\gamma^-}$, then clearly, using again the fact that $\hat{\omega}$ is constant along the flow of y_{ω,λ_i}, one deduces that

$$\|\hat{\omega}(\hat{t})\|_\infty \leq \sup_{s \in [t, \hat{t}]} \|\hat{\omega}_{|\gamma^-}(s)\|_\infty,$$

and hence

$$\begin{aligned} \|\hat{\omega}(\hat{t})\|_\infty - \|\hat{\omega}(t)\|_\infty &\leq \sup_{s \in [t, \hat{t}]} \|\hat{\omega}_{|\gamma^-}(s)\|_\infty - \|\hat{\omega}(t)\|_\infty \\ &\leq \sup_{s \in [t, \hat{t}]} \|\hat{\omega}_{|\gamma^-}(s)\|_\infty - \|\hat{\omega}_{|\gamma^-}(t)\|_\infty. \end{aligned}$$

- Otherwise, it is quite clear that

$$\|\hat{\omega}(\hat{t})\|_\infty = |\hat{\omega}(\hat{t}, \hat{x})| = |\hat{\omega}(t, \Phi^{\omega,\lambda_i}(t, \hat{t}, \hat{x}))| \leq \|\hat{\omega}(t)\|_{C^0(\overline{\Omega})},$$

which is stronger than (5.5).

Now, using (2.22) and the decomposition (4.22) of $F[\omega, \lambda_i]$, one gets on $(0, T) \times \gamma^-$,

$$\|\omega^\flat(t) - \omega^\flat(\hat{t})\|_\infty \leq \max_{A \in \mathcal{A}} |\omega_0(\Phi^{\omega,\lambda_i}(0, t, A)) - \omega_0(\Phi^{\omega,\lambda_i}(0, \hat{t}, A))|, \tag{5.9}$$

and

$$(5.10)\quad \frac{\partial}{\partial t}\left(\omega^\natural + \omega^\sharp\right) = -\sum_{k=1}^{g} \Lambda_k(x)\frac{d}{dt}\lambda_k(t) - M\alpha_{\omega,\lambda_i}(t) \times \left[\omega_0(x) - \sum_{A\in\mathcal{A}} \omega_0(A)\Gamma_A(x) + \sum_{k=1}^{g}\lambda_k^0\Lambda_k(x)\right]\exp\left(-M\int_0^t \alpha_{\omega,\lambda_i}(\tau)d\tau\right).$$

In order to have (5.8), it is sufficient that

$$\|\omega^\flat(t) - \omega^\flat(\hat{t})\|_\infty \le \varepsilon/2 \quad \text{and} \quad \|\omega^\natural(t) + \omega^\sharp(t) - \omega^\natural(\hat{t}) + \omega^\sharp(\hat{t})\|_\infty \le \varepsilon/2$$

and hence, using points (b), (e), and (f) in the definition of X and (3.18), (5.9), and (5.10), that

$$|t - \hat{t}| \le \frac{1}{c}\min(\varepsilon, \Xi_{\overline{\Omega}}[\omega_0](\varepsilon))$$

for some c large enough depending on ω_0 and λ_i^0, but not on ε, $\hat{t}$, or (ω, λ_i). So in all cases, in order to get (5.4) and (5.5), it is sufficient to have

$$|t - \hat{t}| \le \frac{1}{c}\min\left(\varepsilon, \Xi_{\overline{\Omega}}[\omega_0](\varepsilon)\right)$$

for proper c, which allows us to conclude.

This ends the proof that $\mathcal{F}(X) \subset X$. □

5.2. $\mathcal{F}(X)$ is compact in $C^0([0,T]\times\overline{\Omega};\mathbb{R})\times[C^0([0,T];\mathbb{R})]^g$. Consider a sequence $(\omega_n, \lambda_i^n)_{n\ge1}$ in X. Let us prove that one can extract a converging subsequence from $\mathcal{F}(\omega_n, \lambda_i^n)$ in $C^0([0,T]\times\overline{\Omega};\mathbb{R})\times\left[C^0([0,T];\mathbb{R})\right]^g$. We will have to extract subsequences from $(\omega_n, \lambda_i^n)_{n\ge1}$ several times to get the convergence and, in order to avoid too heavy notation, we will continue to write those subsequences (ω_n, λ_i^n) (instead of $(\omega_{\varphi(n)}, \lambda_i^{\varphi(n)})$, for instance). Moreover, we put an index n to objects constructed in section 4.2, corresponding to (ω_n, λ_i^n): each (ω_n, λ_i^n) yields a function α_n by (4.12) and then a vector field y_{ω_n,λ_i^n} on $[0,T]\times\overline{\Omega}$ by (4.13), which in turn yields a vector field $\tilde{y}_{\omega_n,\lambda_i^n}$ by (4.14). To these $\tilde{y}_{\omega_n,\lambda_i^n}$ one can associate a flow Φ_n by (3.6).

Using (3.9), (4.1), and (4.13), one easily gets that for some $C > 0$,

$$q_{\overline{\Omega}}(y_{\omega_n,\lambda_i^n}(t,\cdot) - K\alpha_n(t)\nabla\theta(\cdot)) \le C \quad \forall t\in[0,T],\ \forall n\ge 1,$$

and hence, using (4.1) again, the regularity of the function $\nabla\theta$, and (3.4), that for some $C' > 0$,

$$q_{\overline{B_R}}(\tilde{y}_{\omega_n,\lambda_i^n}(t,\cdot)) \le C' \quad \forall t\in[0,T],\ \forall n\ge 1.$$

Therefore, it follows from the Wolibner–Yudovich (see (3.7)) and Ascoli–Arzela theorems that Φ_n is relatively compact in $C^0([0,T]\times[0,T]\times B_R; B_R)$, say

$$(5.11)\qquad \Phi_n \longrightarrow \overline{\Phi} \quad \text{in } C^0([0,T]\times[0,T]\times B_R; B_R).$$

We now have to prove

$$(5.12)\qquad F(\omega_n, \lambda_i^n) \longrightarrow \overline{F} \quad \text{in } C^0([0,T]\times\overline{\Omega}) \text{ as } n\to+\infty$$

for a certain $\overline{F}$ in $C^0([0,T]\times\overline{\Omega};\mathbb{R})$. To that end, let us first prove that one can get some compactness on the sequence $F(\omega_n,\lambda_i^n)$ on the boundary, precisely

$$F(\omega_n,\lambda_i^n)_{|[0,T]\times\overline{\gamma^-}} \longrightarrow \overline{\omega} \text{ uniformly on } [0,T]\times\overline{\gamma^-} \text{ as } n\to+\infty, \tag{5.13}$$

for some $\overline{\omega}$ in $C^0([0,T]\times\overline{\gamma^-};\mathbb{R})$.

For this, let us decompose $F(\omega_n,\lambda_i^n)_{|[0,T]\times\overline{\gamma^-}}$ into

$$F(\omega_n,\lambda_i^n)=\omega_n^\flat+\omega_n^\sharp+\omega_n^\natural \quad \text{on } [0,T]\times\overline{\gamma^-},$$

as described in (4.22). Then we prove (5.13) in several steps.

- First, it follows from points (b) and (g) in the definition of X and the Ascoli–Arzela theorem that one can extract subsequences s.t.

$$\omega_n^\flat \longrightarrow \overline{\omega}^\flat := \sum_{A\in\mathcal{A}} \overline{\omega}(\cdot,A)\Gamma_A(\cdot) \quad \text{uniformly on } [0,T]\times\overline{\gamma^-} \text{ as } n\to+\infty. \tag{5.14}$$

- From the Ascoli–Arzela theorem and points (b) and (e) in the definition of X, one deduces that the sequence of functions $\Upsilon_n : t \mapsto M\int_0^t \alpha_n(\tau)d\tau$ is relatively compact in $C^0([0,T];\mathbb{R}^+)$ and hence, up to a subsequence, one has

$$\omega_n^\sharp \longrightarrow \overline{\omega}^\sharp := \left[\omega_0(\cdot) - \sum_{A\in\mathcal{A}}\omega_0(A)\Gamma_A(\cdot) + \sum_{i=1}^{g}\lambda_i^0\Lambda_i(\cdot)\right]\exp(-\overline{\Upsilon}), \quad \text{uniformly on } [0,T]\times\overline{\gamma^-} \text{ as } n\to+\infty. \tag{5.15}$$

- Extracting again a subsequence if necessary, one can get from points (e) and (f) in the definition of X that $\lambda_i^n \to \overline{\lambda_i}$ in $C^0([0,T],\mathbb{R})$. Consequently one gets

$$\omega_n^\natural \longrightarrow \overline{\omega}^\natural := \sum_{i=1}^{g}\Lambda_i(\cdot)\left[-\overline{\lambda_i}\right] \quad \text{uniformly on } [0,T]\times\overline{\gamma^-} \text{ as } n\to+\infty. \tag{5.16}$$

We get (5.13) from (5.14), (5.15), and (5.16).

