On the motion of a small body immersed in a two dimensional incompressible perfect fluid.

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April 28, 2011

Abstract

In this paper we prove that the motion of a solid body in a two dimensional incompressible perfect fluid converges, when the body shrinks to a point with fixed mass and circulation, to a variant of the vortex-wave system where the vortex, placed in the point occupied by the shrunk body, is accelerated by a lift force similar to the Kutta-Joukowski force of the irrotational theory.

1 Introduction

In this paper we consider the motion of a small solid body in a planar ideal fluid, and the limit behaviour of the system as the solid body is reduced to a point.

Let us first describe the equations when the solid has a fixed size. Let $\mathcal{S}_0$ be a closed, bounded, connected and simply connected subset of the plane with smooth boundary. We assume that the body initially occupies the domain $\mathcal{S}_0$ and rigidly moves so that at time t it occupies an isometric domain denoted by $\mathcal{S}(t)$. We denote $\mathcal{F}(t) := \mathbb{R}^2 \setminus \mathcal{S}(t)$ the domain occupied by the fluid at time t starting from the initial domain $\mathcal{F}_0 := \mathbb{R}^2 \setminus \mathcal{S}_0$.

The equations modelling the dynamics of the system then read

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 \qquad \text{for } x \in \mathcal{F}(t), \tag{1}$$

$$\operatorname{div} u = 0 \qquad \text{for } x \in \mathcal{F}(t), \tag{2}$$

$$u \cdot n = u_{\mathcal{S}} \cdot n \qquad \text{for } x \in \partial\mathcal{S}(t), \tag{3}$$

$$\lim_{|x|\to\infty} |u| = 0, \tag{4}$$

$$mh''(t) = \int_{\partial\mathcal{S}(t)} pn \, ds, \tag{5}$$

$$\mathcal{J} r'(t) = \int_{\partial\mathcal{S}(t)} (x - h(t))^\perp \cdot pn \, ds, \tag{6}$$

$$u|_{t=0} = u_0 \qquad \text{for } x \in \mathcal{F}_0, \tag{7}$$

$$h(0) = h_0, \; h'(0) = \ell_0, \qquad r(0) = r_0. \tag{8}$$

Here $u = (u_1, u_2)$ and p denote the velocity and pressure fields, m and $\mathcal{J}$ denote respectively the mass and the moment of inertia of the body while the fluid is supposed to be homogeneous of density 1, to simplify the notations. When $x = (x_1, x_2)$ the notation $x^\perp$ stands for $x^\perp = (-x_2, x_1)$, n denotes the unit normal vector pointing outside the fluid, $h'(t)$ is the velocity of the center of mass $h(t)$ of the body and $r(t)$ denotes the angular velocity of the rigid body. Finally we denote by $u_{\mathcal{S}}$ the velocity of the body:

$$u_{\mathcal{S}}(t,x) = h'(t) + r(t)(x - h(t))^\perp. \tag{9}$$

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Since $\mathcal{S}(t)$ is the position occupied by a rigid body there exists a rotation matrix

$$Q(t) := \begin{bmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{bmatrix}, \tag{10}$$

such that the position $\eta(t,x) \in \mathcal{S}(t)$ at the time t of the point fixed to the body with an initial position x is

$$\eta(t,x) := h(t) + Q(t)(x - h_0). \tag{11}$$

The angle θ satisfies

$$\theta'(t) = r(t),$$

and we choose $\theta(t)$ such that $\theta(0) = 0$.

The equations (1) and (2) are the incompressible Euler equations, the condition (3) means that the boundary is impermeable and the equations (5) and (6) are the Newton's balance law for linear and angular momenta.

For the study of ideal flow, an important quantity is the vorticity $w := \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1$, satisfying the transport equation:

$$\frac{\partial w}{\partial t} + (u \cdot \nabla) w = 0 \text{ for } x \in \mathcal{F}(t). \tag{12}$$

One has the following result concerning the Cauchy problem for the above system, the initial position of the solid being given. This result describes weak solutions, extending results concerning the fluid alone. It considers the case when vorticity belongs in L^p as in DiPerna and Majda [7] and includes weak solutions with bounded vorticity as in Yudovich [19].

Theorem 1. *Let $p \in (2, +\infty]$. For any $u_0 \in C^0(\overline{\mathcal{F}_0}; \mathbb{R}^2)$, $(\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$, such that:*

$$\operatorname{div} u_0 = 0 \text{ in } \mathcal{F}_0 \quad \text{and} \quad u_0 \cdot n = (\ell_0 + r_0(x - h_0)^\perp) \cdot n \text{ on } \partial\mathcal{S}_0, \tag{13}$$

$$w_0 := \operatorname{curl} u_0 \in L^p_c(\overline{\mathcal{F}}_0), \tag{14}$$

$$\lim_{|x| \to +\infty} u_0(x) = 0,$$

there exists a solution (h', r, u) of (1)–(8) in $C^1(\mathbb{R}^+; \mathbb{R}^2 \times \mathbb{R}) \times C^0(\mathbb{R}^+, W^{1,p}(\mathcal{F}(t)))$ with $\partial_t u, \nabla p \in L^\infty_{loc}(\mathbb{R}^+, L^q(\mathcal{F}(t)))$ for any $q \in (1, p]$ when $p < \infty$ and in $C^1(\mathbb{R}^+; \mathbb{R}^2 \times \mathbb{R}) \times L^\infty_{loc}(\mathbb{R}^+, \mathcal{LL}(\mathcal{F}(t)))$ with $\partial_t u, \nabla p \in L^\infty_{loc}(\mathbb{R}^+, L^q(\mathcal{F}(t)))$ for any $q \in (1, +\infty)$ when $p = \infty$.

Moreover such a solution satisfies that for all $t > 0$, $w(t) := \operatorname{curl}(u(t)) \in L^p_c(\overline{\mathcal{F}(t)})$, it is energy-conserving in the sense of Proposition 4 and $\|w(t, \cdot)\|_{L^q(\mathcal{F}(t))}$ (for any $q \in [1, p]$), $\int_{\mathcal{F}(t)} w(t,x)\, dx$ and $\int_{\partial\mathcal{S}(t)} u \cdot \tau\, ds$ are preserved over time.

Finally when $p = \infty$, the solution is unique.

This result is proven in [9]. For the sake of self-containedness, we give a short proof of it in appendix. The notation $\mathcal{LL}(\Omega)$ refers to the space of log-Lipschitz functions on Ω, that is the set of functions $f \in L^\infty(\Omega)$ such that

$$\|f\|_{\mathcal{LL}(\Omega)} := \|f\|_{L^\infty(\Omega)} + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|(x - y)(1 + \ln^- |x - y|)|} < +\infty. \tag{15}$$

Above, we used the abuse of notation $L^\infty(\mathbb{R}^+; X(\mathcal{F}(t)))$ (resp. $C^0(\mathbb{R}^+; X(\mathcal{F}(t)))$) where X is a functional space; by this we refer to functions defined for almost each t as a function in the space $X(\mathcal{F}(t))$, and which can be extended as a function in $L^\infty(\mathbb{R}^+; X(\mathbb{R}^2))$ (resp. $C^0(\mathbb{R}^+; X(\mathbb{R}^2))$).

Let us also mention that the existence and uniqueness of finite energy classical solutions to the problem (1)–(8) has been tackled by Ortega, Rosier and Takahashi in [17].

Let us now discuss the main problem considered in this paper, that is the behaviour of this system for a small body. Accordingly, we consider $\mathcal{S}_0$ a fixed domain as above, and define for $\varepsilon > 0$ the resized domain $\mathcal{S}_0^\varepsilon$ given by:

$$\mathcal{S}_0^\varepsilon - h_0 = \varepsilon(\mathcal{S}_0 - h_0).$$

Therefore $\mathcal{S}_0^\varepsilon$ denotes the domain initially occupied by the solid and

$$\mathcal{F}_0^\varepsilon := \mathbb{R}^2 \setminus \mathcal{S}_0^\varepsilon,$$

the one occupied by the fluid. Let $w_0 \in L^p_c(\mathbb{R}^2)$. We fix $\gamma,\ r_0 \in \mathbb{R}$, and $h_0,\ \ell_0 \in \mathbb{R}^2$ independently of ε. Therefore, as we will see in Proposition 3, there exists an unique vector field $u_0^\varepsilon \in C^0(\overline{\mathcal{F}_0^\varepsilon};\mathbb{R}^2)$ such that

$$\begin{cases} \operatorname{div} u_0^\varepsilon = 0,\ \operatorname{curl} u_0^\varepsilon = w_0^\varepsilon \text{ in } \mathcal{F}_0^\varepsilon, \\ u_0^\varepsilon \cdot n = (\ell_0 + r_0(x-h_0)^\perp)\cdot n \text{ on } \partial\mathcal{S}_0^\varepsilon, \\ \lim_{|x|\to\infty} |u_0^\varepsilon(x)| = 0,\ \int_{\partial \mathcal{S}_0^\varepsilon} u_0^\varepsilon \cdot \tau\, ds = \gamma, \end{cases} \tag{16}$$

where

$$w_0^\varepsilon := w_{0|\mathcal{F}_0^\varepsilon}. \tag{17}$$

We will be interested in the limit of the system as $\varepsilon \to 0^+$ in the following particular regime:

$$m_\varepsilon = m \text{ and } \mathcal{J}_\varepsilon = \varepsilon^2 \mathcal{J}_0,$$

where m and $\mathcal{J}_0$ are fixed constant. This is obtained for instance for a homogeneous solid, with a constant mass as $\varepsilon \to 0^+$.

The main goal of this paper is to prove the following theorem.

Theorem 2. *Assume that $p \in (2,+\infty]$. Let be given $h_0 \in \mathbb{R}^2$, $\gamma \in \mathbb{R}$, $(\ell_0, r_0) \in \mathbb{R}^3$, w_0 in $L^p_c(\mathbb{R}^2)$. Consider $T > 0$. For any $\varepsilon \in (0,1]$, we associate u_0^ε by (16)-(17) and consider $(h^\varepsilon, r^\varepsilon, u^\varepsilon)$ a solution of the system (1)–(8) given by Theorem 1.*

Then, up to a subsequence, one has the following:

- *h^ε converges to h weakly-$*$ in $W^{2,\infty}(0,T;\mathbb{R}^2)$,*
- *$\varepsilon\theta^\varepsilon$ converges to 0 weakly-$*$ in $W^{2,\infty}(0,T;\mathbb{R})$,*
- *w^ε converges to w in $C^0([0,T];L^p(\mathbb{R}^2)-w)$ (resp. in $C^0([0,T];L^\infty(\mathbb{R}^2)-w*)$ if $p=+\infty$),*
- *u^ε converges to $\tilde{u} + \dfrac{\gamma}{2\pi}\dfrac{(x-h(t))^\perp}{|x-h(t)|^2}$ in $C^0([0,T];L^q_{loc}(\mathbb{R}^2))$ for $q<2$,*
- *one has*

$$\frac{\partial w}{\partial t} + \operatorname{div}\left(\left[\tilde{u} + \frac{\gamma}{2\pi}\frac{(x-h(t))^\perp}{|x-h(t)|^2}\right] w\right) = 0 \quad \text{in} \quad [0,T]\times\mathbb{R}^2, \tag{18}$$

$$mh''(t) = \gamma\Big(h'(t) - \tilde{u}(t,h(t))\Big)^\perp, \tag{19}$$

$$w|_{t=0} = w_0,\ h(0) = h_0,\ h'(0) = \ell_0, \tag{20}$$

$$\tilde{u}(t,x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{|x-y|^2} w(t,y)\, dy. \tag{21}$$

Remark 1. *Above the convergence of w^ε holds when w^ε is extended by 0 inside the solid. In the same way, the convergence of u^ε holds when extending it for instance by 0 (or by $\ell^\varepsilon + r^\varepsilon(x-h(t))^\perp$) inside the solid.*

Remark 2. *Equation (18) and the w-part of the initial data given in (20) hold in the sense that for any test function $\psi \in C^\infty_c([0,T)\times\mathbb{R}^2)$ we have*

$$\int_0^\infty \int_{\mathbb{R}^2} \psi_t w\ dx\, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla_x \psi \cdot \Big(\tilde{u} + \frac{\gamma}{2\pi}\frac{(x-h(t))^\perp}{|x-h(t)|^2}\Big) w\, dx\, dt + \int_{\mathbb{R}^2} \psi(0,x) w_0(x)\, dx = 0. \tag{22}$$

Equation (18) describes the evolution of the vorticity of the fluid: it is transported by a velocity obtained by the usual Biot-Savart law in the plane, but from a vorticity which is the sum of the fluid vorticity and of a point vortex placed at the (time-dependent) position $h(t)$ where the solid shrinks, with a strength equal to the circulation γ around the body.

Equation (19) means that the shrunk body is accelerated by a lift force similar to the Kutta-Joukowski lift of the irrotational theory: the shrunk body experiments a lift which is proportional to the circulation γ around the body and to the difference between the solid velocity and the virtual fluid velocity obtained by the Biot-Savart law in the plane from the fluid vorticity, up to a rotation of a $\pi/2$ angle. See for instance the textbooks of Childress [4] or Marchioro and Pulvirenti [16] for a discussion of the Kutta-Joukowski force. See also Grotta-Ragazzo, Koiller and Oliva [10], where they consider a similar system of a point mass embedded in an irrotational fluid and driven by Kutta-Joukowski force.

Let us mention that the problem of the limit of the Euler system around a fixed shrinking obstacle, which is tightly connected to ours, has been studied by Iftimie, Lopes-Filho and Nussenzveig-Lopes in [11]. Another result connected to our study is given in Dashti and Robinson [5], where the authors consider the limit of a shrinking ball of fixed density and without rotation in a viscous fluid (modelled by the Navier-Stokes equations).

Remark 3. *A challenging open problem is to extend the previous analysis to the case where the density of the body is fixed as ε goes to zero so that the mass of the body is vanishing as the body is shrinking to a point. Formally the equations (18)–(21) would reduce to the vortex-wave system (for which we refer to [16]), but such a limit is quite singular as the equation* (19) *degenerates into a first order equation as m goes to* 0.

Remark 4. *Since we start with a conservative and reversible system it is expected that the equations* (18)–(21) *should be also conservative and reversible. This is actually the case and we will even see in Section 9 that Equations* (18)–(21) *can be seen as an Hamiltonian system with respect to following renormalized energy*

$$2\mathcal{H} = m|h'(t)|^2 - \int_{\mathbb{R}^2\times\mathbb{R}^2} G(x-y)w(t,x)w(t,y)\,dx\,dy - 2\gamma\int_{\mathbb{R}^2} G(x-h(t))w(t,x)\,dx, \tag{23}$$

where

$$G(x) := \frac{1}{2\pi}\ln|x|. \tag{24}$$

Remark 5. *Following the lines of [11, Section 5.3], one can see that in the above limit equation* (18) *and* (21) *can be rewritten in the following velocity form:*

$$\partial_t\tilde{u} + (\tilde{u}\cdot\nabla)\tilde{u} + \gamma\,\mathrm{div}\left(\tilde{u}\otimes H(x-h(t)) + H(x-h(t))\otimes\tilde{u}\right) - \gamma\tilde{u}(t,h(t))^\perp\delta_{h(t)} = -\nabla p,$$
$$\mathrm{div}\,\tilde{u} = 0 \quad \text{and} \quad \tilde{u}_{|t=0} = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{(x-y)^\perp}{|x-y|^2}w_0(y)\,dy,$$

with

$$H(x) := \frac{1}{2\pi}\frac{x^\perp}{|x|^2}.$$

The structure of the paper is as follows. In Section 2, we give a representation of a velocity field satisfying (16). In Section 3, we discuss a change of variables allowing to rephrase the system in a fixed domain. In Section 4, we give a priori estimates on the system. Section 5 is the central part where we study precisely the effect of pressure on the body as ε tends to 0^+. In Section 6 we prove Theorem 2 by establishing compactness and obtaining the limit equation. Section 7 is devoted to the proof of several technical lemmas. In Section 8 we briefly prove Theorem 1. Finally in Section 9, we prove that the limit system obtained in Theorem 2 has a Hamiltonian structure.

2 Representation of the velocity in the body frame

Without loss of generality and for the rest of the paper, we assume from now on that

$$h_0 = 0,$$

which means that the body is centered at the origin at the initial time $t = 0$.

In this section, we study the elliptic div/curl system which allows to pass from the vorticity to the velocity field, in the body frame. In the whole paper and in this section in particular, we will need some arguments of elementary complex analysis: for the rest of the paper, we identify $\mathbb{C}$ and $\mathbb{R}^2$ through

$$(x_1, x_2) = x_1 + ix_2.$$

2.1 Green's function and Biot-Savart operator

We denote by $G^\varepsilon(x, y)$ the Green's function of $\mathcal{F}_0^\varepsilon$ with Dirichlet boundary conditions. We also introduce the function $K^\varepsilon(x, y) = \nabla^\perp G^\varepsilon(x, y)$ known as the kernel of the Biot-Savart operator $K^\varepsilon[\omega]$ which therefore acts on $\omega \in L^p_c(\overline{\mathcal{F}_0^\varepsilon})$ through the formula

$$K^\varepsilon[\omega](x) = \int_{\mathcal{F}_0^\varepsilon} K^\varepsilon(x, y)\omega(y)\, dy.$$

The following is classical.

Proposition 1. *Let $p \in (2, +\infty)$ (resp. $p = +\infty$). Let $\omega \in L^p_c(\overline{\mathcal{F}_0^\varepsilon})$. Then $K^\varepsilon[\omega]$ is in the Hölder space $C^{1-2/p}(\mathcal{F}_0^\varepsilon)$ of bounded Hölder continuous functions of order $1-2/p$ (resp. in $\mathcal{LL}(\mathcal{F}_0^\varepsilon)$), divergence-free, tangent to the boundary and such that* $\operatorname{curl} K^\varepsilon[\omega] = \omega$*. Moreover, it satisfies*

$$K^\varepsilon[\omega](x) = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } x \to \infty,$$

and is consequently square-integrable, and its circulation around $\partial\mathcal{S}_0^\varepsilon$ is given by

$$\int_{\partial\mathcal{S}_0^\varepsilon} K^\varepsilon[\omega] \cdot \tau\, ds = -\int_{\mathcal{F}_0^\varepsilon} \omega\, dx, \tag{25}$$

where τ is the tangent unit vector field on $\partial\mathcal{S}_0^\varepsilon$.

As we work in the exterior of a single solid, we can have an explicit formula for K^ε in terms of a biholomorphism $\mathcal{F}_0 \to \mathbb{C} \setminus \overline{B}(0, 1)$. To that purpose, let us select the unique such biholomorphism $\mathcal{T} : \mathcal{F}_0 \to \mathbb{C} \setminus \overline{B}(0, 1)$ such that the following development holds for some $(\beta, \tilde{\beta}) \in \mathbb{R}^+_* \times \mathbb{C}$:

$$\mathcal{T}(z) = \beta z + \tilde{\beta} + \mathcal{O}\left(\frac{1}{z}\right) \quad \text{as} \quad z \to +\infty. \tag{26}$$

This is possible since $\mathcal{S}_0$ is a bounded, closed, connected and simply connected domain of the plane (using Riemann's mapping theorem and a conjugation by $z \mapsto 1/z$). Now, as $\mathcal{S}_0^\varepsilon = \varepsilon\mathcal{S}_0$, we can introduce $\mathcal{T}_\varepsilon$ as the biholomorphism from $\mathcal{F}_0^\varepsilon$ to the exterior of the unit ball given by:

$$\mathcal{T}_\varepsilon(z) = \mathcal{T}(z/\varepsilon). \tag{27}$$

In particular $\mathcal{T} = \mathcal{T}_1$.

With these notations we have (see e.g. [11]):

$$G^\varepsilon(x, y) = \frac{1}{2\pi} \ln \frac{|\mathcal{T}_\varepsilon(x) - \mathcal{T}_\varepsilon(y)|}{|\mathcal{T}_\varepsilon(x) - \mathcal{T}_\varepsilon(y)^*||\mathcal{T}_\varepsilon(y)|}, \tag{28}$$

and

$$K^\varepsilon[\omega](x) = \frac{1}{2\pi} D\mathcal{T}_\varepsilon^T(x) \int_{\mathcal{F}_0^\varepsilon} \Big(\frac{\mathcal{T}_\varepsilon(x) - \mathcal{T}_\varepsilon(y)}{|\mathcal{T}_\varepsilon(x) - \mathcal{T}_\varepsilon(y)|^2} - \frac{\mathcal{T}_\varepsilon(x) - \mathcal{T}_\varepsilon(y)^*}{|\mathcal{T}_\varepsilon(x) - \mathcal{T}_\varepsilon(y)^*|^2} \Big)^\perp \omega(y)\, dy,$$

with the notation

$$y^* = \frac{y}{|y|^2}.$$

These explicit formulas will help us to find estimates for the velocity in terms of vorticity estimates.

Let us also introduce the Biot-Savart operator associated to the full plane, that is the operator, denoted $K_{\mathbb{R}^2}$ which maps a vorticity ω to the velocity

$$K_{\mathbb{R}^2}[\omega](x) := \int_{\mathbb{R}^2} H(x-y)\omega(y)\, dy, \tag{29}$$

where H is defined as

$$H(x) := \frac{x^\perp}{2\pi|x|^2}. \tag{30}$$

Proposition 1 is valid on the whole plane (see e.g. [3]). Precisely we have

Proposition 2. *Let $p \in (2,+\infty)$ (resp. $p=+\infty$). There exists a constant $C>0$ such that the following holds true. Let $\omega \in L^p_c(\mathbb{R}^2)$. Then $K_{\mathbb{R}^2}[\omega]$ is bounded, continuous, divergence-free and such that $\operatorname{curl} K_{\mathbb{R}^2}[\omega] = \omega$. Moreover, it satisfies*

$$\|K_{\mathbb{R}^2}[\omega]\|_{W^{1,p}(\mathbb{R}^2)} + \|K_{\mathbb{R}^2}[\omega]\|_{C^{1-2/p}(\mathbb{R}^2)} \leqslant C(\|\omega\|_{L^p(\mathbb{R}^2)} + \|\omega\|_{L^1(\mathbb{R}^2)}),$$
$$\textit{(resp. } \|K_{\mathbb{R}^2}[\omega]\|_{\mathcal{LL}(\mathbb{R}^2)} \leqslant C(\|\omega\|_{L^\infty(\mathbb{R}^2)} + \|\omega\|_{L^1(\mathbb{R}^2)})\textit{)},$$

and

$$K_{\mathbb{R}^2}[\omega](x) = \mathcal{O}\left(\frac{1}{|x|}\right) \quad \textit{as } x \to \infty.$$

We will also use several times the fact that $K_{\mathbb{R}^2}$ commutes with translations and rotations in the plane.

2.2 Harmonic field

To take the velocity circulation around the body into account, the following vector field will be useful. There exists one and only one solution H^ε vanishing at infinity of

$$\begin{aligned} \operatorname{div} H^\varepsilon = 0 \quad &\text{for } x \in \mathcal{F}_0^\varepsilon, \\ \operatorname{curl} H^\varepsilon = 0 \quad &\text{for } x \in \mathcal{F}_0^\varepsilon, \\ H^\varepsilon \cdot n = 0 \quad &\text{for } x \in \partial\mathcal{S}_0^\varepsilon, \\ \int_{\partial\mathcal{S}_0^\varepsilon} H^\varepsilon \cdot \tau\, ds = 1. & \end{aligned}$$

We refer for instance to [12], [11]. This solution is smooth. The vector field H^ε admits a harmonic stream function $\Psi_{H^\varepsilon}(x)$:

$$H^\varepsilon = \nabla^\perp \Psi_{H^\varepsilon},$$

which vanishes on the boundary $\partial\mathcal{S}_0^\varepsilon$, and behaves like $\ln|x|$ as x goes to infinity. In our case, we have

$$\Psi_{H^\varepsilon}(x) = \frac{1}{2\pi}\ln|\mathcal{T}_\varepsilon(x)| \text{ and } H^\varepsilon(x) = \frac{1}{2\pi} D\mathcal{T}_\varepsilon^T(x)\frac{(\mathcal{T}_\varepsilon(x))^\perp}{|\mathcal{T}_\varepsilon(x)|^2}. \tag{31}$$

The scaling law for H^ε is as follows:

$$H^\varepsilon(x) = \frac{1}{\varepsilon} H^1\left(\frac{x}{\varepsilon}\right). \tag{32}$$

We develop the function $H_1^\varepsilon - iH_2^\varepsilon$ in Laurent series. The fact that it is holomorphic (as a function of $z = x_1 + ix_2$), comes from $\operatorname{curl} H = \operatorname{div} H = 0$ which translates into the Cauchy-Riemann equations. One can see that $(H_1^\varepsilon - iH_2^\varepsilon)(z) = a_1^\varepsilon/z + \mathcal{O}(1/z^2)$ as $z \to \infty$ from the behaviour of H^ε at infinity. To identify a_1^ε, we use the fact that

$$\int_{\partial\mathcal{S}_0^\varepsilon} H^\varepsilon \cdot n\, ds = 0 \text{ and } \int_{\partial\mathcal{S}_0^\varepsilon} H^\varepsilon \cdot \tau\, ds = 1.$$

Hence we deduce that

$$(H_1^\varepsilon - iH_2^\varepsilon)(z) = \frac{1}{2i\pi z} + \mathcal{O}(1/z^2) \quad \text{as } z \to \infty. \tag{33}$$

Going back to real variables we have

$$H^1 = \mathcal{O}\left(\frac{1}{|x|}\right) \text{ and } \nabla H^1 = \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{34}$$

Let us also observe other consequences of (33):

$$x^\perp \cdot H^1 = \frac{1}{2\pi} + \mathcal{O}\left(\frac{1}{|x|}\right), \tag{35}$$

and

$$(H^1)^\perp - x^\perp \cdot \nabla H^1 = \mathcal{O}\left(\frac{1}{|x|^2}\right). \tag{36}$$

Remark 6. *In the case of a disk, we have*

$$H^\varepsilon(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} = H^1(x) = H(x).$$

2.3 Kirchoff potentials

Now to lift harmonically the boundary conditions, we will make use of the Kirchoff potentials, which are the solutions $\Phi^\varepsilon := (\Phi_i^\varepsilon)_{i=1,2,3}$ of the following problems:

$$-\Delta\Phi_i^\varepsilon = 0 \quad \text{for } x \in \mathcal{F}_0^\varepsilon, \tag{37}$$

$$\Phi_i^\varepsilon \longrightarrow 0 \quad \text{for } x \to \infty, \tag{38}$$

$$\frac{\partial \Phi_i^\varepsilon}{\partial n} = K_i \quad \text{for } x \in \partial\mathcal{F}_0^\varepsilon, \tag{39}$$

where

$$(K_1, K_2, K_3) := (n_1, n_2, x^\perp \cdot n). \tag{40}$$

Note that K_1, K_2 and K_3 actually depend on ε. Changing variables $y = x/\varepsilon$, we see that

$$\Phi_i^\varepsilon(x) = \varepsilon\Phi_i^1(x/\varepsilon) \quad \text{for } i = 1, 2, \tag{41}$$

$$\Phi_3^\varepsilon(x) = \varepsilon^2\Phi_3^1(x/\varepsilon). \tag{42}$$

The existence of Φ_i^1 is classical; note in particular that for $i = 1, 2, 3$ one has

$$\int_{\partial\mathcal{S}_0} K_i \, ds = 0.$$

Now, Φ_i^1 is the real part of a holomorphic function admitting a development in Laurent series; hence $\Phi_i^1(x) = \mathcal{O}(1/|x|)$ at infinity. For what concerns $\nabla\Phi_i^1$, we see that $\partial_1\Phi_i^1 - i\partial_2\Phi_i^1$ is holomorphic and admits a development in Laurent series, which is the derivative with respect to z of the former. We deduce that

$$\Phi_i^1(x) = \mathcal{O}\left(\frac{1}{|x|}\right) \text{ and } \nabla\Phi_i^1(x) = \mathcal{O}\left(\frac{1}{|x|^2}\right) \quad \text{as } |x| \to +\infty. \tag{43}$$

and consequently that $\nabla\Phi_i^\varepsilon$, for $i = 1, 2, 3$, are in $L^2(\mathcal{F}_0^\varepsilon)$.