Furthermore, using point (h) and the Ascoli–Arzela theorem one can extract a converging subsequence from $\|\omega_n(\cdot)\|_{L^\infty(\Omega)}$:

$$\|\omega_n(t)\|_{L^\infty(\Omega)} \longrightarrow \overline{N}(t) \text{ uniformly on } [0,T].$$

This yields, as $n\to+\infty$,

$$\begin{aligned}\alpha_n(t) &:= \max(|\lambda_1^n(t)|,\dots,|\lambda_g^n(t)|,\|\omega_n(t)\|_\infty)\\ &\longrightarrow \overline{\alpha}(t) := \max(|\overline{\lambda}_1(t)|,\dots,|\overline{\lambda}_g(t)|,\overline{N}(t)) \ \text{ in } C^0([0,T];\mathbb{R}^+).\end{aligned} \tag{5.17}$$

Let us prove that this implies that $\overline{\Phi}$ satisfies the conclusions of Proposition 3.5. We proceed exactly as for [4, equation (3.57)ff]. Let us recall the argument. Define

$$\check{y}_{\omega_n,\lambda_i^n} := y_{\omega_n,\lambda_i^n} - K\alpha_n(t)\nabla\theta.$$

Hence $\check{y}_{\omega_n,\lambda_i^n}$ satisfies (3.8) for (ω_n, λ_i^n). From (4.1) and usual elliptic estimates concerning (3.8), one deduces that, for any $r \in (2, +\infty)$,

$$\check{y}_{\omega_n,\lambda_i^n} \text{ is bounded in } C^0([0,T], W^{1,r}(\Omega;\mathbb{R}^2)),$$
$$\frac{\partial}{\partial t}\check{y}_{\omega_n,\lambda_i^n} \text{ is bounded in } L^\infty([0,T], H^{-1}(\Omega;\mathbb{R}^2)).$$

Using [8, Appendix C, Lemma C1] with $X = W^{1,r}(\Omega;\mathbb{R}^2)$ and $Y = H^{-1}(\Omega;\mathbb{R}^2)$, one deduces that $\check{y}_{\omega_n,\lambda_i^n}$ is relatively compact in $C^0([0,T], W^{1,r}(\Omega;\mathbb{R}^2) - w)$ and hence, using the Rellich–Kondrakov theorem, relatively compact in $C(\Omega_T)$. Hence using (5.17), the sequence y_{ω_n,λ_i^n} is itself relatively compact in $C(\Omega_T)$.

Hence, up to a subsequence, one has

$$y_{\omega_n,\lambda_i^n} \longrightarrow \overline{Y} \quad \text{in } C^0(\Omega_T;\mathbb{R}^2).$$

As in [4, equation (3.60)], we get that $\overline{\Phi} = \Phi^{\pi(\overline{Y})}$: this is a consequence of the definition of the flow and of the dominated convergence theorem.

Now let us observe that $\overline{Y}$ satisfies the assumptions of Proposition 3.5. This is a consequence of the fact that the sequence (ω_n, λ_i^n) satisfies them and of the fact that

$$\| \operatorname{curl} \overline{Y} \|_\infty \leq \liminf_{n\to+\infty} \|\omega_n\|_\infty.$$

Note that by (5.17) and by (4.10)–(4.11), one has $\overline{\alpha} > 0$.

Let us now show that, together with (5.11), this yields a convergence for $F(\omega_n, \lambda_i^n)$ in $[0,T] \times \overline{\Omega}$. The flow $\overline{\Phi}$ yields functions $\overline{s}$ and $\overline{a}$ as for (4.16) with Φ^{ω,λ_i} replaced by $\overline{\Phi}$. Then one can define $\overline{F}$ by

$$\overline{F}(t,x) := \overline{\omega}(\overline{s}, \overline{a}), \tag{5.18}$$

where we extend the definition of $\overline{\omega}$ on $\{0\} \times \overline{\Omega}$ by ω_0. Note that in this setting, the function $\overline{\omega}$ is well-defined and continuous in $(\{0\} \times \overline{\Omega}) \cup ([0,T] \times \overline{\gamma^-})$.

We have determined the potential limit $\overline{F}$; it remains to prove (5.12). Toward this end, let us prove the following equivalent assertion (by using a compactness argument):

(5.19)
$$\forall \varepsilon > 0 \text{ and } \forall (t,x) \in [0,T] \times \overline{\Omega}, \exists N \in \mathbb{N} \text{ and } \exists \mathcal{V} \text{ a vicinity of } (t,x) \text{ in } [0,T] \times \overline{\Omega}$$
$$\text{such that } \forall n \geq N, \text{ one has } \|F(\omega_n, \lambda_i^n) - \overline{F}\|_{C^0(\mathcal{V})} \leq \varepsilon.$$

To prove (5.19), we fix $\varepsilon > 0$ and $(t,x) \in [0,T] \times \overline{\Omega}$, and discuss them relative to the location of $\overline{a}(t,x)$.

Case α: $\overline{a}(t,x)$ in $\Omega \cup (\partial\Omega \backslash \overline{\gamma^-})$. Then, by continuity of $\overline{\Phi}$, one has, for any (t',x') in a neighborhood $\mathcal{V}_1$ of (t,x) in $[0,T] \times \overline{\Omega}$, that $\overline{a}(t',x') \in \Omega \cup (\partial\Omega \backslash \overline{\gamma^-})$. Then by (5.11) we get that for n large enough, $a_n(t',x') \in \Omega \cup (\partial\Omega \backslash \overline{\gamma^-})$ for $(t',x') \in \mathcal{V}_1$. So on $\mathcal{V}_1$, for such n, the expression of $F(\omega_n, \lambda_i^n)$ is $\omega_0(\Phi_n(t,0,x))$. So enlarging N and reducing $\mathcal{V}_1$ if necessary, using (5.18), we get the conclusion of (5.19) in this case.

Cases β and δ: $\overline{a}(t,x)$ in $\overline{\gamma^-} \backslash \operatorname{Supp}(\Gamma)$. In this case—remember that $\overline{\Phi}$ satisfies the conclusions of Proposition 3.5—one can show that (5.1)–(5.2) is true for $\overline{s}$, exactly as in Case β in section 5.1. Hence, $(\overline{s}, \overline{a})$ is continuous in a neighborhood of (t,x). Let us distinguish two subcases.

– *Subcase* (i): Suppose $\overline{s}(t,x) > 0$. In some neighborhood $\mathcal{V}_1$ of (t,x) in $[0,T] \times \overline{\Omega}$, one has $\overline{s}(t',x') > 0$. We introduce a neighborhood $\mathcal{W}$ of $(\overline{s}(t,x), \overline{a}(t,x))$ in $[0,T] \times (\overline{\gamma^-} \setminus \mathrm{Supp}(\Gamma))$, small enough that on it,

$$|\overline{\omega}(\tau,y) - \overline{\omega}(\overline{s}(t,x), \overline{a}(t,x))| \leq \varepsilon/2$$

(and $\tilde{\mathcal{W}}$ containing $\mathcal{W}$ in which this is valid with ε). Reducing $\mathcal{V}_1$ if necessary, we have

$$(\overline{s}(t',x'), \overline{a}(t',x')) \in \mathcal{W}$$

for (t',x') in $\mathcal{V}_1$. Using (3.15)–(3.16) and (5.11), one gets that for N large enough, one has

$$(s_n(t',x'), a_n(t',x')) \in \tilde{\mathcal{W}}$$

for (t',x') in $\mathcal{V}_1$, which yields the conclusion of (5.19).

– *Subcase* (ii): Suppose $\overline{s}(t,x) = 0$. Let us call $\mathcal{W}$ a vicinity of $(0, \overline{a}(t,x))$ in $(\{0\} \times \overline{\Omega}) \cup ([0,T] \times \overline{\gamma^-})$ such that for (τ,y) in $\mathcal{W}$ one has $|\overline{\omega}(\tau,y) - \overline{\omega}(0, \overline{a}(t,x))| \leq \varepsilon/2$. For (t',x') in a certain neighborhood $\mathcal{V}$ of (t,x) in $[0,T] \times \overline{\Omega}$ and N large enough, we have, for all $n \geq N$, either $s_n(t',x') \in pr_1(\mathcal{W})$ or $a_n(t',x') \in pr_2(\mathcal{W})$, because otherwise, we could find a subsequence for which $\Phi_n(\cdot,t',x')$ meets $\overline{\gamma^-} \setminus pr_2(\mathcal{W})$ for any n, which would be in contradiction with (5.11). With (5.13), this yields again the conclusion of (5.19).

Case γ: $\overline{a}(t,x)$ in $\mathrm{Supp}(\Gamma_A)$ for some $A \in \mathcal{A}$. We divide again into subcases:

– *Subcase* (i): Suppose $\overline{a}(t,x) \neq A$. Then one can reproduce the proof of the previous Cases β and δ if we take care that $\mathcal{W}$ stays at positive distance from $[0,T] \times \{A\}$.