Remark 7. *In the case of a disk, we have*

$$(\Phi_1^1, \Phi_2^1) = 2\pi(H^1)^\perp, \quad \Phi_3 = 0.$$

2.4 Velocity decomposition

Using the functions defined above, we deduce the following proposition.

Proposition 3. *Let $p > 2$. Let be given ω in $L^p_c(\mathcal{F}^\varepsilon_0)$, ℓ in $\mathbb{R}^2$, r and γ in $\mathbb{R}$. Then there is a unique solution v in $W^{1,p}(\mathcal{F}^\varepsilon_0)$ when $p < +\infty$ (resp. in $\mathcal{LL}(\mathcal{F}^\varepsilon_0)$ when $p = +\infty$) of*

$$\begin{cases} \operatorname{div} v = 0, \quad \text{for } x \in \mathcal{F}^\varepsilon_0, \\ \operatorname{curl} v = \omega \quad \text{for } x \in \mathcal{F}^\varepsilon_0, \\ v \cdot n = \left(\ell + r x^\perp\right) \cdot n \quad \text{for } x \in \partial\mathcal{S}^\varepsilon_0, \\ v \longrightarrow 0 \quad \text{as } x \to \infty, \\ \int_{\partial\mathcal{S}^\varepsilon_0} v \cdot \tau \, ds = \gamma. \end{cases} \tag{44}$$

Moreover v is given by

$$v = K^\varepsilon[\omega] + (\gamma + \alpha) H^\varepsilon + \ell_1 \nabla \Phi^\varepsilon_1 + \ell_2 \nabla \Phi^\varepsilon_2 + r \nabla \Phi^\varepsilon_3, \tag{45}$$

with

$$\alpha := \int_{\mathcal{F}^\varepsilon_0} \omega \, dx. \tag{46}$$

Proof of Proposition 3. The existence comes from the above paragraphs. The uniqueness can be easily deduced from [12, Lemma 2.14]. □

Let us also introduce the following variant of the Biot-Savart operator K^ε, the so-called hydrodynamic Biot-Savart operator K^ε_H which can here be deduced from K^ε by the formula

$$K^\varepsilon_H = K^\varepsilon + \alpha H^\varepsilon.$$

As a consequence it satisfies

$$\begin{aligned} \operatorname{div} K^\varepsilon_H[\omega] = 0, \quad \text{for} \quad x \in \mathcal{F}^\varepsilon_0, \\ \operatorname{curl} K^\varepsilon_H[\omega] = \omega, \quad \text{for} \quad x \in \mathcal{F}^\varepsilon_0, \\ K^\varepsilon_H[\omega] \cdot n = 0, \quad \text{for} \quad x \in \partial\mathcal{S}^\varepsilon_0, \\ \int_{\partial\mathcal{S}^\varepsilon_0} K^\varepsilon_H[\omega] \cdot \tau \, ds = 0. \end{aligned}$$

Then v can be decomposed as

$$v = K^\varepsilon_H[\omega] + \gamma H^\varepsilon + \ell_1 \nabla \Phi^\varepsilon_1 + \ell_2 \nabla \Phi^\varepsilon_2 + r \nabla \Phi^\varepsilon_3. \tag{47}$$

We also introduce the hydrodynamic Green function G^ε_H as

$$\begin{aligned} G^\varepsilon_H(x,y) &:= G^\varepsilon(x,y) + \Psi_{H^\varepsilon}(x) + \Psi_{H^\varepsilon}(y) \\ &= \frac{1}{2\pi} \ln \frac{|\mathcal{T}_\varepsilon(x) - \mathcal{T}_\varepsilon(y)||\mathcal{T}_\varepsilon(x)|}{|\mathcal{T}_\varepsilon(x) - \mathcal{T}_\varepsilon(y)^*|}. \end{aligned} \tag{48}$$

Consequently one has

$$K^\varepsilon_H[\omega](x) = \int_{\mathcal{F}^\varepsilon_0} \nabla^\perp G^\varepsilon_H(x,y) \omega(y) \, dy.$$

3 Equations in the body frame

3.1 Velocity equation

In order to transfer the equations in the body frame we apply the following isometric change of variable:

$$\begin{cases} v^\varepsilon(t,x) = Q^\varepsilon(t)^T \; u^\varepsilon(t, Q^\varepsilon(t)x + h^\varepsilon(t)), \\ q^\varepsilon(t,x) = p^\varepsilon(t, Q^\varepsilon(t)x + h^\varepsilon(t)), \\ \ell^\varepsilon(t) = Q^\varepsilon(t)^T \; (h^\varepsilon)'(t). \end{cases} \tag{49}$$

so that the equations (1)-(8) become

$$\frac{\partial v^\varepsilon}{\partial t} + \left[(v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp)\cdot\nabla\right] v^\varepsilon + r^\varepsilon (v^\varepsilon)^\perp + \nabla q^\varepsilon = 0 \qquad x \in \mathcal{F}_0^\varepsilon, \tag{50}$$

$$\operatorname{div} v^\varepsilon = 0 \qquad x \in \mathcal{F}_0^\varepsilon, \tag{51}$$

$$v^\varepsilon \cdot n = \left(\ell^\varepsilon + r^\varepsilon x^\perp\right)\cdot n \qquad x \in \partial\mathcal{S}_0^\varepsilon, \tag{52}$$

$$m(\ell^\varepsilon)'(t) = \int_{\partial\mathcal{S}_0^\varepsilon} q^\varepsilon n \, ds - m r^\varepsilon (\ell^\varepsilon)^\perp \tag{53}$$

$$\mathcal{J}_\varepsilon (r^\varepsilon)'(t) = \int_{\partial\mathcal{S}_0^\varepsilon} x^\perp \cdot q^\varepsilon n \, ds \tag{54}$$

$$v^\varepsilon(0,x) = v_0^\varepsilon(x) \qquad x \in \mathcal{F}_0^\varepsilon, \tag{55}$$

$$\ell^\varepsilon(0) = \ell_0, \ r^\varepsilon(0) = r_0. \tag{56}$$

3.2 Vorticity equation

We define

$$\begin{aligned} \omega^\varepsilon(t,x) &:= w^\varepsilon(t, Q^\varepsilon(t)x + h^\varepsilon(t)) \\ &= \operatorname{curl} v^\varepsilon(t,x). \end{aligned} \tag{57}$$

Taking the curl of the equation (50) we get

$$\partial_t \omega^\varepsilon + \left[(v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp)\cdot\nabla\right]\omega^\varepsilon = 0 \text{ for } x \in \mathcal{F}_0^\varepsilon. \tag{58}$$

Due to the conservations mentioned in Theorem 1, we have

$$\begin{aligned} \gamma &= \int_{\partial\mathcal{S}_0^\varepsilon} v^\varepsilon\cdot\tau\, ds = \int_{\partial\mathcal{S}^\varepsilon(t)} u^\varepsilon\cdot\tau\, ds = \int_{\partial\mathcal{S}_0^\varepsilon} u_0^\varepsilon\cdot\tau\, ds, \\ \alpha^\varepsilon &= \int_{\mathcal{F}_0^\varepsilon} \omega^\varepsilon(t,x)\, dx = \int_{\mathcal{F}^\varepsilon(t)} w^\varepsilon(t,x)\, dx = \int_{\mathcal{F}_0^\varepsilon} w_0^\varepsilon(x)\, dx. \end{aligned} \tag{59}$$

Now using Section 2, we can recover the velocity of the fluid from the vorticity, the velocity of the rigid body and the circulation of the flow around the solid through the following formula:

$$v^\varepsilon = K_H^\varepsilon[\omega^\varepsilon] + \gamma H^\varepsilon + \ell_1^\varepsilon \nabla\Phi_1^\varepsilon + \ell_2^\varepsilon\nabla\Phi_2^\varepsilon + r^\varepsilon\nabla\Phi_3^\varepsilon. \tag{60}$$

Also, introducing $\tilde v^\varepsilon$ by

$$\tilde v^\varepsilon := v^\varepsilon - \gamma H^\varepsilon, \tag{61}$$

we have

$$\tilde v^\varepsilon = K_H^\varepsilon[\omega^\varepsilon] + \ell_1^\varepsilon \nabla\Phi_1^\varepsilon + \ell_2^\varepsilon\nabla\Phi_2^\varepsilon + r^\varepsilon\nabla\Phi_3^\varepsilon. \tag{62}$$

4 A priori estimates

The goal of this section is to derive a priori bounds on the solutions given by Theorem 1, independently of ε (see Proposition 6 below). In particular, p is fixed in $(2,+\infty]$. We also give a result on an approximation of the velocity as $\varepsilon \to 0^+$ (Proposition 7), which will be useful in the sequel.

4.1 Vorticity

Due to Theorem 1, the generalized enstrophies are conserved when time proceeds, in particular, we have for any $t > 0$,

$$\|\omega^\varepsilon(t,\cdot)\|_{L^p(\mathcal{F}_0^\varepsilon)} = \|w_0^\varepsilon\|_{L^p(\mathcal{F}_0^\varepsilon)} \leqslant \|w_0\|_{L^p(\mathbb{R}^2)}, \quad \|\omega^\varepsilon(t,\cdot)\|_{L^1(\mathcal{F}_0^\varepsilon)} = \|w_0^\varepsilon\|_{L^1(\mathcal{F}_0^\varepsilon)} \leqslant \|w_0\|_{L^1(\mathbb{R}^2)}. \tag{63}$$

4.2 Energy

Let us introduce the matrix

$$\mathcal{M}^\varepsilon := \mathcal{M}_1^\varepsilon + \mathcal{M}_2^\varepsilon, \tag{64}$$

where

$$\mathcal{M}_1^\varepsilon := \begin{bmatrix} m\,\mathrm{Id}_2 & 0 \\ 0 & \mathcal{J}_\varepsilon \end{bmatrix}, \quad \text{and } \mathcal{M}_2^\varepsilon := \left[\int_{\mathcal{F}_0^\varepsilon} \nabla\Phi_a^\varepsilon \cdot \nabla\Phi_b^\varepsilon \, dx\right]_{a,b\in\{1,2,3\}} = \left[\varepsilon^{2+\delta_{a,3}+\delta_{b,3}} \int_{\mathcal{F}_0} \nabla\Phi_a^1 \cdot \nabla\Phi_b^1\right]_{a,b\in\{1,2,3\}}. \tag{65}$$

The matrix $\mathcal{M}^\varepsilon$ is symmetric and positive definite. The matrix $\mathcal{M}_2^\varepsilon$ actually encodes the phenomenon of added mass, which, loosely speaking, measures how much the surrounding fluid resists the acceleration as the body moves through it.

Using this added mass matrix, one can deduce a conserved quantity.

Proposition 4. *The following quantity is conserved along the motion:*

$$2\mathcal{H}^\varepsilon = X^T \mathcal{M}^\varepsilon X - \int_{\mathcal{F}_0^\varepsilon \times \mathcal{F}_0^\varepsilon} G_H^\varepsilon(x,y)\omega^\varepsilon(x)\omega^\varepsilon(y)\,dx\,dy - 2\gamma \int_{\mathcal{F}_0^\varepsilon} \omega^\varepsilon(x)\Psi_{H^\varepsilon}(x)\,dx,$$

where

$$X := \begin{pmatrix} \ell_1^\varepsilon \\ \ell_2^\varepsilon \\ r^\varepsilon \end{pmatrix}.$$

The proof of Proposition 4 is given in Section 7.

We will need the following technical lemma in order to derive some uniform estimates from Proposition 4.

Lemma 1. *Let f in $L^1(\mathbb{R}^2) \cap L^p(\mathbb{R}^2)$. We denote by*

$$\rho_f := \inf\{d > 1 \ / \ \mathrm{Supp}(f) \subset B(0,d)\}. \tag{66}$$

Then there exists $C > 0$ such that

$$\int_{\mathbb{R}^2} \Big|\ln|x-y| f(x)\Big|\,dx \leqslant C\|f\|_{L^p} + \ln(2\rho_f)\|f\|_{L^1},$$

for any $y \in B(0,\rho_f)$.

Proof. We fix $y \in B(0,\rho_f)$ and we decompose the integral:

$$\begin{aligned} \int_{\mathbb{R}^2} \Big|\ln|x-y| f(x)\Big|\,dx &= \int_{|x-y|\leqslant 1} \Big|\ln|x-y| f(x)\Big|\,dx + \int_{|x-y|\geqslant 1} \Big|\ln|x-y| f(x)\Big|\,dx \\ &\leqslant \|f\|_{L^p} \|\ln|\cdot|\|_{L^{p'}(B(0,1))} + \ln(2\rho_f)\|f\|_{L^1}, \end{aligned}$$

where

$$p' := \frac{p}{p-1}.$$

This ends the proof. ∎

As a consequence we have the following result.

Proposition 5. *One has the following estimate for some constant $C = C(m, \mathcal{J}_0, \|w_0\|_{L^1\cap L^p}, |\ell_0|, |r_0|, |\gamma|, \rho_{w_0})$, depending only on these values and the geometry for $\varepsilon = 1$:*

$$|\ell^\varepsilon(t)| + |\varepsilon r^\varepsilon(t)| \leqslant C[1 + \ln(\rho^\varepsilon(t))], \tag{67}$$

where

$$\rho^\varepsilon(t) := \rho_{\omega^\varepsilon(t,\cdot)} = \inf\{d > 1 \ / \ \mathrm{Supp}(\omega^\varepsilon(t,\cdot)) \subset B(0,d)\}.$$

Proof. We first add a constant in time to $\mathcal{H}^\varepsilon$, in order to get a quantity which is bounded with respect to ε:

$$\hat{\mathcal{H}}^\varepsilon := \mathcal{H}^\varepsilon - \frac{1}{2}\ln(\varepsilon)(\alpha^\varepsilon)^2 - \ln(\varepsilon)\gamma\alpha^\varepsilon,$$

where α^ε is given by (59). We decompose

$$\begin{aligned} 2\hat{\mathcal{H}}^\varepsilon &= m|\ell^\varepsilon|^2 + \mathcal{J}_0(\varepsilon r^\varepsilon)^2 + (X^\varepsilon)^T\mathcal{M}_2^\varepsilon X^\varepsilon \\ &\quad - \int_{\mathcal{F}_0^\varepsilon\times\mathcal{F}_0^\varepsilon}\Big(G_H^\varepsilon(x,y)+\ln(\varepsilon)\Big)\omega^\varepsilon(x)\omega^\varepsilon(y)\,dx\,dy - 2\gamma\int_{\mathcal{F}_0^\varepsilon}\omega^\varepsilon(x)\Big(\Psi_{H^\varepsilon}(x)+\ln(\varepsilon)\Big)\,dx, \end{aligned}$$

and we denote the last two terms as follows:

$$2\hat{\mathcal{H}}^\varepsilon =: m|\ell^\varepsilon|^2 + \mathcal{J}_0(\varepsilon r^\varepsilon)^2 + (X^\varepsilon)^T\mathcal{M}_2^\varepsilon X^\varepsilon - R_1^\varepsilon - 2\gamma R_2^\varepsilon. \tag{68}$$

We begin by estimating R_1^ε using (28) and (48):

$$R_1^\varepsilon = \frac{1}{2\pi}\int_{\mathcal{F}_0^\varepsilon\times\mathcal{F}_0^\varepsilon}\Big(\ln\Big|\varepsilon\mathcal{T}\Big(\frac{x}{\varepsilon}\Big)-\varepsilon\mathcal{T}\Big(\frac{y}{\varepsilon}\Big)\Big| - \ln\Big|\varepsilon\mathcal{T}\Big(\frac{x}{\varepsilon}\Big)-\varepsilon\mathcal{T}\Big(\frac{y}{\varepsilon}\Big)^*\Big| + \ln\Big|\varepsilon\mathcal{T}\Big(\frac{x}{\varepsilon}\Big)\Big|\Big)\omega^\varepsilon(x)\omega^\varepsilon(y)\,dx\,dy.$$

Next we make the change of variables $X = \varepsilon\mathcal{T}(\frac{x}{\varepsilon})$ and $Y = \varepsilon\mathcal{T}(\frac{y}{\varepsilon})$ to obtain

$$R_1^\varepsilon = \frac{1}{2\pi}\int_{B(0,\varepsilon)^c\times B(0,\varepsilon)^c}\Big(\ln|X-Y| - \ln|X-\varepsilon^2 Y^*| + \ln|X|\Big)f^\varepsilon(X)f^\varepsilon(Y)\,dX\,dY, \tag{69}$$

where

$$f^\varepsilon(z) := \omega^\varepsilon\Big(\varepsilon\mathcal{T}^{-1}\Big(\frac{z}{\varepsilon}\Big)\Big)\,|\det(D\mathcal{T})^{-1}|\Big(\frac{z}{\varepsilon}\Big).$$

Changing variables back, we note that

$$\|f^\varepsilon\|_{L^1(B(0,\varepsilon)^c)} = \|\omega^\varepsilon\|_{L^1(\mathcal{F}_0^\varepsilon)} \leqslant \|w_0\|_{L^1(\mathbb{R}^2)}.$$

Moreover, as $D\mathcal{T}^{-1}$ is bounded (due to (26) and the regularity of $\mathcal{S}_0$), we have that

$$\|f^\varepsilon\|_{L^p(B(0,\varepsilon)^c)} \leqslant C\|\omega^\varepsilon\|_{L^p(\mathcal{F}_0^\varepsilon)} \leqslant C\|w_0\|_{L^p(\mathbb{R}^2)}.$$

As $Y^*\in B(0,1/\varepsilon)$ for $Y\in B(0,\varepsilon)^c$, we have that $\varepsilon^2Y^*\in B(0,\varepsilon)\subset B(0,\rho_{f^\varepsilon})$. Using Lemma 1 and the fact that in (69) it is enough to consider $Y\in \operatorname{Supp}(f^\varepsilon)\subset B(0,\rho_{f^\varepsilon})$, we deduce that

$$|R_1^\varepsilon| \leqslant 3(C\|w_0\|_{L^p(\mathbb{R}^2)} + \ln(2\rho_{f^\varepsilon})\|w_0\|_{L^1(\mathbb{R}^2)})\|w_0\|_{L^1(\mathbb{R}^2)}.$$

Thanks to the behavior (26) at infinity of $\mathcal{T}$, we know that there exists $C_0\geqslant\beta$ such that $\mathcal{T}(B(0,d))\subset B(0,C_0 d)$ for any $d>1$. Then,

$$\operatorname{Supp}(f^\varepsilon(t)) = \varepsilon\mathcal{T}\Big(\frac{\operatorname{Supp}(\omega^\varepsilon(t))}{\varepsilon}\Big)\subset B(0,C_0\rho^\varepsilon),$$

which involves that $\rho_{f^\varepsilon}\leqslant C_0\rho^\varepsilon$, and we finally obtain

$$|R_1^\varepsilon(t)| \leqslant C_1[1+\ln(\rho^\varepsilon(t))]. \tag{70}$$

Using the same reasoning on R_2^ε as for R_1^ε, we obtain

$$R_2^\varepsilon = \frac{1}{2\pi}\int_{\mathcal{F}_0^\varepsilon}\ln\Big|\varepsilon\mathcal{T}\Big(\frac{x}{\varepsilon}\Big)\Big|\omega^\varepsilon(x)\,dx.$$

Hence we also deduce

$$|R_2^\varepsilon(t)| \leqslant C_2[1+\ln(\rho^\varepsilon(t))]. \tag{71}$$

Finally, we use that $\hat{\mathcal{H}}^\varepsilon$ is constant in time, and putting together (68), (70) and (71), we get:

$$\begin{aligned} m|\ell^\varepsilon|^2(t) + \mathcal{J}_0(\varepsilon r^\varepsilon(t))^2 &\leqslant m|\ell^\varepsilon|^2(t) + \mathcal{J}_0(\varepsilon r^\varepsilon(t))^2 + (X^\varepsilon)^T\mathcal{M}_2^\varepsilon X^\varepsilon(t) \\ &\leqslant m|\ell_0|^2 + \mathcal{J}_0(\varepsilon r_0)^2 + X_0^T\mathcal{M}_2^\varepsilon X_0 - R_1^\varepsilon(0) - 2\gamma R_2^\varepsilon(0) + R_1^\varepsilon(t) + 2\gamma R_2^\varepsilon(t) \\ &\leqslant C[1+\ln(\rho^\varepsilon(t))]. \end{aligned}$$

Above we used the notation $X_0 = (\ell_0, r_0)$ and the boundedness of $\mathcal{M}_2^\varepsilon$ with respect to ε which is a consequence of (65). This concludes the proof of Proposition 5. □

4.3 Velocity

We will use the following lemma (see [11, Theorem 4.1] and [13, Lemma 3.5]).

Lemma 2. *There exists a constant $C > 0$ which depends only on the shape of the solid for $\varepsilon = 1$ such that for any ω smooth enough,*

$$\|K_H^\varepsilon[\omega]\|_{L^\infty(\mathcal{F}_0^\varepsilon)} \leqslant C\|\omega\|_{L^1(\mathcal{F}_0^\varepsilon)}^{1-\frac{p'}{2}} \|\omega\|_{L^p(\mathcal{F}_0^\varepsilon)}^{\frac{p'}{2}}. \tag{72}$$

Combining with the conservation laws (63), the decomposition (62) and the scaling laws (41)-(42) we obtain that for any $t > 0$,

$$\|\tilde{v}^\varepsilon(t,\cdot)\|_{L^\infty(\mathcal{F}_0^\varepsilon)} \leqslant C\left(|\ell^\varepsilon(t,\cdot)| + |\varepsilon r^\varepsilon(t,\cdot)| + \|w_0\|_{L^1(\mathbb{R}^2)}^{1-\frac{p'}{2}} \|w_0\|_{L^p(\mathbb{R}^2)}^{\frac{p'}{2}}\right). \tag{73}$$

We will also use that combining Proposition 2 with (63) yields that for $p < +\infty$ (respectively for $p = +\infty$), $\|K_{\mathbb{R}^2}[\omega^\varepsilon(t,\cdot)]\|_{C^{1-2/p}(\mathbb{R}^2)}$ (resp. $\|K_{\mathbb{R}^2}[\omega^\varepsilon(t,\cdot)]\|_{\mathcal{LL}(\mathbb{R}^2)}$) is bounded independently of t and of ε, where ω^ε is extended by 0 inside $\mathcal{S}_0^\varepsilon$.

4.4 Support of the vorticity

We start with the following result about ρ^ε.

Lemma 3. *For all $t \geqslant 0$,*

$$\rho^\varepsilon(t) \leqslant \rho^\varepsilon(0) + \int_0^t \|v^\varepsilon - \ell^\varepsilon\|_{L^\infty(\mathbb{R}^2\setminus B(0,1))}.$$

This lemma will be proven in Section 8.3, together with the properties of solutions given by Theorem 1.

Let us now deduce an estimate on ρ in terms of the time and of the initial data. We note that $\rho^\varepsilon(0)$ does not depend on ε. We also see that, due to (32) and (34), $\|H^\varepsilon\|_{L^\infty(\mathbb{R}^2\setminus B(0,1))}$ is bounded independently of ε. It follows from (61) and (73) that

$$\rho^\varepsilon(t) \leqslant \rho^\varepsilon(0) + C\int_0^t \left(|\ell^\varepsilon(\tau,\cdot)| + |\varepsilon r^\varepsilon(\tau,\cdot)| + \|w_0\|_{L^1(\mathbb{R}^2)}^{1-\frac{p'}{2}} \|w_0\|_{L^p(\mathbb{R}^2)}^{\frac{p'}{2}} + |\gamma|\right) d\tau.$$

Using (67), it follows that for some constants C_1, $C_2 > 0$ depending only on m, $\mathcal{J}_0$, ℓ_0, r_0, w_0, ρ_{w_0} and the geometry for $\varepsilon = 1$, we have:

$$\rho^\varepsilon(t) \leqslant C_1 + C_2 \int_0^t (1 + \ln(\rho^\varepsilon(\tau)))\, d\tau.$$

Using Gronwall's lemma, we deduce that for any $T > 0$, ρ^ε is bounded on $[0,T]$ independently of ε.

4.5 Main a priori bounds

Gathering the previous estimates, we deduce the following.

Proposition 6. *For all $T > 0$, $\|\rho^\varepsilon\|_{L^\infty(0,T)}$, $\|\ell^\varepsilon\|_{L^\infty(0,T)}$, $\|\varepsilon r^\varepsilon\|_{L^\infty(0,T)}$, $\|\tilde{v}^\varepsilon\|_{L^\infty(0,T;L^\infty(\mathcal{F}_0^\varepsilon))}$ and $\|K_{\mathbb{R}^2}[\omega^\varepsilon]\|_{L^\infty(0,T;C^{1-2/p}(\mathbb{R}^2))}$ if $p < +\infty$ (resp. $\|K_{\mathbb{R}^2}[\omega^\varepsilon]\|_{L^\infty(0,T;\mathcal{LL}(\mathbb{R}^2))}$ if $p = +\infty$) are bounded independently of $\varepsilon > 0$.*

4.6 Approximation of the velocity

We will also use that

$$\check{v}^\varepsilon := K_{\mathbb{R}^2}[\omega^\varepsilon] + \sum_{i=1}^2 (\ell^\varepsilon - K_{\mathbb{R}^2}[\omega^\varepsilon]_{|x=0})_i \nabla\Phi_i^\varepsilon + r^\varepsilon\nabla\Phi_3^\varepsilon, \tag{74}$$

is a good approximation of $\tilde{v}^\varepsilon$ (which was introduced in (61)). More precisely we have the following estimate:

Proposition 7. *As ε approaches 0^+, we have:*

$$\|\check{v}^\varepsilon - \tilde{v}^\varepsilon\|_{L^\infty(0,T;L^2(\partial\mathcal{S}_0^\varepsilon))} + \|\check{v}^\varepsilon - \tilde{v}^\varepsilon\|_{L^\infty(0,T;H_\varepsilon^{1/2}(\mathcal{F}_0^\varepsilon))} = o(\varepsilon^{1/2}), \tag{75}$$

where

$$\|\cdot\|_{H_\varepsilon^{1/2}(\mathcal{F}_0^\varepsilon)} := \|\cdot\|_{\dot{H}^{1/2}(\mathcal{F}_0^\varepsilon)} + \varepsilon^{-1/2}\|\cdot\|_{L^2(\mathcal{F}_0^\varepsilon)}.$$

Proof. One checks that

$$\begin{cases} \operatorname{curl}(\check{v}^\varepsilon - \tilde{v}^\varepsilon) = 0, \quad \text{for } x \in \mathcal{F}_0^\varepsilon, \\ \operatorname{div}(\check{v}^\varepsilon - \tilde{v}^\varepsilon) = 0, \quad \text{for } x \in \mathcal{F}_0^\varepsilon, \\ \int_{\partial\mathcal{S}_0^\varepsilon} (\check{v}^\varepsilon - \tilde{v}^\varepsilon)\cdot\tau\, ds = 0, \\ (\check{v}^\varepsilon - \tilde{v}^\varepsilon)\cdot n = g^\varepsilon, \quad \text{for } x \in \partial\mathcal{S}_0^\varepsilon, \\ \check{v}^\varepsilon - \tilde{v}^\varepsilon \to 0 \quad \text{as } x \to \infty, \end{cases} \tag{76}$$

with

$$g^\varepsilon := (K_{\mathbb{R}^2}[\omega^\varepsilon] - K_{\mathbb{R}^2}[\omega^\varepsilon]_{|x=0})\cdot n. \tag{77}$$

As a consequence there exists Ψ^ε such that

$$\check{v}^\varepsilon - \tilde{v}^\varepsilon = \nabla\Psi^\varepsilon, \tag{78}$$

and

$$\begin{cases} \Delta\Psi^\varepsilon = 0, \quad \text{for } x \in \mathcal{F}_0^\varepsilon, \\ \partial_n\Psi^\varepsilon = g^\varepsilon, \quad \text{for } x \in \partial\mathcal{S}_0^\varepsilon, \\ \Psi^\varepsilon \to 0 \quad \text{as } x \to \infty. \end{cases} \tag{79}$$

We now use a dilatation argument and the following classical result (see for instance [12]).