– *Subcase* (ii): Suppose $\overline{a}(t,x) = A$ and $\overline{s}(t,x) > 0$. Note that in this case, $\overline{F}(t,x) = \omega_0(\overline{\Phi}(0,t,x))$ (thanks to (4.19) and (5.14)–(5.16)).
We fix $\mathcal{W}_1$ as an open vicinity of $(\overline{s}(t,x), \overline{a}(t,x))$ in $[0,T] \times \overline{\gamma^-}$ and $\mathcal{W}_2$ as an open vicinity of $(0, \overline{\Phi}(0,t,x))$ in $\{0\} \times \overline{\Omega}$, small enough such that, on both $\mathcal{W}_1$ and $\mathcal{W}_2$, we have $|\overline{\omega}(t',x') - \overline{\omega}(0, \overline{a}(t,x))| \leq \varepsilon/2$. Reduce them so that they are disjoint (this is possible thanks to (3.18)). Let us prove that for (t',x') in some neighborhood of (t,x) and for n large enough, we have

$$(s_n(t',x'), a_n(t',x')) \in \mathcal{W}_1 \cup \mathcal{W}_2. \tag{5.20}$$

If not, we would have an increasing sequence of integers $\varphi(n)$ and a sequence of points (t'_n, x'_n) converging to (t,x), for which

$$(s_{\varphi(n)}(t'_n,x'_n), a_{\varphi(n)}(t'_n,x'_n)) \notin \mathcal{W}_1 \cup \mathcal{W}_2.$$

By compactness of $[0,T] \times \overline{\Omega}$, one would have, up to a subsequence,

$$(s_{\varphi(n)}(t'_n,x'_n), a_{\varphi(n)}(t'_n,x'_n)) \to (\hat{s}, \hat{a}) \notin \mathcal{W}_1 \cup \mathcal{W}_2.$$

This would be in contradiction with $(t'_n, x'_n) \to (t,x)$ and (5.11):

– Suppose indeed that $\hat{s} > 0$. By continuity of $\overline{\Phi}$ one has

$$\overline{\Phi}(s_{\varphi(n)}(t'_n, x'_n), t'_n, x'_n) \to \overline{\Phi}(\hat{s}, t, x) \quad \text{as } n \to +\infty.$$

This involves $\hat{s} = \overline{s}$, because for $\tau \neq s$, we have $\overline{\Phi}(\tau, t, x) \notin \overline{\gamma}$. Consequently, we have $\hat{a} \in \overline{\gamma^-} \backslash pr_2(\mathcal{W}_1)$. But the trajectory from x to $\overline{\Phi}(0, t, x)$ has no such point, so this is impossible.

– Suppose now that $\hat{s} = 0$. Then by (5.11) one should have

$$a_{\varphi(n)}(t'_n, x'_n) \sim \overline{\Phi}(0, t'_n, x'_n) \to \overline{\Phi}(0, t, x)$$

and hence $(\hat{s}, \hat{a}) \in \mathcal{W}_2$.

Now (5.20) gives again the conclusion in (5.19).

– *Subcase* (iii): Suppose $\overline{a}(t, x) = A$ and $\overline{s}(t, x) = 0$. Again, we have $\overline{F}(t, x) = \omega_0(\overline{\Phi}(0, t, x))$. We fix $\mathcal{W}$ as a vicinity of $(0, A)$ in $(\{0\} \times \overline{\Omega}) \cup ([0, T] \times \overline{\gamma^-})$ on which again $|\overline{\omega}(t', x') - \overline{\omega}(0, \overline{a}(t, x))| \leq \varepsilon/2$ occurs. Then again, as in Subcase (ii) one can see that, for (t', x') in a small neighborhood of (t, x) and n large enough, one has

$$(s_n(t', x'), a_n(t', x')) \in \mathcal{W},$$

which leads to the conclusion.

So in all cases (5.19) is obtained; thus we get (5.12), and then the relative compactness of the sequence $G_i(\omega_n, \lambda_1^n, \dots, \lambda_g^n)$ follows. This concludes this section.

5.3. $\mathcal{F}$ is continuous for the $C^0([0,T] \times \overline{\Omega}; \mathbb{R}) \times [C^0([0,T]; \mathbb{R})]^g$ topology. Using the previous section; we see that it is enough to prove that if $(\omega_n, \lambda_i^n) \to (\overline{\omega}, \overline{\lambda}_i)$ for the C^0 topology, then $F[\omega_n, \lambda_i^n] \to F(\overline{\omega}, \overline{\lambda})$ pointwise. This is essentially the same argument as in the previous section; we do not repeat it. Now, as the convergence of $F(\omega_n, \lambda_i^n)$ is established, obtaining the convergence of $G_k(\omega_n, \lambda_i^n)$, $k = 1, \dots, g$, is straightforward.

5.4. Conclusion. Hence we get by the Leray–Schauder fixed point theorem a fixed point $(\omega^*, \lambda_i^*) \in X$ of the operator $\mathcal{F}$ described in section 4.2.

It follows from the construction that on $[0, T] \times \Omega$, one has

$$\partial_t F(\omega^*, \lambda_i^*) + \operatorname{div}(y_{\omega^*, \lambda_i^*} F(\omega^*, \lambda_i^*)) = 0 \tag{5.21}$$

and

$$F(\omega^*, \lambda_i^*)_{|t=0} = \omega_0. \tag{5.22}$$

Let us assume for the moment that the following lemma is proven.

LEMMA 5.2. *If M has been chosen large enough (depending on Ω, Σ, θ, and K), then for any $k \in \{1, \dots, g\}$ and all $t \in [0, T]$,*

$$\left| \lambda_k^0 + \int_0^t \int_{\Gamma_k} y_{\omega^*, \lambda_i^*}(\sigma, x).n(x) \omega^*(\sigma, x) dx d\sigma \right| \leq \mathcal{M}_{\omega_0, \lambda_i^0}. \tag{5.23}$$

If (5.23) is true, then by (1.8) one has

$$\lambda_k^* = G_k(\omega^*, \lambda_1^*, \dots, \lambda_g^*)(t) = \lambda_k^0 + \int_0^t \int_{\Gamma_k} y_{\omega^*, \lambda_i^*}(\sigma, x).n(x) \omega^*(\sigma, x) dx d\sigma. \tag{5.24}$$

Hence $(\omega^*, \lambda_1^*, \dots, \lambda_g^*)$ satisfies (1.6)–(1.8). Moreover, this fixed point satisfies the initial conditions (2.30). That $F(\omega^*, \lambda_i^*)$ satisfies the boundary condition (2.27)–(2.28) is a clear consequence of the construction of $\mathcal{F}$. So it remains only to prove Lemma 5.2.

Proof of Lemma 5.2. In the proof of Lemma 5.2, we will not use the specific form of T (in particular, Remark 8). This will be useful in section 6. Denote $y^* := y_{\omega^*, \lambda_i^*}$, $\alpha^* := \alpha_{\omega^*, \lambda_i^*}$, and $\Phi^* := \Phi^{\omega^*, \lambda_i^*}$. For any $k \in \{1, \dots, g\}$, we have (using (2.22), (2.23), and (4.22))

$$\begin{aligned}
&\int_0^t \int_{\Gamma_k} y^*(\sigma, x).n(x) F[\omega^*, \lambda_i^*](\sigma, x) dx d\sigma \\
&\quad = K \int_0^t \int_{\Gamma_k} \alpha^*(\sigma) \nabla\theta(x).n(x) \omega^*(\sigma, x) dx d\sigma \\
&\quad = K \int_0^t \alpha^*(\sigma) \left[-\lambda_k^*(\sigma) + \int_{\Gamma_k \cap \gamma^-} \nabla\theta(x).n(x) \omega^{*\sharp}(x) \right. \\
&\quad \left. + \int_{\Gamma_k \cap \gamma^+} \nabla\theta(x).n(x) \omega^*(\sigma, x) \right] dx d\sigma \\
&\quad = -K \int_0^t \alpha^*(\sigma) \lambda_k^*(\sigma) dx d\sigma \\
&\quad + K \int_0^t \alpha^*(\sigma) \int_{\Gamma_k \cap \gamma^-} \nabla\theta(x).n(x) \left[\omega_0(x) + \lambda_k^0 \Lambda_k(x) \right] \\
&\quad \cdot \exp\left(-M \int_0^\sigma \alpha^*(\tau) d\tau \right) dx d\sigma \\
&\quad + K \int_0^t \int_{\Gamma_k \cap \gamma^+} \alpha^*(\sigma) \nabla\theta(x).n(x) \omega^*(\sigma, x) dx d\sigma. \qquad (5.25)
\end{aligned}$$