Lemma 4. *There exists $C > 0$ such that for any g in $L^2(\partial\mathcal{S}_0)$ satisfying*

$$\int_{\partial\mathcal{S}_0} g(s)ds = 0, \tag{80}$$

there is only one solution Ψ in $H^{\frac{3}{2}}(\mathcal{F}_0)$ solution of

$$\begin{cases} \Delta\Psi = 0, \quad \textit{for } x \in \mathcal{F}_0, \\ \partial_n\Psi = g, \quad \textit{for } x \in \partial\mathcal{S}_0, \\ \Psi \to 0 \quad \textit{as } x \to \infty. \end{cases} \tag{81}$$

given as the potential layer

$$\Psi(x) = -\int_{\partial\mathcal{S}_0} G_{\mathcal{F}_0}(x,y)g(y)ds(y),$$

where $G_{\mathcal{F}_0}$ stands for the Green's function associated to the exterior domain $\mathcal{F}_0$ with Dirichlet boundary condition, and

$$\|\Psi\|_{H^{3/2}(\mathcal{F}_0)} \leqslant C\|g\|_{L^2(\partial\mathcal{S}_0)}. \tag{82}$$

Note that by a classical trace lemma, one has for some constant $C > 0$:

$$\|\nabla\Psi\cdot\tau\|_{L^2(\partial\mathcal{S}_0)} \leqslant C\|\Psi\|_{H^{3/2}(\mathcal{F}_0)}.$$

Now we use the change of variables:

$$\Psi^\varepsilon(x) = \varepsilon\Psi(x/\varepsilon), \quad g^\varepsilon(x) = g(x/\varepsilon),$$

with

$$\|\nabla\Psi^\varepsilon\cdot\tau\|_{L^2(\partial\mathcal{S}_0^\varepsilon)} = \sqrt{\varepsilon}\|\nabla\Psi\cdot\tau\|_{L^2(\partial\mathcal{S}_0)}, \quad \|\nabla\Psi^\varepsilon\|_{H_\varepsilon^{1/2}(\mathcal{F}_0^\varepsilon)} = \sqrt{\varepsilon}\|\nabla\Psi\|_{H_1^{1/2}(\mathcal{F}_0)} \text{ and } \|g^\varepsilon\|_{L^2(\partial\mathcal{S}_0^\varepsilon)} = \sqrt{\varepsilon}\|g\|_{L^2(\partial\mathcal{S}_0)}.$$

We apply Lemma 4 on (Ψ, g) (we note that g satisfies (80) because $\operatorname{div}(K_{\mathbb{R}^2}[\omega^\varepsilon] - K_{\mathbb{R}^2}[\omega^\varepsilon]_{|x=0}) = 0$), and infer that for $0 < \varepsilon \leqslant 1$,

$$\|(\check{v}^\varepsilon - \tilde{v}^\varepsilon)\cdot\tau\|_{L^\infty(0,T;L^2(\partial\mathcal{S}_0^\varepsilon))} + \|\check{v}^\varepsilon - \tilde{v}^\varepsilon\|_{L^\infty(0,T;H^{1/2}_\varepsilon(\mathcal{F}_0^\varepsilon))} \leqslant C\|g^\varepsilon\|_{L^\infty(0,T;L^2(\partial\mathcal{S}_0^\varepsilon))}.$$

Finally we use the uniform Hölder estimate on $K_{\mathbb{R}^2}[\omega^\varepsilon]$ given by Proposition 6 to deduce

$$\begin{aligned}\|(\check{v}^\varepsilon - \tilde{v}^\varepsilon)\cdot n\|_{L^\infty(0,T;L^2(\partial\mathcal{S}_0^\varepsilon))} = \|g^\varepsilon\|_{L^\infty(0,T;L^2(\partial\mathcal{S}_0^\varepsilon))} &\leqslant C\|K_{\mathbb{R}^2}[\omega^\varepsilon] - K_{\mathbb{R}^2}[\omega^\varepsilon]_{|x=0}\|_{L^\infty(0,T;L^2(\partial\mathcal{S}_0^\varepsilon))} \\ &= o(\varepsilon^{1/2}).\end{aligned}$$

This gives the desired conclusion and ends the proof of Proposition 7. □

5 Pressure force

The aim of this section is to study the pressure force/torque acting on the body:

$$F^\varepsilon(t) := \left(\int_{\partial\mathcal{S}_0^\varepsilon} q^\varepsilon n\, ds,\ \int_{\partial\mathcal{S}_0^\varepsilon} q^\varepsilon x^\perp\cdot n\, ds\right).$$

To convert the previous boundary integrals into distributed integrals, we first use Green's formula and the functions Φ_i^ε defined in (37)-(39) to write

$$F^\varepsilon(t) \quad = \quad \left(\int_{\mathcal{F}_0^\varepsilon} \nabla q^\varepsilon(x)\cdot\nabla\Phi_i^\varepsilon dx\right)_{i=1,2,3}.$$

That there is no contribution coming from infinity is justified by (136).

Using the following equality for two vector fields a and b in $\mathbb{R}^2$:

$$\nabla(a\cdot b) = a\cdot\nabla b + b\cdot\nabla a - (a^\perp \operatorname{curl} b + b^\perp \operatorname{curl} a), \tag{83}$$

the equation (50) reads as follows

$$\frac{\partial v^\varepsilon}{\partial t} + [v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp]^\perp\omega^\varepsilon + \nabla\frac{1}{2}(v^\varepsilon)^2 - \nabla((\ell^\varepsilon + r^\varepsilon x^\perp)\cdot v^\varepsilon) + \nabla q^\varepsilon = 0. \tag{84}$$

Plugging the decomposition (61) into the previous equation, we find

$$\frac{\partial v^\varepsilon}{\partial t} + [v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp]^\perp\omega^\varepsilon + \nabla(\mathcal{Q}^\varepsilon + q^\varepsilon) = 0, \tag{85}$$

$$\mathcal{Q}^\varepsilon := \frac{1}{2}|\tilde{v}^\varepsilon|^2 + \gamma(\tilde{v}^\varepsilon - (\ell^\varepsilon + r^\varepsilon x^\perp))\cdot H^\varepsilon + \frac{1}{2}\gamma^2|H^\varepsilon|^2 - (\ell^\varepsilon + r^\varepsilon x^\perp)\cdot\tilde{v}^\varepsilon. \tag{86}$$

Using (85)-(86) we get the following decomposition of F^ε, for $i = 1,2,3$:

$$-F_i^\varepsilon(t) \quad = \quad A_i^\varepsilon + B_i^\varepsilon + C_i^\varepsilon,$$

where

$$\begin{aligned} A_i^\varepsilon &:= \int_{\mathcal{F}_0^\varepsilon} \partial_t v^\varepsilon\cdot\nabla\Phi_i^\varepsilon(x)\, dx, \\ B_i^\varepsilon &:= \int_{\mathcal{F}_0^\varepsilon} \omega^\varepsilon[v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp]^\perp\cdot\nabla\Phi_i^\varepsilon(x)\, dx, \\ C_i^\varepsilon &:= \int_{\partial\mathcal{S}_0^\varepsilon} \mathcal{Q}^\varepsilon K_i\, ds. \end{aligned}$$

For the last term, we used again Green's formula. We underline that there is no contribution from the infinity since each term in $\mathcal{Q}^\varepsilon$ is (at least) bounded as $|x| \to +\infty$, while the normal derivative of Φ_i^ε over large circles satisfies $\partial_n\Phi_i^\varepsilon = \mathcal{O}(1/|x|^2)$.

In the rest of this section, we study the limit as ε goes to zero of all these terms.

5.1 Treatment of the first term

Let us examine A_i^ε for $i = 1, 2, 3$. As $\operatorname{div} v^\varepsilon = 0$ for all time, using (37)-(38) we deduce that

$$A_i^\varepsilon = \int_{\partial \mathcal{S}_0^\varepsilon} \partial_t v^\varepsilon \cdot n \, \Phi_i^\varepsilon(x) \, ds.$$

Now using the boundary condition (52), (65) and Green's formula we obtain

$$(A_i^\varepsilon)_{i=1,2,3} = \mathcal{M}_2^\varepsilon \begin{pmatrix} \ell_1^\varepsilon \\ \ell_2^\varepsilon \\ r^\varepsilon \end{pmatrix}' (t).$$

These terms will be put on the left hand side of the solid equations (see also (145) in the Section concerning the Cauchy problem).

5.2 Limit for the second term

The second term will have no contribution in the limit, as the following proposition shows.

Proposition 8. *As $\varepsilon \to 0^+$, one has:*

$$B_i^\varepsilon \longrightarrow 0 \ \text{ for } i = 1, 2 \text{ and } \frac{B_3^\varepsilon}{\varepsilon} \longrightarrow 0 \text{ for } i = 3.$$

Proof of Proposition 8. According to (61), we cut B_i^ε in two parts: $B_i^\varepsilon = \hat{B}_i^\varepsilon + \check{B}_i^\varepsilon$ with

$$\hat{B}_i^\varepsilon := \int_{\mathcal{F}_0^\varepsilon} \omega^\varepsilon [\tilde{v}^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp]^\perp \cdot \nabla \Phi_i^\varepsilon(x) \, dx, \tag{87}$$

$$\check{B}_i^\varepsilon := \int_{\mathcal{F}_0^\varepsilon} \gamma \omega^\varepsilon [H^\varepsilon]^\perp \cdot \nabla \Phi_i^\varepsilon(x) \, dx. \tag{88}$$

The following estimates are uniform with respect to $t \in [0, T]$.

1. Let us begin with $i = 1, 2$. In this case we have

$$\begin{aligned} |\hat{B}_i^\varepsilon| &= \left| \int_{\mathcal{F}_0^\varepsilon \cap B(0,\rho^\varepsilon)} [\tilde{v}^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp]^\perp \omega^\varepsilon \cdot \nabla \Phi_i^\varepsilon(x) \, dx \right| \\ &\leqslant \|\tilde{v}^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp\|_{L^\infty(\mathcal{F}_0^\varepsilon \cap B(0,\rho^\varepsilon))} \|\omega^\varepsilon\|_{L^p} \|\nabla \Phi_i^\varepsilon\|_{L^{p'}(\mathcal{F}_0^\varepsilon \cap B(0,\rho^\varepsilon))}, \end{aligned}$$

when $p < +\infty$. In the case $p = +\infty$, we have in particular $w_0 \in L^3_c(\mathcal{F}_0)$ and use the inequalities written here for $p = 3$. Using (41) and recalling (43), we see that for $r > 1$

$$\|\nabla \Phi_i^\varepsilon\|_{L^r(\mathcal{F}_0^\varepsilon \cap B(0,\rho^\varepsilon))} = \varepsilon^{2/r} \|\nabla \Phi_i^1\|_{L^r(\mathcal{F}_0 \cap B(0,\rho^\varepsilon/\varepsilon))} \leqslant \varepsilon^{2/r} \|\nabla \Phi_i^1\|_{L^r(\mathcal{F}_0)}. \tag{89}$$

On the other side, due to (73) and to Proposition 6, we have

$$|\rho^\varepsilon| + \|\tilde{v}^\varepsilon\|_{L^\infty(\mathcal{F}_0^\varepsilon \cap B(0,\rho^\varepsilon))} + |\ell^\varepsilon| + \varepsilon |r^\varepsilon| \leqslant C.$$

It follows that

$$\hat{B}_i^\varepsilon = \mathcal{O}\left(\varepsilon^{\frac{2}{p'} - 1}\right) \to 0 \text{ as } \varepsilon \to 0^+.$$

Concerning $\check{B}_i^\varepsilon$, we write

$$|\check{B}_i^\varepsilon| \leqslant |\gamma| \|H^\varepsilon\|_{L^q(\mathcal{F}_0^\varepsilon \cap B(0,\rho^\varepsilon))} \|\omega^\varepsilon\|_{L^p} \|\nabla \Phi_i^\varepsilon\|_{L^r(\mathcal{F}_0^\varepsilon \cap B(0,\rho^\varepsilon))},$$

with $\frac{1}{q}=\frac{1}{2}(\frac{1}{p'}+\frac{1}{2})$ and $\frac{1}{r}=\frac{1}{2}(\frac{1}{p'}-\frac{1}{2})$. We use the fact that $\|H^\varepsilon\|_{L^q(\mathcal{F}_0^\varepsilon\cap B(0,\rho^\varepsilon))}$ is bounded independently of ε (see (32) and (34) and observe that $q<2$) and once again (89). Hence we also have

$$\check{B}_i^\varepsilon \longrightarrow 0 \text{ as } \varepsilon\to 0^+.$$

2. Let us now turn to the case $i=3$. In that case, the scaling of $\nabla\Phi_3^\varepsilon$ is not the same. But using (42), we see that the situation is actually better, in the sense that using the same estimates as before, we get an additional power of ε. Then (88) follows.

□

5.3 Limit for the third term

We decompose C_i^ε into

$$\begin{aligned}
C_{i,a}^\varepsilon &= \frac{1}{2}\int_{\partial\mathcal{S}_0^\varepsilon}|\tilde{v}^\varepsilon|^2K_i\,ds, &(90)\\
C_{i,b}^\varepsilon &= \gamma\int_{\partial\mathcal{S}_0^\varepsilon}(\tilde{v}^\varepsilon-(\ell^\varepsilon+r^\varepsilon x^\perp))\cdot H^\varepsilon K_i\,ds, &(91)\\
C_{i,c}^\varepsilon &= \frac{\gamma^2}{2}\int_{\partial\mathcal{S}_0^\varepsilon}|H^\varepsilon|^2K_i\,ds, &(92)\\
C_{i,d}^\varepsilon &= -\int_{\partial\mathcal{S}_0^\varepsilon}(\ell^\varepsilon+r^\varepsilon x^\perp)\cdot\tilde{v}^\varepsilon K_i\,ds. &(93)
\end{aligned}$$

1. We first tackle the terms $C_{i,a}^\varepsilon$ and $C_{i,d}^\varepsilon$ which are the easiest ones. One easily sees that for $i=1,2$:

$$|C_{i,a}^\varepsilon|+|C_{i,d}^\varepsilon|\leqslant C\varepsilon(\|\tilde{v}^\varepsilon\|_{L^\infty}^2+|\ell^\varepsilon|^2+|\varepsilon r^\varepsilon|^2),$$

and that for $i=3$:

$$|C_{3,a}^\varepsilon|+|C_{3,d}^\varepsilon|\leqslant C\varepsilon^2(\|\tilde{v}^\varepsilon\|_{L^\infty}^2+|\ell^\varepsilon|^2+|\varepsilon r^\varepsilon|^2).$$

We conclude with Proposition 6 that these terms tend to zero as $\varepsilon\to 0^+$, as $\mathcal{O}(\varepsilon^2)$ when $i=3$.