Put

$$\mathcal{C}_0 = -\int_{\gamma^-} \nabla\theta(x).n(x) \left| \omega_0(x) + \sum_{i=1}^g \lambda_i^0 \Lambda_i(x) \right| dx \qquad (5.26)$$

and

$$\mathcal{C}_1 = \int_{\gamma^-} |\nabla\theta(x).n(x)| dx. \qquad (5.27)$$

We will show that (5.23) is valid provided M is large enough (in terms of Ω, Σ, θ, and K) to satisfy

$$\frac{K\mathcal{C}_0}{M} < \frac{\max(|\lambda_1^0|, \dots, |\lambda_g^0|, \|\omega_0\|_\infty)}{4} \quad \text{and} \quad \frac{K(2\mathcal{C}_1 + \mathcal{T}(\Gamma))}{M} < \frac{1}{4}, \qquad (5.28)$$

which we suppose from now on. In fact, the most problematic term in (5.25) is the last one. To estimate λ_k^*, we thus introduce the following function:

$$h(t) = K \int_0^t \int_{\bigcup_{k=1}^g (\Gamma_k \cap \gamma^+)} \alpha^*(\sigma) \nabla\theta(x) \cdot n(x) |\omega^*(\sigma, x)| dx d\sigma.$$

In order to estimate $h(t)$, let us consider $\tilde{\omega} := \tilde{F}[\omega^*, \lambda_i^*]$. As y^* satisfies the assumptions of Proposition 3.5, one easily sees (using (3.17)) that $\tilde{\omega} = \omega^*$ on $[0,T] \times \cup_{k=1}^{g}(\Gamma_k \cap \gamma^+)$. Also, by (4.23), one has

$$\tilde{\omega}(t,x) = \left(\omega_0(x) - \sum_{A\in\mathcal{A}} \omega_0(A)\Gamma_A(x)\right) \exp\left(-M\int_0^t \alpha^*(\tau)d\tau\right) + \sum_{A\in\mathcal{A}} \omega^*(t,A)\Gamma_A(x) \quad \text{on } [0,T]\times\overline{\gamma^-}. \tag{5.29}$$

But as $\tilde{\omega}$ satisfies

$$\partial_t\tilde{\omega} + \operatorname{div}(y^*\tilde{\omega}) = 0,$$

one deduces that

$$\frac{d}{dt}\left(\int_\Omega |\tilde{\omega}|\right)(t) = -\int_{\partial\Omega}(y^*\cdot n)|\tilde{\omega}|(t) = -K\int_{\partial\Omega}\alpha^*(t)\nabla\theta(x)\cdot n(x)|\tilde{\omega}|(t,x)dx.$$

Consequently, one gets

$$\begin{aligned} h(t) &\le K\int_0^t\int_{\gamma^+}\alpha^*(\sigma)\nabla\theta(x)\cdot n(x)|\tilde{\omega}(\sigma,x)|d\sigma dx \\ &\le \int_\Omega |\tilde{\omega}(0,\cdot)| - K\int_0^t\int_{\gamma^-}\alpha^*(\sigma)\nabla\theta(x)\cdot n(x)|\tilde{\omega}(\sigma,x)|d\sigma dx. \end{aligned}$$

Using (5.29), one gets

$$\begin{aligned} h(t) \le \int_\Omega |\omega(0,\cdot)| &+ 2K\mathcal{C}_1\|\omega_0\|_\infty \int_0^t \alpha^*(\sigma)\exp\left(-M\int_0^\sigma \alpha^*(\tau)d\tau\right)d\sigma \\ &- K\sum_{A\in\mathcal{A}}\int_0^t\int_{\gamma^-}\alpha^*(\sigma)\nabla\theta(x)\cdot n(x)|\omega^*(\sigma,A)\Gamma_A(x)|d\sigma dx, \end{aligned}$$

and hence

$$h(t) \le \int_\Omega |\omega(0,\cdot)| + 2K\frac{\mathcal{C}_1\|\omega_0\|_\infty}{M} + K\mathcal{T}(\Gamma)\max_{A\in\mathcal{A}}\int_0^t \alpha^*(\sigma)|\omega^*(\sigma,A)|d\sigma. \tag{5.30}$$

Let us now concentrate on the last term. For $A\in\mathcal{A}$ and $\sigma\in[0,T]$, let us define

$$\begin{aligned} &\mathfrak{s}(\sigma,A) := \min\left\{t\in[0,\sigma] \;\Big/\; \Phi^*([t,\sigma],\sigma,A)\subset\overline{\Omega}\right\}, \\ &\mathfrak{a}(\sigma,A) := \Phi^*(\mathfrak{s}(\sigma,A),\sigma,A). \end{aligned} \tag{5.31}$$

Using (3.20), (2.28), and the fact that ω^* is constant along the flow, one deduces that for any $A\in\mathcal{A}$ and any σ, one has

$$\omega(\sigma,A) = \omega_0(\mathfrak{a}(\sigma,A))\exp\left(-M\int_0^{\mathfrak{s}(\sigma,A)}\alpha^*(\tau)d\tau\right). \tag{5.32}$$

To estimate the last term in (5.30), we fix $A\in\mathcal{A}$.

- If t is such that

$$\int_0^t \alpha^*(\sigma)d\sigma \leq \frac{V(\theta)}{\kappa K}, \tag{5.33}$$

then using (5.32), one easily gets

$$\int_0^t \alpha^*(\sigma)|\omega(\sigma, A)|d\sigma \leq \frac{V(\theta)}{\kappa K}\|\omega_0\|_\infty.$$

(When considering a time T with the specific form (4.9), one could prove that on such a time interval we always have (5.33).)
- If not, call ξ the (unique) time for which

$$\int_0^\xi \alpha^*(\sigma)d\sigma = \frac{V(\theta)}{\kappa K}.$$

Let us prove that for $\sigma > \xi$, one has

$$\int_{\mathfrak{s}(\sigma,A)}^\sigma \alpha^*(\tau)d\tau \leq \frac{V(\theta)}{\kappa K}\left(= \int_0^\xi \alpha^*(\tau)d\tau\right). \tag{5.34}$$

This results from the fact that if one defines

$$\mu(t) = \theta(\Phi^*[t, \mathfrak{s}(\sigma, A), A]),$$

then one has (in the classical sense) for any $t \in [\mathfrak{s}(\sigma, A), \sigma]$

$$\begin{aligned}\frac{d\mu}{dt} &= y^*(t, \Phi^*[t, \mathfrak{s}(t, A), A]) \cdot \nabla\theta(\Phi^*[t, \mathfrak{s}(t, A), A])\\ &\geq \kappa K \alpha^*(t).\end{aligned}$$

Integrating this inequality between $\mathfrak{s}(\sigma, A)$ and σ yields (5.34). In particular, as a consequence of (5.34), one gets that for $\sigma > \xi$, one has $\mathfrak{s}(\sigma, A) > 0$. Consequently, using (5.32),

$$\begin{aligned}\int_0^t \alpha^*(\sigma)|\omega(\sigma, A)|d\sigma &= \int_0^\xi \alpha^*(\sigma)|\omega(\sigma, A)|d\sigma + \int_\xi^t \alpha^*(\sigma)|\omega(\sigma, A)|d\sigma\\ &\leq \frac{V(\theta)}{\kappa K}\|\omega_0\|_\infty\\ &\quad + \int_\xi^t \alpha^*(\sigma)\|\omega_0\|_\infty \exp\left(-M\int_0^{\mathfrak{s}(\sigma,A)} \alpha^*(\tau)d\tau\right)d\sigma.\end{aligned}$$

Now using (5.34), one gets

$$\begin{aligned}\int_0^{\mathfrak{s}(\sigma,A)} \alpha^*(\tau)d\tau &= \int_0^\sigma \alpha^*(\tau)d\tau - \int_{\mathfrak{s}(\sigma,A)}^\sigma \alpha^*(\tau)d\tau\\ &\geq \int_0^\sigma \alpha^*(\tau)d\tau - \int_0^\xi \alpha^*(\tau)d\tau.\end{aligned}$$

Hence,

$$\int_{\xi}^{t} \alpha^*(\sigma)\|\omega_0\|_\infty \exp\left(-M\int_0^{\mathfrak{s}(\sigma,A)} \alpha^*(\tau)d\tau\right) d\sigma \\ \leq \int_{\xi}^{t} \alpha^*(\sigma)\|\omega_0\|_\infty \exp\left(-M\int_{\xi}^{\sigma} \alpha^*(\tau)d\tau\right) d\sigma \leq \frac{\|\omega_0\|_\infty}{M}.$$

Finally, in all cases we get

$$h(t) \leq \int_\Omega |\omega_0| + \left(\frac{2K\mathcal{C}_1}{M} + \mathcal{T}(\Gamma)\left[\frac{V(\theta)}{\kappa} + \frac{K}{M}\right]\right)\|\omega_0\|_\infty.$$

Let us go back to λ_k^*. At times t for which (5.23) is valid in $[0,t]$ (this is at least the case for times in a neighborhood of 0), one has (5.24) and consequently, one gets

$$\begin{aligned}|\lambda_k^*(t)| &\leq |\lambda_k^0| + h(t) + K\int_0^t \left[-\alpha^*(s)\lambda_k^*(s) + \mathcal{C}_0\alpha^*(s)\exp\left(-M\int_0^s \alpha^*(\tau)d\tau\right)\right] ds \\ &\leq |\lambda_k^0| + \int_\Omega |\omega(0,\cdot)| + \frac{V(\theta)\mathcal{T}(\Gamma)}{\kappa}\|\omega_0\|_\infty + K\frac{(2\mathcal{C}_1 + \mathcal{T}(\Gamma))\|\omega_0\|_\infty + \mathcal{C}_0}{M} \\ &\quad -K\int_0^t \alpha^*(s)\lambda_k^*(s)ds.\end{aligned}$$