2. We turn to $C_{i,c}^\varepsilon$. We will make use of the following classical Blasius' lemma (see for instance [16] and [4, Problem 4.3]), which we prove in the appendix for the sake of self-containedness.

Lemma 5. *Let $\mathcal{C}$ be a smooth Jordan curve, $f:=(f_1,f_2)$ and $g:=(g_1,g_2)$ two smooth tangent vector fields on $\mathcal{C}$. Then*

$$\int_{\mathcal{C}}(f\cdot g)n\,ds = i\left(\int_{\mathcal{C}}(f_1-if_2)(g_1-ig_2)\,dz\right)^*, \tag{94}$$

$$\int_{\mathcal{C}}(f\cdot g)(x^\perp\cdot n)\,ds = Re\left(\int_{\mathcal{C}}z(f_1-if_2)(g_1-ig_2)\,dz\right). \tag{95}$$

where $(\cdot)^*$ denotes the complex conjugation.

We now apply Lemma 5 and use (33) and Cauchy's Residue Theorem. We deduce directly that $C_{1,c}^\varepsilon=C_{2,c}^\varepsilon=C_{3,c}^\varepsilon=0$.

3. Let us finally turn to the main term, that is $C_{i,b}^\varepsilon$. Let us prove the following.

Proposition 9. *One has for $i=1,2$*

$$C_{i,b}^\varepsilon=\gamma(K_{\mathbb{R}^2}[\omega^\varepsilon](t,0)-\ell^\varepsilon)^\perp+\varepsilon r^\varepsilon\gamma\xi+o(1), \tag{96}$$

and

$$C_{3,b}^{\varepsilon} = \gamma\varepsilon\,\zeta \cdot (K_{\mathbb{R}^2}[\omega^{\varepsilon}](t,0) - \ell^{\varepsilon}) + o(\varepsilon), \tag{97}$$

where ξ *and* ζ *are defined in* $\mathbb{R}^2 \simeq \mathbb{C}$ *by*

$$\xi := \left(\int_{\partial\mathcal{S}_0} \overline{z}(H_1^1 - iH_2^1)\,dz \right)^*,$$
$$\zeta := \int_{\partial\mathcal{S}_0} (H_1^1 - iH_2^1) z\,dz.$$

Proof of Proposition 9. We introduce

$$\overline{v}^{\varepsilon}(t,x) := K_{\mathbb{R}^2}[\omega^{\varepsilon}](t,0) + \sum_{i=1}^{2} (\ell_i^{\varepsilon}(t) - K_{\mathbb{R}^2}[\omega^{\varepsilon}]_i(t,0)) \nabla\Phi_i^{\varepsilon}(x) + r^{\varepsilon}(t)\nabla\Phi_3^{\varepsilon}(x) \text{ in } \mathbb{R}^+ \times \mathcal{F}_0^{\varepsilon},$$

which will give a good approximation of $\check{v}^{\varepsilon}$ on $\partial\mathcal{S}_0^{\varepsilon}$ (compare to (74)). Here ω^{ε} is again extended by 0 inside $\mathcal{S}_0^{\varepsilon}$. Note in particular that one has

$$\overline{v}^{\varepsilon} \cdot n = \tilde{v}^{\varepsilon} \cdot n = (\ell^{\varepsilon} + r^{\varepsilon} x^{\perp}) \cdot n \text{ on } \partial\mathcal{S}_0^{\varepsilon}. \tag{98}$$

The two steps in estimating $C_{i,b}^{\varepsilon}$ consists in computing the integral $C_{i,b}^{\varepsilon}$ in (91) when $\tilde{v}^{\varepsilon}$ is replaced with $\overline{v}^{\varepsilon}$, and then to show that the error of this replacement is small as $\varepsilon \to 0^+$.

a. Denote

$$\hat{C}_{i,b}^{\varepsilon} = \gamma \int_{\partial\mathcal{S}_0^{\varepsilon}} (\overline{v}^{\varepsilon} - (\ell^{\varepsilon} + r^{\varepsilon} x^{\perp})) \cdot H^{\varepsilon} K_i\,ds,$$

and

$$\underline{v}^{\varepsilon}(t,x) := \overline{v}^{\varepsilon}(t,\varepsilon x) - (\ell^{\varepsilon} + \varepsilon r^{\varepsilon} x^{\perp}) \text{ in } \mathbb{R}^+ \times \mathcal{F}_0. \tag{99}$$

By a direct scaling argument (see (32)), we deduce that

$$\hat{C}_{i,b}^{\varepsilon} = \hat{C}_{i,b} \text{ with } \hat{C}_{i,b} := \gamma \int_{\partial\mathcal{S}_0} \underline{v}^{\varepsilon} \cdot H^1 n_i\,ds \text{ for } i = 1,2,$$

and

$$\hat{C}_{3,b}^{\varepsilon} = \varepsilon \hat{C}_{3,b} \text{ with } \hat{C}_{3,b} := \gamma \int_{\partial\mathcal{S}_0} \underline{v}^{\varepsilon} \cdot H^1 (x^{\perp} \cdot n)\,ds.$$

We remark that $\overline{v}^{\varepsilon} - (\ell^{\varepsilon} + r^{\varepsilon} x^{\perp})$ is a smooth vector field, tangent to $\partial\mathcal{S}_0^{\varepsilon}$, so that $\underline{v}^{\varepsilon}$ is tangent to $\partial\mathcal{S}_0$. Hence, we get by Lemma 5 that

$$(\hat{C}_{1,b}, \hat{C}_{2,b}) = \gamma \int_{\partial\mathcal{S}_0} (\underline{v}^{\varepsilon} \cdot H^1) n\,ds = i\gamma \left(\int_{\partial\mathcal{S}_0} (\underline{v}_1^{\varepsilon} - i\underline{v}_2^{\varepsilon})(H_1^1 - iH_2^1)\,dz \right)^*, \tag{100}$$

$$\hat{C}_{3,b} = \gamma \int_{\partial\mathcal{S}_0} \underline{v}^{\varepsilon} \cdot H^1 (x^{\perp} \cdot n)\,ds = \gamma \operatorname{Re} \left(\int_{\partial\mathcal{S}_0} (\underline{v}_1^{\varepsilon} - i\underline{v}_2^{\varepsilon})(H_1^1 - iH_2^1) z\,dz \right). \tag{101}$$

Let us denote

$$\underline{v}_{\infty}^{\varepsilon} := K_{\mathbb{R}^2}[\omega^{\varepsilon}](t,0) - \ell^{\varepsilon}.$$

For what concerns $(H_1^{\varepsilon} - iH_2^{\varepsilon})$ we have (33). Concerning $\underline{v}^{\varepsilon}$, due to (43) and (99), and using

$$(x^{\perp})_1 - i(x^{\perp})_2 = -i(x_1 + ix_2)^*,$$

we have that

$$\underline{v}_1^{\varepsilon} - i\underline{v}_2^{\varepsilon} = i\varepsilon r^{\varepsilon} \overline{z} + \underline{v}_{\infty,1}^{\varepsilon} - i\underline{v}_{\infty,2}^{\varepsilon} + \mathcal{O}(1/|z|^2).$$

• We first study $(\hat{C}_{1,b},\, \hat{C}_{2,b})$. Using Cauchy's residue theorem, we deduce that for $i = 1, 2$:

$$i \left(\int_{\partial \mathcal{S}_0} (\underline{v}_1^\varepsilon - i \underline{v}_2^\varepsilon - i \varepsilon r^\varepsilon \overline{z})(H_1^1 - i H_2^1)\, dz \right)^* \quad = \quad i(\underline{v}_{\infty,1}^\varepsilon - i \underline{v}_{\infty,2}^\varepsilon)^* \tag{102}$$

$$= \quad {\underline{v}_\infty^\varepsilon}^\perp. \tag{103}$$

With the definition of ξ we deduce

$$(\hat{C}_{1,b},\, \hat{C}_{2,b}) = \gamma {\underline{v}_\infty^\varepsilon}^\perp + \varepsilon r^\varepsilon \gamma \xi. \tag{104}$$

• We now consider $\hat{C}_{3,b}$. We will use the following lemma, proved in the appendix.

Lemma 6.

$$Im \left(\int_{\partial \mathcal{S}_0} \overline{z} (H_1^1 - i H_2^1) z\, dz \right) = 0.$$

It follows from this lemma that the term $r^\varepsilon x^\perp$ does not intervene in $\hat{C}_{3,b}$. By Cauchy's residue theorem and using the definition of ζ we deduce that

$$\mathrm{Re} \left(\int_{\partial \mathcal{S}_0} (\underline{v}_1^\varepsilon - i \underline{v}_2^\varepsilon + \varepsilon r^\varepsilon \overline{z})(H_1^1 - i H_2^1) z\, dz \right) = \zeta \cdot \underline{v}_\infty^\varepsilon.$$

b. Let us now establish that

$$\int_{\partial \mathcal{S}_0^\varepsilon} (\overline{v}^\varepsilon - \tilde{v}^\varepsilon) \cdot H^\varepsilon n_i\, ds = o(1), \tag{105}$$

and that

$$\int_{\partial \mathcal{S}_0^\varepsilon} (\overline{v}^\varepsilon - \tilde{v}^\varepsilon) \cdot H^\varepsilon (x^\perp \cdot n)\, ds = o(\varepsilon). \tag{106}$$

On one side, it is straightforward using the Hölder estimate on $K_{\mathbb{R}^2}[\omega^\varepsilon]$ given by Proposition 6 to infer that

$$|\check{v}^\varepsilon - \overline{v}^\varepsilon| = o(1) \;\text{ uniformly on } (0,T) \times \partial \mathcal{S}_0^\varepsilon.$$

One the other side from Proposition 7 we have that $\|\check{v}^\varepsilon - \tilde{v}^\varepsilon\|_{L^\infty(0,T;L^2(\partial \mathcal{S}_0^\varepsilon))} = o(\varepsilon^{1/2})$. Estimates (105) and (106) follow by the Cauchy-Schwarz inequality. □

5.4 Conclusion

Putting together all the results established in this section, we can state the following proposition.

Proposition 10. *The pressure force/torque can be written:*

$$\begin{pmatrix} F_1^\varepsilon \\ F_2^\varepsilon \\ F_3^\varepsilon \end{pmatrix} = -\mathcal{M}_2^\varepsilon \begin{pmatrix} \ell^\varepsilon \\ r^\varepsilon \end{pmatrix}' + \gamma \begin{pmatrix} (\ell^\varepsilon - K_{\mathbb{R}^2}[\omega^\varepsilon](t,0))^\perp - \varepsilon r^\varepsilon \xi \\ \varepsilon\, \zeta \cdot (\ell^\varepsilon - K_{\mathbb{R}^2}[\omega^\varepsilon](t,0)) \end{pmatrix} + \begin{pmatrix} R_1^\varepsilon \\ R_2^\varepsilon \\ \varepsilon R_3^\varepsilon \end{pmatrix},$$

with

$$R_i^\varepsilon \longrightarrow 0 \;\; \text{in } L^\infty(0,T) \;\; \text{as} \;\; \varepsilon \to 0^+.$$

6 Passage to the limit

6.1 Compactness

1. *Compactness for the solid velocity.* We begin by obtaining compactness on the solid linear and angular velocities in the original frame. Using Proposition 10 and (53)-(54), we obtain

$$\mathcal{M}^\varepsilon \begin{pmatrix} \ell^\varepsilon \\ r^\varepsilon \end{pmatrix}' = \gamma \begin{pmatrix} (\ell^\varepsilon - K_{\mathbb{R}^2}[\omega^\varepsilon](t,0))^\perp - \varepsilon r^\varepsilon \xi \\ \varepsilon\, \zeta \cdot (\ell^\varepsilon - K_{\mathbb{R}^2}[\omega^\varepsilon](t,0)) \end{pmatrix} + \begin{pmatrix} -m r^\varepsilon (\ell^\varepsilon)^\perp \\ 0 \end{pmatrix} + \begin{pmatrix} R_1^\varepsilon \\ R_2^\varepsilon \\ \varepsilon R_3^\varepsilon \end{pmatrix}.$$

We multiply by $\mathcal{M}_1^\varepsilon(\mathcal{M}^\varepsilon)^{-1}$; using (65), Proposition 6 and (63), and simplifying by ε the second equation we deduce that

$$m(\ell^\varepsilon)' = \gamma(\ell^\varepsilon - K_{\mathbb{R}^2}[\omega^\varepsilon](t,0))^\perp - (\varepsilon r^\varepsilon)\gamma\xi - mr^\varepsilon(\ell^\varepsilon)^\perp + \tilde{R}_1^\varepsilon \tag{107}$$

$$\mathcal{J}_0(\varepsilon r^\varepsilon)' = \gamma\zeta\cdot(\ell^\varepsilon - K_{\mathbb{R}^2}[\omega^\varepsilon](t,0)) + \tilde{R}_2^\varepsilon, \tag{108}$$

with

$$\tilde{R}_1^\varepsilon, \tilde{R}_2^\varepsilon \longrightarrow 0 \text{ in } L^\infty(0,T) \text{ as } \varepsilon \to 0^+. \tag{109}$$

Going back to the original velocity by using (49) and (57) we deduce:

$$m(h^\varepsilon)'' = \gamma((h^\varepsilon)' - K_{\mathbb{R}^2}[w^\varepsilon](t,h^\varepsilon))^\perp - \gamma(\varepsilon r^\varepsilon)Q^\varepsilon(t)\xi + Q^\varepsilon(t)\tilde{R}_1^\varepsilon, \tag{110}$$

$$\mathcal{J}_0(\varepsilon r^\varepsilon)' = \gamma\zeta\cdot Q^\varepsilon(t)^T((h^\varepsilon)' - K_{\mathbb{R}^2}[w^\varepsilon](t,h^\varepsilon)) + \tilde{R}_2^\varepsilon, \tag{111}$$

We used the fact that the Biot-Savart law in the plane (29) commutes with translations and rotations.