Hence, with $\alpha^*(t) \geq 0$, we get

$$|\lambda_k^*(t)| \leq |\lambda_k^0| + \int_\Omega |\omega(0,\cdot)| + \frac{V(\theta)\mathcal{T}(\Gamma)}{\kappa}\|\omega_0\|_\infty + K\frac{(2\mathcal{C}_1 + \mathcal{T}(\Gamma))\|\omega_0\|_\infty + \mathcal{C}_0}{M}. \tag{5.35}$$

Using (5.28), one gets

$$\begin{aligned}|\lambda_k(t)| &< \left(\frac{3}{2} + |\Omega| + \frac{V(\theta)\mathcal{T}(\Gamma)}{\kappa}\right)\max(|\lambda_1^0|,\dots,|\lambda_g^0|,\|\omega_0\|_\infty) \\ &< \mathcal{M}_{\omega_0,\lambda_i^0}.\end{aligned} \tag{5.36}$$

Hence (5.23) propagates during the whole time interval $[0,T]$.

So at this point, we have proven that for any (ω_0,λ_i^0), there exists a local solution of the closed-loop system.

6. End of the proof. To finish the proof, we still have to establish two propositions:

- any maximal solution of the closed-loop system is global,
- for any global solution of the closed-loop system, 0 is asymptotically stable.

6.1. Maximal solutions are global solutions. Consider a maximal solution (ω,λ_i) of the closed-loop system, say it is defined on $[0,T^*)$, with T^* maximal. Let us prove that $T^* = +\infty$. Toward this end, let us suppose by contradiction that $T^* < +\infty$, and prove that

$$(\omega(t),\lambda_i(t)) \longrightarrow (\omega(T^*),\lambda_i(T^*)) \text{ as } t \to T^{*-} \tag{6.1}$$

in $C^0(\overline{\Omega};\mathbb{R}) \times \mathbb{R}^g$. Using again the local existence result, this yields a contradiction. This is done as in [4, Proposition 3.4]. We first establish the following lemma.

LEMMA 6.1. *Let $T > 0$ and let $(\omega, \lambda_i) \in C^0([0,T] \times \overline{\Omega}) \times C^0([0,T])^g$ be a solution of the closed-loop system. Then one has for $(t,x) \in \Omega_T$ and any $s \in [0,t]$*

$$\omega(t,x) = \omega(s_{\omega,\lambda_i}(t,x), a_{\omega,\lambda_i}(t,x)), \tag{6.2}$$

and for any t in $[0,T]$, one has

$$\begin{aligned} &\max(\|\omega(t)\|_\infty, |\lambda_1|(t), \ldots, |\lambda_g|(t)) \\ &\quad \leq \left(3 + |\Omega| + \frac{V(\theta)\mathcal{T}(\Gamma)}{\kappa}\right)(1 + \|\Lambda\|_\infty) \max(\|\omega_0\|_\infty, |\lambda_1^0|, \ldots, |\lambda_g^0|). \end{aligned} \tag{6.3}$$

Proof of Lemma 6.1. First, such a solution satisfies

$$\partial_t \omega + \operatorname{div}(y_{\omega,\lambda_i}\omega) = 0.$$

A classical regularization argument shows that ω is constant along the flow of y_{ω,λ_i}, which yields (6.2).

We suppose that $(\omega(t), \lambda_1(t), \ldots, \lambda_g(t))$ does not vanish. If it does, then using the definition of the feedback and the fact that the vorticity is constant along the flow, $(\omega(t), \lambda_1(t), \ldots, \lambda_g(t))$ stays null. From now on, we work on the initial interval where $(\omega(t), \lambda_1(t), \ldots, \lambda_g(t))$ is not zero.

Now, the "λ_i" part in (6.3) can be reproduced from what was already done in (5.36), because we did not use the particular form of T but only (2.27)–(2.28), the fact that the vorticity follows the flow, and the fact that the velocity satisfies Proposition 3.5.

It remains to prove the "ω" part of (6.3). Having proved the estimate on the λ_i, this is done as for point (b) in the proof of $\mathcal{F}(X) \subset X$ (see section 5.1), except that now the estimate

$$|\omega(\cdot, A)| \leq \|\omega_0\|_\infty \quad \text{for any } A \in \mathcal{A}$$

comes now from (5.32) (and not from the choice of T).

Having proved Lemma 6.1, we get Hölder estimates on the flow from (3.7), which can consequently be extended on $[0, T^*]$, and then we get (6.1) approximately as for the continuity of $F(\omega, \lambda_i)$ in section 5.1 (we omit the details). Hence, using again the local existence theorem, we find a contradiction to $T^* < +\infty$.

6.2. 0 is asymptotically stable. Now that we have proved (6.3), it remains to prove (2.32). We consider again a global solution (ω, λ_i) of the closed-loop system; let us show that $\|(\omega, \lambda_i)(t)\|_\infty \to 0$ as $t \to +\infty$.

We suppose that $(\omega(t, \cdot), \lambda_i(t))$ never vanishes. If it does vanish for some $T > 0$, it follows from (2.26) and from the fact that the vorticity is constant along the flow of y_{ω,λ_i} that (ω, λ_i) is null in the neighborhood in time of $+\infty$; hence the result is valid.

This is done in several steps. First, we prove that $\omega(t, \cdot) \to 0$ on the entering zone γ^- and in a second step that this convergence holds in the rest of the domain. The convergence to zero of $\lambda_1, \ldots, \lambda_g$ is proved in the same step.

Again, we denote

$$\alpha(t) := \max\left(|\lambda_1(t)|, \ldots, |\lambda_g(t)|, \|\omega(t)\|_\infty\right).$$

We first prove the following lemma.

LEMMA 6.2. *Let $(\omega, \lambda_i) \in C^0([0,+\infty) \times \overline{\Omega}) \times C^0([0,+\infty))^g$ be a global solution of the closed-loop system. Then it satisfies*

$$\|\omega(t)\|_{C^0(\gamma^- \setminus [\mathrm{Supp}(\Lambda) \cup \mathrm{Supp}(\Gamma)])} \longrightarrow 0 \text{ as } t \to +\infty. \tag{6.4}$$

Proof of Lemma 6.2. Consider $x \in \gamma^- \setminus (\mathrm{Supp}(\Gamma) \cup \mathrm{Supp}(\Lambda))$. If $\omega(t_0, x) = 0$ for some time t_0, then it follows from (2.28) that $\omega(t,x) = 0$ for all t. Let us suppose that $\omega_0(x) \neq 0$. Then it follows from (2.28) that on $\gamma^- \setminus [\mathrm{Supp}(\Lambda) \cup \mathrm{Supp}(\Gamma)]$,

$$\partial_t |\omega(t,x)| \leq -M |\omega(t,x)|^2;$$

hence

$$|\omega(t,x)| \leq \frac{|\omega_0(x)|}{1 + M|\omega_0(x)| t},$$

and hence $\omega(t,x) \to 0$ as $t \to +\infty$. One sees that the estimate is uniform and hence that (6.4) holds.

Now, we have the following lemma.

LEMMA 6.3. *Let $(\omega, \lambda_i) \in C^0([0,+\infty) \times \overline{\Omega}) \times C^0([0,+\infty))^g$ be a global solution of the closed-loop system. Then it satisfies*

$$\|\omega(t)\|_{C^0(\mathrm{Supp}(\Gamma))} \longrightarrow 0 \text{ as } t \to +\infty. \tag{6.5}$$

Proof of Lemma 6.3. Let us first prove that for any $A \in \mathcal{A}$, one has

$$\omega(t, A) \longrightarrow 0 \text{ as } t \to +\infty. \tag{6.6}$$

It follows from (3.20) that

$$\Phi^{\omega,\lambda_i}(\mathfrak{s}(t,A), t, A) \notin \mathrm{Supp}(\Gamma) \cup \mathrm{Supp}(\Lambda),$$

with $\mathfrak{s}(t,A)$ given by (5.31). Now, we fix $\varepsilon > 0$ and let t_0 be a time such that for $t \geq t_0$, one has

$$\|\omega(t)\|_{C^0(\overline{\gamma^-} \setminus [\mathrm{Supp}(\Lambda) \cup \mathrm{Supp}(\Gamma)])} \leq \varepsilon.$$

Then if t_1 is such that $|\omega(t_1, A)| > \varepsilon$, one deduces that $\mathfrak{s}(t_1, A) \leq t_0$. Hence, using $\Phi^{\omega,\lambda_i}(s, t_1, A) \in \overline{\Omega}$ for $t_0 \leq s \leq t_1$ and the fact that the vorticity is constant along the flow, one gets

$$\|\omega(s)\|_{C^0(\overline{\Omega})} \geq \varepsilon \text{ for } t_0 \leq s \leq t_1.$$

But (3.14) implies that, in the classical sense,

$$\frac{d}{ds}\Big[\theta(\Phi^{\omega,\lambda_i}(s,t_0,x))\Big] = y_{\omega,\lambda_i}(s, \Phi^{\omega,\lambda_i}(s,t_0,x)) \cdot \nabla\theta(\Phi^{\omega,\lambda_i}(s,t_0,x)) \geq \kappa K \alpha(s) \geq \kappa K \varepsilon. \tag{6.7}$$

With the boundedness of θ in $\overline{\Omega}$, one sees that $|t_1 - t_0|$ must be bounded, which gives (6.6).

Now, consider $x \in \mathrm{Supp}(\Gamma_A)$ for a certain $A \in \mathcal{A}$ and t large enough. Then, (6.5) follows from the fact, due to (2.28), that for $t \in [0,T]$ and $x \in \mathrm{Supp}(\Gamma_A)$, one has

$$(6.8)\quad \omega(t,x) = (\omega_0(x) - \omega_0(A)\Gamma_A(x)) \exp\left(-M\int_0^t \alpha(\tau)d\tau\right) + \omega(t,A)\Gamma_A(x).$$

Indeed, given any $\varepsilon > 0$, for t_0 large enough, one has $|\omega_2(s,x)| \leq \varepsilon$ on $\mathrm{Supp}(\Gamma_A)$ for any $s \geq t_0$ (with the notation of ω_1 and ω_2 in (2.27)). One gets for any $s \geq t_0$ that

$$|\alpha(s)| \geq K\max(0, |\omega_1(s,x)| - |\omega_2(s,x)|) \geq K\max(0, |\omega_1(s,x)| - \varepsilon).$$

Then

- if $|\omega_1(t,x)| \leq 2\varepsilon$, then, because ω_1 has the form

$$\omega_1(s,x) = \left(\omega_0(x) - \sum_{A\in\mathcal{A}} \omega_0(A)\Gamma_A(x)\right)\exp\left(-M\int_0^s \alpha(\tau)d\tau\right),$$

  this inequality stays valid for $s \geq t$, or
- if $|\omega_1(t,x)| \geq 2\varepsilon$, then for $s \geq t$ such that this is still valid, one has $|\alpha(s)| \geq \varepsilon$; then with (6.8), one sees that $|\omega_1(s,x)|$ decreases until it reaches the previous situation.

Then, we establish the following lemma.

LEMMA 6.4. *Let $(\omega, \lambda_i) \in C^0([0,+\infty) \times \overline{\Omega}) \times C^0([0,+\infty))^g$ be a global solution of the closed-loop system. Then it satisfies*

$$(6.9)\qquad \|\omega(t)\|_{C^0(\gamma^+\cap\Gamma_k)} \longrightarrow 0 \text{ as } t \to +\infty \quad \forall k = 1,\dots,g.$$

Proof of Lemma 6.4. The limit (6.9) follows from (3.17), (6.2), and Lemmas 6.2 and 6.3. Indeed, we introduce t_0 such that for $t \geq t_0$, one has $|\omega(t,\cdot)| \leq \varepsilon$ on $\overline{\gamma^-}\setminus\mathrm{Supp}(\Lambda)$. Suppose that we could find, for times arbitrarily large, some points in $\cup_{i=1}^g(\overline{\gamma}^+ \cap \Gamma_k)$ for which $|\omega(t,x)| > \varepsilon$ and hence, by (3.17), such that $s_{\omega,\lambda_i}(t,x) \leq t_0$. This contradicts (6.7) and the boundedness of θ in $\overline{\Omega}$.

Now, we have the following lemma.

LEMMA 6.5. *Let $(\omega, \lambda_i) \in C^0([0,+\infty) \times \overline{\Omega}) \times C^0([0,+\infty))^g$ be a global solution of the closed-loop system. Then it satisfies*

$$(6.10)\qquad \|\omega(t)\|_{C^0(\mathrm{Supp}(\Lambda))} \longrightarrow 0 \text{ as } t \to +\infty.$$

Proof of Lemma 6.5. Fix $k \in \{1,\dots,g\}$. It follows from (1.8), (2.22), (2.23), (2.27), and (2.28) that λ_k satisfies

$$(6.11)\quad \frac{d}{dt}\lambda_k(t) = -K\alpha(t)\lambda_k(t) + K\alpha(t)\int_{\Gamma_k\cap\gamma^-} \nabla\theta(x)\cdot n(x)\omega_1(t,x)dx + K\alpha(t)\int_{\Gamma_k\cap\gamma^+} \nabla\theta(x)\cdot n(x)\omega(t,x)dx.$$

But ω_1 converges uniformly to 0 (this is proved exactly as Lemma 6.2), and by Lemma 6.4, the second integral in (6.11) converges to 0 (remember that $\alpha(t)$ is bounded thanks to (6.3)). Hence, given $\varepsilon > 0$, there exists t_0 such that for $t \geq t_0$,

$$\left|\int_{\Gamma_k\cap\gamma^-} \nabla\theta(x)\cdot n(x)\omega_1(t,x)dx\right| + \left|\int_{\Gamma_k\cap\gamma^+} \nabla\theta(x)\cdot n(x)\omega(t,x)dx\right| \leq \varepsilon.$$

Consequently, for $t \geq t_0$, if $\lambda_k(t) \geq 2\varepsilon$, using $|\lambda_k(t)| \leq \alpha(t)$, one gets

$$\frac{d}{dt}\lambda_k(t) \leq -\varepsilon K \lambda_k(t),$$

and if $\lambda_k(t) \leq -2\varepsilon$, one gets

$$\frac{d}{dt}\lambda_k(t) \geq -\varepsilon K \lambda_k(t).$$

This yields

$$\lambda_i(t) \longrightarrow 0 \text{ as } t \to +\infty \text{ for } i = 1, \dots, g. \tag{6.12}$$

Then having proved (6.12), (6.10) follows from the same procedure as the one at the end of Lemma 6.3.

These lemmas allow us to establish the following proposition.

PROPOSITION 6.6. *Let* $(\omega, \lambda_i) \in C^0([0, +\infty) \times \overline{\Omega}) \times C^0([0, +\infty))^g$ *be a global solution of the closed-loop system. Then it satisfies*

$$\max(\|\omega(t)\|_{C^0(\overline{\Omega})}, |\lambda_1(t)|, \dots, |\lambda_g(t)|) \longrightarrow 0 \text{ as } t \to +\infty. \tag{6.13}$$

Proof of Proposition 6.6. The $\lambda_i(t)$-part is precisely (6.12). For the ω-part, consider $\tau(\varepsilon)$ such that for $t \geq \tau(\varepsilon)$, one has

$$\|\omega_{|(\gamma^+ \cup \gamma^-)}(t, \cdot)\|_{L^\infty} \leq \varepsilon \quad \text{and} \quad |\lambda_i(t)| \leq \varepsilon \quad \forall i = 1, \dots, g.$$

Suppose that for any $\tilde{\tau}$, one can find $t \geq \tilde{\tau}$ for which $\|\omega(t, \cdot)\|_{C^0(\overline{\Omega})} > \varepsilon$. Then there is some $x \in \overline{\Omega}$ for which $|\omega(t, x)| > \varepsilon$, and hence by (6.2) one has $s_{\omega,\lambda_i}(t, x) \leq \tau(\varepsilon)$ and hence $\alpha(t) \geq \varepsilon$ on $[\tau(\varepsilon), t]$. Consequently, there exists $x_0 \in \overline{\Omega}$ such that $|\omega(\tau(\varepsilon), x_0)| > \varepsilon$ and for which

$$\Phi^{\omega,\lambda_i}([\tau(\varepsilon), t], \tau(\varepsilon), x_0) \subset \overline{\Omega}.$$

With (6.7), this contradicts the fact that θ is bounded in $\overline{\Omega}$.