Thanks to Proposition 6 we have that $K_{\mathbb{R}^2}[\omega^\varepsilon](t,0)$ is bounded in $L^\infty(0,T)$ as $\varepsilon \to 0^+$. Now since the right hand sides of (108) and (110) are bounded in $L^\infty(0,T)$ (due to Proposition 6), we infer that for some subsequence (ε_n), $\varepsilon_n \to 0^+$ of the parameter ε, we have

$$h^{\varepsilon_n} \overset{w*}{\rightharpoonup} h \text{ in } W^{2,\infty}(0,T), \tag{112}$$

$$\varepsilon_n r^{\varepsilon_n} \overset{w*}{\rightharpoonup} R \text{ in } W^{1,\infty}(0,T). \tag{113}$$

2. *Compactness for the fluid velocity.* Let us now obtain some compactness for the fluid vorticity in the original frame, and for the velocity it generates via the Biot-Savart law. We obtain a convergence along a subsequence of (ε_n); to simplify the notations we will still call it (ε_n).

We extend $w^\varepsilon(t,\cdot)$ by 0 inside $\mathcal{S}^\varepsilon(t)$. Using the a priori estimate (63), we deduce that, up to a subsequence of (ε_n), one has, for some $w \in L^\infty(0,T;L^p(\mathbb{R}^2))$:

$$w^{\varepsilon_n} \overset{w*}{\rightharpoonup} w \text{ weakly in } L^\infty(0,T;L^p(\mathbb{R}^2)) \text{ as } n \to +\infty. \tag{114}$$

Also, using (63) and Proposition 2, we deduce that $K_{\mathbb{R}^2}[w^\varepsilon]$ is bounded in $L^\infty(0,T;W^{1,p}(\mathbb{R}^2))$ as $\varepsilon \to 0^+$ for $p < +\infty$ (resp. in $L^\infty(0,T;\mathcal{LL}(\mathbb{R}^2))$ if $p = +\infty$). We extend ω^ε by 0 inside $\mathcal{S}^\varepsilon(t)$. Then it is not difficult to check that from (52) and (58) that

$$\partial_t\omega^\varepsilon + \operatorname{div}((v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp)\omega^\varepsilon) = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^2).$$

Going back to the original variables we infer

$$\partial_t w^\varepsilon + \operatorname{div}(u^\varepsilon w^\varepsilon) = 0 \text{ in } \mathcal{D}'((0,T)\times\mathbb{R}^2). \tag{115}$$

In particular, $\partial_t w^\varepsilon$ is bounded in $L^\infty(0,T;W^{-1,p}(\mathbb{R}^2))$. Hence we deduce by [15, Appendix C] that the convergence (114) can be improved into

$$w^{\varepsilon_n} \longrightarrow w \text{ in } C^0([0,T];L^p(\mathbb{R}^2)-w) \text{ (resp. in } C^0([0,T];L^\infty(\mathbb{R}^2)-w*) \text{ if } p=+\infty) \text{ as } n\to+\infty\ . \tag{116}$$

Actually, [15, Appendix C] considers only the case $p < +\infty$ since it proves the compactness of a sequence in $C^0([0,T];X-w)$ for X a reflexive separable Banach space. However, the generalization to $C^0([0,T];L^\infty(\mathbb{R}^2)-w*)$ is straightforward using the separability of $L^1(\mathbb{R}^2)$.

Now using Proposition 2 and the Ascoli-Arzelà theorem, we see that $K_{\mathbb{R}^2}$ is a compact operator form $L^p(\mathbb{R}^2)$ to $L^\infty_{loc}(\mathbb{R}^2)$, so one deduces that

$$K_{\mathbb{R}^2}[w^{\varepsilon_n}] \longrightarrow K_{\mathbb{R}^2}[w] \text{ in } C^0([0,T];L^\infty_{loc}(\mathbb{R}^2)) \text{ as } n\to+\infty. \tag{117}$$

6.2 Characterization of the limit of the fluid velocity

1. *Convergence of* u^ε.

• We extend u^ε by $(h^\varepsilon)' + r^\varepsilon (x-h^\varepsilon)^\perp$ inside $\mathcal{S}^\varepsilon(t)$. We define $\tilde{u}^\varepsilon$ by the relation

$$\tilde{v}^\varepsilon(t,x) = Q^\varepsilon(t)^T \; \tilde{u}^\varepsilon(t, Q^\varepsilon(t)x + h^\varepsilon(t)), \tag{118}$$

so that

$$\tilde{u}^\varepsilon := u^\varepsilon - \gamma Q^\varepsilon H^\varepsilon((Q^\varepsilon)^T(x-h^\varepsilon(t))), \tag{119}$$

Using (41) and (42) we deduce that

$$\nabla \Phi_i^\varepsilon \longrightarrow 0 \;\text{ for } i=1,2 \text{ and } \frac{1}{\varepsilon}\nabla\Phi_3^\varepsilon \longrightarrow 0 \;\text{ in } L^2(\mathbb{R}^2) \text{ as } \varepsilon \to 0^+.$$

Here we extended $\nabla\Phi_i^\varepsilon$ inside $\mathcal{S}_0^\varepsilon$ by the basis vector e_i for $i=1$ or 2, and by $x^\perp$ for $i=3$. Consequently, from Proposition 6 and (74) we deduce that

$$\check{v}^\varepsilon - K_{\mathbb{R}^2}[\omega^\varepsilon] \longrightarrow 0 \;\text{ in } L^\infty(0,T;L^2(\mathbb{R}^2)) \text{ as } \varepsilon \to 0^+. \tag{120}$$

Gathering (75) and (120) we obtain that

$$K_{\mathbb{R}^2}[\omega^\varepsilon] - \tilde{v}^\varepsilon \longrightarrow 0 \;\text{ in } L^\infty(0,T;L^2(\mathbb{R}^2)) \text{ as } \varepsilon \to 0^+.$$

Using (118), (57) and the fact that the Biot-Savart law in the plane commutes with translations and rotations, we infer

$$K_{\mathbb{R}^2}[w^\varepsilon] - \tilde{u}^\varepsilon \longrightarrow 0 \;\text{ in } L^\infty(0,T;L^2(\mathbb{R}^2)) \text{ as } \varepsilon \to 0^+. \tag{121}$$

• Now we use the fact that, since $p' < 2$,

$$H^\varepsilon \longrightarrow H \;\text{ in } L^{p'}_{loc}(\mathbb{R}^2) \;\text{ as } \varepsilon \to 0^+,$$

where H is given by (30), and where as usual we extend H^ε by 0 inside $\mathcal{F}_0^\varepsilon$, see for instance [13, Lemma 3.11]. Since H is invariant by rotation, it follows from an easy change of variable that

$$Q^{\varepsilon_n} H^{\varepsilon_n}((Q^{\varepsilon_n})^T(\cdot - h^{\varepsilon_n}(t))) \longrightarrow H(\cdot - h(t)) \;\text{ in } L^{p'}_{loc}(\mathbb{R}^2) \;\text{ as } n \to +\infty, \tag{122}$$

no matter the rotation matrix Q^{ε_n}.

• Using (117), (121) and (122), we finally deduce that

$$u^{\varepsilon_n}(x) \longrightarrow K_{\mathbb{R}^2}[w] + \gamma H(\cdot - h(t)) \;\text{ in } L^\infty(0,T;L^{p'}_{loc}(\mathbb{R}^2)) \;\text{ as } n \to +\infty. \tag{123}$$

2. *Fluid equation in the limit.*
Let us show that u and w satisfy (22). Since u^ε and w^ε satisfy (115), it can be easily seen that for any test function $\psi \in C_c^\infty([0,T)\times\mathbb{R}^2)$,

$$\int_0^\infty \int_{\mathbb{R}^2} \psi_t w^{\varepsilon_n} \, dx\, dt + \int_0^\infty \int_{\mathbb{R}^2} \nabla\psi \cdot u^{\varepsilon_n} w^{\varepsilon_n} \, dx\, dt + \int_{\mathbb{R}^2} \psi(0,x) w_0(x)\, dx = 0. \tag{124}$$

The convergence as $n \to +\infty$ of the first term of (124) is a direct consequence of (114). For what concerns the second one, it is a matter of weak/strong convergence since u^{ε_n} converges strongly in $L^\infty(0,T;L^{p'}_{loc}(\mathbb{R}^2))$ (according to (122) and (123)) while for w^ε we have (114).

6.3 Characterization of the limit of the solid velocity

We will use the following lemmata, proven in the appendix.

Lemma 7. *Let $(\rho_n)_{n\in\mathbb{N}} \in W^{1,\infty}(0,T)^{\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}} \in (\mathbb{R}_*^+)^{\mathbb{N}}$ such that $(\varepsilon_n\rho_n)$ is bounded in $L^\infty(0,T)$ and*

$$\varepsilon_n \longrightarrow 0 \quad as \quad n \to +\infty. \tag{125}$$

Let

$$\alpha_n(t) := \int_0^t \rho_n. \tag{126}$$

Then

$$\varepsilon_n\rho_n \exp(i\alpha_n) \xrightarrow{w*} 0 \quad in \quad L^\infty(0,T) \quad as \quad n \to +\infty. \tag{127}$$

Lemma 8. *Let $(\rho_n)_{n\in\mathbb{N}} \in W^{1,\infty}(0,T)^{\mathbb{N}}$ and $(\varepsilon_n)_{n\in\mathbb{N}} \in (\mathbb{R}_*^+)^{\mathbb{N}}$ satisfying* (125)*, and let α_n be defined by* (126)*. Let $(w_n)_{n\in\mathbb{N}} \in L^\infty(0,T)^{\mathbb{N}}$ such that*

$$w_n \longrightarrow w \quad in \quad L^\infty(0,T) \quad as \quad n \to +\infty, \tag{128}$$

and suppose that

$$\varepsilon_n\rho_n \xrightarrow{w*} \overline{\rho} \quad in \quad W^{1,\infty}(0,T) \quad as \quad n \to +\infty, \tag{129}$$

and

$$\varepsilon_n\rho_n'(t) = Re[w_n(t)\exp(-i\alpha_n(t))]. \tag{130}$$

Then $\overline{\rho}$ is constant on $[0,T]$.

Now let us establish the behavior of the solid in the limit with the help of these lemmas. First, using (112), (117) and the uniform estimates on $K_{\mathbb{R}^2}[w^\varepsilon]$ in $C^{1-2/p}(\mathbb{R}^2)$ if $p < +\infty$ (resp. $\mathcal{LL}(\mathbb{R}^2)$ if $p = +\infty$) given by Proposition 6, we deduce

$$(h^{\varepsilon_n})' - K_{\mathbb{R}^2}[w^\varepsilon](t,h^{\varepsilon_n}) \longrightarrow h' - K_{\mathbb{R}^2}[w](t,h) \quad \text{in} \quad L^\infty(0,T) \quad \text{as} \quad n \to +\infty. \tag{131}$$

Now, we rephrase (111) with the complex variable:

$$\mathcal{J}_0(\varepsilon_n r^{\varepsilon_n})' = \text{Re}\Big\{\gamma\overline{\zeta}\exp(-i\theta^{\varepsilon_n})\Big[((h^{\varepsilon_n})' - K_{\mathbb{R}^2}[w^\varepsilon](t,h^{\varepsilon_n}))_1 + i((h^{\varepsilon_n})' - K_{\mathbb{R}^2}[w^\varepsilon](t,h^{\varepsilon_n}))_2\Big]\Big\} + \tilde{R}_2^{\varepsilon_n}.$$

We apply Lemma 8 with $\rho_n = r^{\varepsilon_n}$, $\alpha_n = \theta^{\varepsilon_n}$ and

$$w_n := \gamma\overline{\zeta}\Big[((h^{\varepsilon_n})' - K_{\mathbb{R}^2}[w^\varepsilon](t,h^{\varepsilon_n}))_1 + i((h^{\varepsilon_n})' - K_{\mathbb{R}^2}[w^\varepsilon](t,h^{\varepsilon_n}))_2\Big] + \exp(i\theta^{\varepsilon_n})\tilde{R}_2^{\varepsilon_n}.$$

The assumption on w_n comes directly from (109) and (131). We deduce that the function R defined in (113) is constant. Taking into account that the initial data r_0 is independent of ε we therefore deduce that $\varepsilon_n\theta^{\varepsilon_n}$ converges to 0 weakly-$*$ in $W^{2,\infty}(0,T;\mathbb{R}^2)$.

We apply Lemma 7 on (110) to get rid of the second term in the right hand side and we arrive to (19).

Remark 8. *Actually, we do not need to apply Lemma 7 since we know that $\varepsilon r^\varepsilon$ converges to 0 weakly-$*$ in $W^{1,\infty}(0,T)$. It can be noted however that using this lemma, Theorem 2 can be extended in a straightforward manner to the situation where r_0 depends on ε as follows:*

$$\varepsilon r_0^\varepsilon \longrightarrow R_0 \;\; as \;\; \varepsilon \to 0^+.$$

In that case, one deduces that $\varepsilon_n r^{\varepsilon_n}$ converges to R_0 weakly-$*$ in $W^{1,\infty}(0,T)$, but due to Lemma 7, no additional term appears in (19).

7 Technical results

7.1 Proof of Proposition 4

This is proven in [9]; we recall it here for the sake of completeness. As we consider ε fixed here, we omit the ε in the notations. In particular, here H stands for H^ε and G for G^ε.