7. Appendix.

7.1. Proof of Corollary 2.2. We reduce Σ a little in order to keep some kind of margin. Introduce $\tilde{\theta}$ as in Proposition 2.1. We describe a procedure that allows us to slightly modify $\tilde{\theta}$ to get rid of problematic points "E," while preserving (2.11)–(2.16). The idea is the following: consider an E point as in (2.17); by $\tilde{\Phi}$ it is first transported along $\partial\Omega \backslash (\overline{\gamma^+(\tilde{\theta})} \cup \overline{\gamma^-(\tilde{\theta})})$ (remember (2.12) and (2.13)); call γ_E the corresponding connected component of $\partial\Omega \backslash (\overline{\gamma^+(\tilde{\theta})} \cup \overline{\gamma^-(\tilde{\theta})})$. Consider t_E the biggest positive time for which $\tilde{\Phi}((0, t), 0, E) \subset \overline{\gamma_E}$. There are two cases:

- If $\tilde{\Phi}(t_E, 0, E) \in \overline{\gamma^+(\tilde{\theta})}$, it is clear from (2.12)–(2.13) that this point is in $\partial\gamma^+(\tilde{\theta})$, pointing inside $\gamma^+(\tilde{\theta})$. And consequently for t just after t_E, one has $\tilde{\Phi}(t, 0, E) \in B_R \backslash \overline{\Omega}$, so the E point under consideration satisfies (2.17).
- If $\tilde{\Phi}(t_E, 0, E) \in \overline{\gamma^-(\tilde{\theta})}$, we consider the following time t'_E:

$$t'_E = \sup\Big\{ t \in (t_E, +\infty),\ \tilde{\Phi}((t_E, t), 0, E) \in \Omega \Big\}.$$

It is indeed quite clear that for times $t > t_E$ with $t - t_E$ small, $\tilde{\Phi}(t, 0, E) \in \Omega$, and it follows from $|\nabla\tilde{\theta}|(x) > 0$ in $\overline{\Omega}$ that the preceding set is bounded from above. Then $\tilde{\Phi}(t'_E, 0, E) \in \overline{\gamma^+(\tilde{\theta})}$ (for it cannot be in $\partial\Omega\backslash(\overline{\gamma^-(\tilde{\theta})} \cup \overline{\gamma^+(\tilde{\theta})})$ because of the uniqueness of the flow and it cannot be in $\overline{\gamma^-(\tilde{\theta})}$ because points in $\overline{\gamma^-(\tilde{\theta})}$ come from $(B_R\backslash\overline{\Omega}) \cup \partial\Omega$ by the flow of $\nabla\theta$).
If $\tilde{\Phi}(t'_E, 0, E) \in \gamma^+(\tilde{\theta})$, then (2.17) is valid for this E; we now suppose that $E_2 := \tilde{\Phi}(t'_E, 0, E) \in \partial\gamma^+(\tilde{\theta})$.
We consider a small connected neighborhood $\mathcal{U}$ of E_2 in $\partial\Omega$; thanks to the margin we kept on Σ, one can require $\mathcal{U} \subset \Sigma$. We also consider a point $F \in \gamma^+(\tilde{\theta})$ and $\mathcal{V}_F$ a small connected neighborhood of F in $\gamma^+(\tilde{\theta})$, not touching $\partial\gamma^+(\tilde{\theta})$ or $\mathcal{U}$.
We introduce a function on $\partial\Omega$, say ψ, supported in $\mathcal{U} \cup \mathcal{V}_F$, nonnegative in $\mathcal{U}$, nonpositive in $\mathcal{V}_F$, and such that

$$\int_{\partial\Omega} \psi = 0 \quad \text{and} \quad \psi(E_2) = 1.$$

Then we define $\hat{\theta} \in C^\infty(\Omega; \mathbb{R})$ by

$$\begin{cases} \Delta\hat{\theta} = 0 & \text{in } \Omega, \\ \partial_n\hat{\theta} = \psi & \text{on } \partial\Omega, \\ \int_\Omega \hat{\theta} = 0. \end{cases} \tag{7.1}$$

Using elliptic estimates and Lemma 3.3, one sees that $\tilde{\theta} + \varepsilon\hat{\theta}$ still satisfies (2.11)–(2.16) for $\varepsilon > 0$ small enough. Let us particularly emphasize that, for $\varepsilon > 0$ small enough, one has $\partial_n(\tilde{\theta} + \varepsilon\hat{\theta}) > 0$ on $\mathcal{V}_F$. The E considered now satisfies (2.17). The procedure has not added an E point, but it has slightly moved the frontier of $\gamma^+(\tilde{\theta})$: introduce

$$\gamma^+(\tilde{\theta} + \varepsilon\hat{\theta}) := \{x \in \partial\Omega \ / \ \partial_n(\tilde{\theta} + \varepsilon\hat{\theta}) > 0\}.$$

Then $E_2 \in \gamma^+(\tilde{\theta} + \varepsilon\hat{\theta})$, and the new frontier of γ^+ is now given by

$$\partial\gamma^+(\tilde{\theta} + \varepsilon\hat{\theta}) = \partial\gamma^+(\tilde{\theta}) \cup \{E_3\} \setminus \{E_2\},$$

where E_3 is the point in $\partial\mathcal{U}$ that does not belong to $\gamma^+(\tilde{\theta})$.
Now if E_2 satisfied (2.17), then E_3 also does for ε small enough: we have two cases:

- $\nabla\tilde{\theta}$ is pointing inside $\gamma^+(\tilde{\theta})$ at E_2. This case is in fact not possible because of the definition of E_2 and t'_E: points in $\partial\gamma^+(\tilde{\theta})$ at which $\nabla\tilde{\theta}$ is pointing inside $\gamma^+(\tilde{\theta})$ come from $\partial\Omega$ when following the flow.
- $\nabla\tilde{\theta}$ is pointing outside $\gamma^+(\tilde{\theta})$ at E_2. Then using (2.13), one sees that the trajectory of E_2 under the flow of $\nabla\tilde{\theta}$ follows the connected component of E_2 in $\partial\Omega\backslash[\gamma^+(\tilde{\theta}) \cup \gamma^-(\tilde{\theta})]$. In particular, this trajectory meets E_3. But for ε small enough, the trajectories under the flow of $\nabla(\tilde{\theta} + \varepsilon\hat{\theta})$ are almost the same as the ones in the flow of $\nabla\tilde{\theta}$ (as seen by Lemma 3.3 and elliptic estimates). This yields the conclusion.

So one can get rid of problematic points one after another.

7.2. Proof of Proposition 3.5.

- *Proof of* (3.14). Introduce $\hat{y}$ as the solution of

$$\begin{cases} \operatorname{curl} \hat{y}(t,x) = \omega(t,x) \quad \text{for } (t,x) \in \Omega_T, \\ \operatorname{div} \hat{y}(t,x) = 0 \quad \text{for } (t,x) \in \Omega_T, \\ \hat{y}(t,x) \cdot n(x) = 0 \quad \text{for } (t,x) \in \Sigma_T, \\ \displaystyle\int_{\Gamma_i} \hat{y}(t,x) \cdot \vec{\tau}(x) dx = \lambda_i(t) \quad \text{for } t \in [0,T], \text{ for } i = 1, \dots, g. \end{cases}$$

Of course, one has $y = \hat{y} + K\alpha(t)\nabla\theta(x)$. Now (3.9) involves

$$\|\hat{y}(t)\|_{L^\infty(\Omega)} \le C_{\mathcal{LL}} \max\left(|\lambda_1(t)|, \dots, |\lambda_g(t)|, \|\omega(t)\|_\infty\right) \ \forall t \in [0,T].$$

(The constant $C_{\mathcal{LL}}$ does not depend on t.) Hence

$$y(t,x) \cdot \nabla\theta(x) \ge K\alpha(t)|\nabla\theta(x)|^2 - C_{\mathcal{LL}}\|\nabla\theta\|_\infty \alpha(t). \tag{7.2}$$

Equation (2.13) and the compactness of $\overline{\Omega}$ allow us to introduce

$$\underline{m} := \min_{x \in \overline{\Omega}} |\nabla\theta(x)| > 0.$$

One easily deduces from (7.2) that (3.14) holds if $\overline{K} \ge 2C_{\mathcal{LL}}\|\nabla\theta\|_\infty/\underline{m}^2$ and $\kappa = \underline{m}^2/2$ (which we suppose in what follows).

- *Proof of* (3.17). Property (3.17) will essentially follow from Gronwall's inequality (3.11), from (2.13), and from (2.16).
We extend the definition of α and (ω, λ^i) for times $t \ge T$ by $\alpha(T)$ and $(\omega, \lambda^i)(T)$, respectively.
We write $\Theta^\alpha(t,x) := K\alpha(t)\theta(x)$. We consider Φ, Φ^y, and Φ^α the respective flows of $\pi(\nabla\theta)$, $\pi(y(t,x))$, and $\pi(\nabla\Theta^\alpha(t,x))$.
First, by a compactness argument and using (2.13), one sees that there exist $T_\theta > 0$ and $d_\theta > 0$ such that

$$\forall x \in \overline{\Omega}, \ \exists t \in [0, T_\theta] \text{ such that } \operatorname{dist}(\Phi(t,0,x), \overline{\Omega}) \ge d_\theta.$$

It suffices, for instance, to observe that

$$\frac{d}{dt}\theta(\Phi(t,0,x)) = |\nabla\theta(\Phi(t,0,x))|^2$$

if x and t are such that $\Phi(t,0,x) \in \overline{\Omega}$, and to use (2.13) and the boundedness of θ on $\overline{\Omega}$.
Hence

$$\forall x \in \overline{\Omega}, \ \exists \mathcal{T}(x) \text{ such that } \int_0^{\mathcal{T}(x)} K\alpha(\tau)d\tau \le T_\theta \\ \text{and such that } \operatorname{dist}(\Phi^\alpha(\mathcal{T}(x),0,x), \overline{\Omega}) \ge d_\theta.$$

Now by Lemma 3.3 one has

$$|\Phi^y(t,0,x) - \Phi^\alpha(t,0,x)| \\ \le \exp\left(K\|\pi(\nabla\theta)\|_{\mathcal{L}ip} \int_0^t \alpha(\tau)d\tau\right) \|\pi(\hat{y})\|_{L^1([0,T],L^\infty(B_R))}.$$

Consequently, one sees that for $x \in \overline{\Omega}$ and t such that $0 \leq t \leq \mathcal{T}(x)$,

$$\begin{aligned}|\Phi^y(t,0,x) - \Phi^\alpha(t,0,x)| &\leq \exp(T_\theta \|\pi(\nabla\theta)\|_{\mathcal{L}ip(B_R)}) \|\hat{y}\|_{L^1([0,t],L^\infty_x)} \\ &\leq C_{\mathcal{LL}} \exp(T_\theta \|\pi(\nabla\theta)\|_{\mathcal{L}ip(B_R)}) \|(\omega,\lambda_i)\|_{L^1([0,t],L^\infty_x)}.\end{aligned}$$

Now one deduces from (3.12) that for $0 \leq t \leq \mathcal{T}(x)$,

$$\|(\omega,\lambda_i)\|_{L^1([0,t],L^\infty_x)} \leq \int_0^t \alpha(\tau)d\tau \leq \frac{T_\theta}{K}.$$

This yields

$$|\Phi^y(t,0,x) - \Phi^\alpha(t,0,x)| \leq \frac{C_{\mathcal{LL}} \exp(T_\theta \|\pi(\nabla\theta)\|_{\mathcal{L}ip(B_R)}) T_\theta}{K}. \tag{7.3}$$

Consequently, if K is large enough (in terms of only θ), one has

$$\forall x \in \overline{\Omega},\ \exists \mathcal{T}(x) \text{ such that } \int_0^{\mathcal{T}(x)} K\alpha(\tau)d\tau \leq T_\theta$$
$$\text{and such that } \operatorname{dist}(\Phi^y(\mathcal{T}(x),0,x),\overline{\Omega}) \geq d_\theta/2,$$

with (7.3) valid between times 0 and $\mathcal{T}(x)$. With (2.18), this gives (3.17) for K large enough.

- *Proof of* (3.18). This is due to the uniqueness of the flow: on $\overline{\gamma_A}$, $y(s,x)$ is of the form $\lambda(s,x)\vec{\tau}(x)$, the sign of $\lambda(s,x)$ being constant in such a way that the direction of $y(s,A)$ is pointing inside γ^- (indeed, thanks to (3.13) and (3.14), $y(s,x)$ has the same direction as $\nabla\theta(x)$ on $\partial\Omega$). So one finds a local in time backward solution of (3.6) inside γ_A. This solution does not go outside γ_A for times $\tau \in [0,t)$ if $t-\tau$ is small enough so that

$$c_\pi(K\|\nabla\theta\|_\infty + C_{\mathcal{LL}}) \left(\int_\tau^t \alpha(s)ds \right) \leq \ell/2, \tag{7.4}$$

because the velocity is estimated by

$$\begin{aligned}|\pi[y](s,x)| &\leq \|\pi[\hat{y}](s,\cdot)\|_{L^\infty(B_R)} + K\alpha(s)\|\pi[\nabla\theta]\|_{L^\infty(B_R)} \\ &\leq c_\pi(C_{\mathcal{LL}} + K\|\nabla\theta\|_\infty)\alpha(s)\end{aligned} \tag{7.5}$$

and because of the definition of ℓ.

- *Proof of* (3.15)–(3.16). This is mutatis mutandis [4, Lemma 3.3]. Let us treat separately the points in γ^- and the B points.
  - Let us consider (3.16) for a point $B \in \mathcal{B}$. Using again (3.13) and (3.14), one sees that, as for $s \in [0,T]$, $y(s,B)$ is tangent to $\partial\Omega$ and by (3.14) pointing outside γ^-, there is a solution for the flow starting from B and that stays inside $\partial\Omega \setminus (\overline{\gamma^-} \cup \overline{\gamma^-})$ at least for small times. So the uniqueness of the flow gives (3.16).
  - Consider $x \in \gamma^-$. Let us use the coordinates in the reference frame given by $(\vec{\tau}(x), n(x))$. Then using (3.9) and (3.13), we see that for any $\varepsilon > 0$, one finds some neighborhood $\mathcal{U}$ of (t,x) in $(0,T] \times B_R$ for which one has for any $(\tilde{t},\tilde{x})$ in $\mathcal{U}$,

$$|\{\pi[y](\tilde{t},\tilde{x}) - K\alpha(\tilde{t})\pi[\nabla\theta](\tilde{x})\} \cdot n(x)| \leq \varepsilon.$$

(And the left-hand side is null when $\tilde{x}$ is on γ^-.) Hence for suitable ε and $\mathcal{U}$ the second coordinate of $y(\tilde{t},\tilde{x})$ is positive in a neighborhood of (t,x) in $[0,T]\times B_R$. Using (3.6), one deduces (3.15) and (3.16) except for B points.
Before dealing with (3.15) for points $B\in\mathcal{B}$, let us make (3.15) more precise for points in γ^-. We consider $\tau<t$ sufficiently close to t for (7.4) to hold. Consider $\underline{s}$ the smallest time $s\in[\tau,t]$ such that $\Phi^y((s,t),t,E)\subset B_R\backslash\overline{\Omega}$, and let us suppose $\underline{s}>\tau$. One can estimate the velocity by (7.5) and consequently in our case, one has $\Phi^y(\underline{s},t,x)\notin\overline{\gamma^+}$. Certainly, one also has that $\Phi^y(\underline{s},t,x)\notin\Omega\cup\gamma^-\cup\mathcal{B}$ because of (3.16) that we just proved for points in $\gamma^-\cup\mathcal{B}$. This point can thus only be in $\partial\Omega\backslash(\overline{\gamma^+}\cup\gamma^-\cup\mathcal{B})$ (unless it is in $B_R\backslash\overline{\Omega}$). Hence it must lie in γ_B for the other components are too far because of (7.4). But by uniqueness of the flow, this is not possible. Consequently one has, for any $s\in[\tau,t]$ with τ satisfying (7.4),

$$\Phi^y(s,t,x)\in B_R\backslash\overline{\Omega}. \tag{7.6}$$

– Now, let us deal with (3.15) for B points. We see, using Remark 6(i) and the same procedure as for the proof of (3.17), that for some ν, one has $\Phi^y(t-\nu,t,B)\in B_R\backslash\overline{\Omega}$. Now we claim that, at least if ν has been chosen small enough, $\Phi^y(s,t,B)\in B_R\backslash\overline{\Omega}$ for any s in $[t-\nu,t)$. Indeed, when considering a sequence of points x_n in γ^- converging to B, by continuity of the flow we have that $\Phi^y(s,t,x_n)$ is converging to $\Phi^y(s,t,B)$. Using (7.6), one gets that $\Phi^y(s,t,B)\notin\Omega$ for $s\in[t-\nu,t)$. But $\Phi^y(s,t,B)$ cannot belong to γ^- by (3.15), which is already established for points in γ^-; nor can it belong to γ_B by uniqueness of the flow (and the other components of $\partial\Omega\backslash(\gamma^+\cup\gamma^-)$ are too far by the choice of ν). Hence, one must have $\Phi^y(s,t,B)\in B_R\backslash\overline{\Omega}$.

- *Proof of* (3.19). This is a consequence of the continuity of the flow. Suppose indeed by contradiction that for some ω, $(\lambda_i)_{i=1,\dots,g}$, and α, one has (3.19) not satisfied by an A point. Then for some $\tau>t$, $\Phi^y(\tau,t,A)\in B_R\backslash\overline{\Omega}$ (note indeed that this point cannot be in $\overline{\gamma_A}$ by uniqueness of the flow or in other components of $\partial\Omega\backslash(\gamma^+\cup\gamma^-)$ by the choice of τ (which is not too far from t), and if this point is in γ^-, we conclude by (3.15) that there indeed exists a point $\Phi^y(\tau-\nu,t,A)$ in $B_R\backslash\overline{\Omega}$). We look at the trajectories starting from x in γ^- close to A. By (3.16) they are inside Ω for small time and, in fact, by the same argument as previously, in Ω as long as $\tau-t$ is small enough so that

$$c_\pi(K\|\nabla\theta\|_\infty+C_{\mathcal{L}\mathcal{L}})\left(\int_t^\tau\alpha(s)ds\right)\leq\ell/2.$$

So our assumption would be in contradiction with the continuity of the flow.
- *Proof of* (3.20). This follows again from the procedure of the proof of (3.17) and the choices of $\mathrm{Supp}(\Gamma_A)$ and $\mathrm{Supp}(\Lambda_i)$ (which are at positive distance from $\underline{A}$).

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