1. We first give another form of the above Hamiltonian. Let us prove that

$$2\mathcal{H} = m|\ell(t)|^2 + \mathcal{J}r(t)^2 + \int_{\mathcal{F}_0} (|\hat{v}(t,\cdot)|^2 + 2(\gamma+\alpha)\hat{v}(t,\cdot)\cdot H)\,dx, \tag{132}$$

where α is given by (46) and

$$\hat{v} := v - (\gamma+\alpha)H. \tag{133}$$

Note in particular that

$$\hat{v}(x) = \mathcal{O}(1/|x|^2) \ \text{ as } \ |x| \to +\infty. \tag{134}$$

Let us denote

$$\Psi(x) := \int_{\mathcal{F}_0} G(x,y)\omega(y)dy$$

which is a stream function of $K[\omega]$ vanishing on the boundary $\mathcal{S}_0$:

$$K[\omega] = \nabla^\perp \Psi.$$

Let us also denote

$$\nabla\Phi := \ell_1\nabla\Phi_1 + \ell_2\nabla\Phi_2 + r\nabla\Phi_3,$$

so that

$$\hat{v} = K[\omega] + \nabla\Phi. \tag{135}$$

Then we compute

$$\int_{\mathcal{F}_0} |\hat{v}|^2 dx = \int_{\mathcal{F}_0} \nabla^\perp\Psi\cdot\hat{v} + \int_{\mathcal{F}_0} \nabla^\perp\Psi\cdot\nabla\Phi + \int_{\mathcal{F}_0} \nabla\Phi\cdot\nabla\Phi.$$

First, integrating by parts yields

$$\int_{\mathcal{F}_0} \nabla^\perp\Psi\cdot\hat{v} = -\int_{\mathcal{F}_0\times\mathcal{F}_0} G(x,y)\omega(x)\omega(y)\,dx\,dy,$$

$$\int_{\mathcal{F}_0} \nabla^\perp\Psi\cdot\nabla\Phi = 0,$$

$$\int_{\mathcal{F}_0} \hat{v}\cdot H = -\int_{\mathcal{F}_0} \omega(x)\Psi_H(x)dx.$$

There is no boundary terms since Ψ and Ψ_H vanish on the boundary $\mathcal{S}_0$, and $\nabla\Phi$ and $\hat{v}$ decrease also like $1/|x|^2$ at infinity.

Also, by definition, we have

$$\int_{\mathcal{F}_0} \nabla\Phi\cdot\nabla\Phi = X^T\mathcal{M}_2X.$$

This proves (132) by using (48).

2. We use (50), the fact that $\partial_t v = \partial_t \hat{v} \in L^\infty(0,T;L^q(\mathcal{F}_0))$ for any q in $(1,p]$ (resp. in $(1,+\infty)$ if $p=+\infty$) and we notice that H and v are in $L^\infty(0,T;L^p(\mathcal{F}_0))$; this allows to write

$$\begin{aligned}\mathcal{H}'(t) &= m\ell\cdot\ell'(t) + \mathcal{J}rr'(t) + \int_{\mathcal{F}_0}(\partial_t\hat{v}\cdot\hat{v} + (\gamma+\alpha)\partial_t\hat{v}\cdot H),\\ &= m\ell\cdot\ell'(t) + \mathcal{J}rr'(t) + \int_{\mathcal{F}_0}\partial_t v\cdot v,\\ &= m\ell\cdot\ell'(t) + \mathcal{J}rr'(t) - \int_{\mathcal{F}_0}(\left[(v-\ell-rx^\perp)\cdot\nabla\right]v + rv^\perp + \nabla q)\cdot v.\end{aligned}$$

Then

$$\mathcal{H}'(t) = I_1 + I_2 + I_3,$$

where

$$I_1 := m\ell\cdot\ell'(t) + \mathcal{J}rr'(t) - \int_{\mathcal{F}_0}\nabla q\cdot v, \quad I_2 := -\int_{\mathcal{F}_0}(v-\ell)\cdot\nabla v\cdot v, \quad I_3 := -r\int_{\mathcal{F}_0}[v^\perp - (x^\perp\cdot\nabla)v]\cdot v.$$

Let us justify that each integral above is convergent. For I_2 the integrability is clear. For I_3, we write $I_3 = I_4+I_5+I_6$, with

$$I_4 := -r\int_{\mathcal{F}_0}\hat{v}^\perp\cdot v, \quad I_5 := r\int_{\mathcal{F}_0}x^\perp\cdot\nabla\hat{v}\cdot v, \quad I_6 := (\gamma+\alpha)r\int_{\mathcal{F}_0}(x^\perp\cdot\nabla H - H^\perp)\cdot v.$$

The integrability of I_4 is clear, for I_5 we use that $\hat{v}$ decreases like $1/|x|^2$ at infinity, so that $x^\perp\cdot\nabla\hat{v}$ is integrable and for I_6 we use (36). It remains to justify the integral in I_1. We analyze the decay of the pressure at infinity, observing that (50) reads as follows:

$$-\nabla q = \partial_t\hat{v} + (v-\ell)\cdot\nabla v - rx^\perp\cdot\nabla\hat{v} + r\hat{v}^\perp + (\alpha+\gamma)r[H^\perp - (x^\perp\cdot\nabla)H], \tag{136}$$

so that ∇q decreases like $1/|x|^2$ at infinity. Integrating along rays yields that the pressure decreases like $1/|x|$ at infinity. This allows to integrate by parts both I_1 and I_2.

Now using that (52) and then the equations (53)–(54), we obtain that $I_1 = 0$. For what concerns I_2 we get that

$$I_2 = -\frac{1}{2}\int_{\partial\mathcal{S}_0}|v|^2(v-\ell)\cdot n.$$

For what concerns I_3, we consider $R>0$ large in order that $\mathcal{S}_0\subset B(0,R)$, and consider the same integral as I_3, over $\mathcal{F}_0\cap B(0,R)$. Integrating by parts we obtain

$$\int_{\mathcal{F}_0\cap B(0,R)}[v^\perp - (x^\perp\cdot\nabla)v]\cdot v = -\int_{\partial\mathcal{S}_0}(x^\perp\cdot n)\frac{|v|^2}{2} - \int_{S(0,R)}(x^\perp\cdot n)\frac{|v|^2}{2},$$

where we denote by n also the unit outward normal on the circle $S(0,R)$. Of course $x^\perp\cdot n = 0$ on $S(0,R)$, so letting $R\to+\infty$, we end up with

$$I_3 = \frac{1}{2}\int_{\partial\mathcal{S}_0}(rx^\perp\cdot n)|v|^2.$$

Using (52) we deduce $I_2 + I_3 = 0$, so in total we get $\mathcal{H}'(t) = 0$.

7.2 Proofs of Lemmas 5 and 6

Proof of Lemma 5. By polarization, it is sufficient to consider the case where $f = g$. Let us consider $\gamma = (\gamma_1,\gamma_2)$: $[0,1]\to\mathbb{R}^2$ a smooth parameterization of the Jordan curve $\mathcal{C}$. On one side, one has

$$\int_{\mathcal{C}}(f\cdot f)n\,ds = \int_0^1\left(f_1(\gamma(t))^2 + f_2(\gamma(t))^2\right)\begin{pmatrix}-\gamma_2'(t)\\ \gamma_1'(t)\end{pmatrix}dt. \tag{137}$$

On the other side, one has

$$\int_{\mathcal{C}} (f_1(z) - if_2(z))^2\,dz = \int_{\mathcal{C}} \Big(f_1(\gamma(t)) - if_2(\gamma(t))\Big)\Big[\big(f_1(\gamma(t)) - if_2(\gamma(t))\big)(\gamma_1'(t) + i\gamma_2'(t))\Big]\,dt. \tag{138}$$

But since f is tangent to $\mathcal{C}$, one sees that the expression inside the brackets in (138) is real, and hence is equal to its complex conjugate. It follows that

$$\int_{\mathcal{C}} (f_1(z) - if_2(z))^2\,dz = \int_{\mathcal{C}} \big|f_1(\gamma(t)) - if_2(\gamma(t))\big|^2 (\gamma_1'(t) - i\gamma_2'(t))\,dt,$$

and (94) follows.

The proof of (95) is analogous: using again

$$(f_1(\gamma(t)) - if_2(\gamma(t)))(\gamma_1'(t) + i\gamma_2'(t)) = (f_1(\gamma(t)) + if_2(\gamma(t)))(\gamma_1'(t) - i\gamma_2'(t)),$$

we deduce

$$\int_{\mathcal{C}} (f_1(z) - if_2(z))^2 z\,dz = \int_{\mathcal{C}} |f_1(\gamma(t)) - if_2(\gamma(t))|^2 (\gamma_1(t) + i\gamma_2(t))(\gamma_1'(t) - i\gamma_2'(t))\,dt,$$

so that

$$\begin{aligned} \mathrm{Re}\left(\int_{\mathcal{C}} (f_1(z) - if_2(z))^2 z\,dz\right) &= \int_{\mathcal{C}} \Big(f_1(\gamma(t))^2 + f_2(\gamma(t))^2\Big)\big(\gamma_1(t)\gamma_1'(t) + \gamma_2(t)\gamma_2'(t)\big)\,dt \\ &= \int_{\mathcal{C}} (f\cdot f)(x^\perp \cdot n)\,ds. \end{aligned}$$

□

Proof of Lemma 6. Parameterizing $\partial\mathcal{S}_0$ by $\gamma = (\gamma_1,\gamma_2) : [0,1] \to \mathbb{R}^2$ as previously we have

$$\int_{\partial\mathcal{S}_0} \overline{z}(H_1 - iH_2)z\,dz = \int_{\partial\mathcal{S}_0} (\gamma_1^2(t) + \gamma_2^2(t))\Big[(H_1(\gamma(t)) - iH_2(\gamma(t)))(\gamma_1'(t) + i\gamma_2'(t))\Big]\,dt.$$

Since H is tangent to $\partial\mathcal{S}_0$, the imaginary part of the bracket above is zero, so the integral is real. □

7.3 Proofs of Lemmas 7 and 8

Proof of Lemma 7. We first note that the sequence $(\varepsilon_n\rho_n\exp(i\alpha_n))_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T)$, so it suffices to prove that the convergence (127) takes place in the sense of distributions. Given $\varphi \in C_0^\infty((0,T))$, we see that

$$\begin{aligned} \int_a^b \varepsilon_n\rho_n(t)\exp(i\alpha_n(t))\varphi(t)\,dt &= -i\varepsilon_n \int_a^b i\alpha_n'(t)\exp(i\alpha_n(t))\varphi(t)\,dt \\ &= i\varepsilon_n \int_a^b \exp(i\alpha_n(t))\varphi'(t)\,dt, \end{aligned}$$

which yields the desired convergence. □

Proof of Lemma 8. Let us define the following open subset of $(0,T)$:

$$A := \{x \in (0,T) \;/\; \overline{\rho}(x) \neq 0\}. \tag{139}$$

• It is classical that, by the Lipschitz character of the function $\overline{\rho}$ only, one has

$$\overline{\rho}' = 0 \;\text{ a. e. on }\; (0,T)\setminus A.$$

• Now let us consider what happens on A. Define

$$\Theta(t) := \int_0^t \overline{\rho}.$$

Consider a connected component of A, say (a, b). Let us show that, due to the nonstationary phase $\Theta' \neq 0$ in (a, b), one has

$$\exp(-i\alpha_n) \stackrel{w*}{\rightharpoonup} 0 \text{ in } L^\infty(a,b) \text{ as } n \to +\infty. \tag{140}$$

Of course, it is sufficient to prove this convergence in the sense of $\mathcal{D}'(a,b)$. Hence, let us consider $\varphi \in C_0^\infty((a,b))$, say $\mathrm{Supp}(\varphi) \subset [\tilde{a}, \tilde{b}] \subset (a,b)$. Note that by (126) and (129), one has

$$\varepsilon_n \alpha_n \stackrel{w*}{\rightharpoonup} \Theta \text{ in } W^{2,\infty}([0,T]).$$

Consequently there exists $\kappa > 0$ such that for all n large enough, one has

$$\varepsilon_n |\alpha_n'(t)| \geqslant \kappa > 0 \text{ and } \varepsilon_n |\alpha_n'(t)| + \varepsilon_n |\alpha_n''(t)| \leqslant \kappa^{-1} \text{ on } [\tilde{a}, \tilde{b}].$$

For such n one has

$$\begin{aligned}
\int_a^b \exp(-i\alpha_n(t))\varphi(t)\,dt &= \int_{\tilde{a}}^{\tilde{b}} \alpha_n'(t)\exp(-i\alpha_n(t))\frac{\varphi(t)}{\alpha_n'(t)}\,dt \\
&= -i\int_{\tilde{a}}^{\tilde{b}} \exp(-i\alpha_n(t))\frac{\varphi'(t)\alpha_n'(t) - \varphi(t)\alpha_n''(t)}{(\alpha_n')^2(t)}\,dt \\
&= -i\varepsilon_n\int_{\tilde{a}}^{\tilde{b}} \exp(-i\alpha_n(t))\frac{\varphi'(t)(\varepsilon_n\alpha_n)'(t) - \varphi(t)(\varepsilon_n\alpha_n)''(t)}{(\varepsilon_n\alpha_n')^2(t)}\,dt,
\end{aligned}$$

and (140) follows.

Hence, by weak/strong convergence we deduce from (128), (130) and (140) that for any $\varphi \in C_0^\infty((a,b);\mathbb{R})$,

$$\varepsilon_n \langle \rho_n', \varphi \rangle_{L^\infty \times L^1} = \mathrm{Re}\big(\langle \exp(-i\alpha_n), w_n\varphi\rangle_{L^\infty\times L^1}\big) \longrightarrow 0 \text{ as } n \to +\infty.$$

Consequently, on each connected component (a, b) of A, we obtain that

$$\overline{\rho}' = 0 \text{ a. e. on } (a,b).$$

Of course there is at most a countable quantity of such connected components. Hence we obtain that $\overline{\rho}' = 0$ a.e. on $(0, T)$ and the conclusion follows. □

8 Appendix. Proof of Theorem 1

We first prove a result of global in time existence and uniqueness similar to the celebrated result by Yudovich about a fluid alone. We recall that the space $\mathcal{LL}$ was defined in (15).

Theorem 3. *For any $u_0 \in C^0(\overline{\mathcal{F}_0};\mathbb{R}^2)$, $(\ell_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}$, such that:*

$$\operatorname{div} u_0 = 0 \text{ in } \mathcal{F}_0 \text{ and } u_0 \cdot n = (\ell_0 + r_0 x^\perp)\cdot n \text{ on } \partial\mathcal{S}_0, \tag{141}$$

$$w_0 := \operatorname{curl} u_0 \in L_c^\infty(\overline{\mathcal{F}_0}), \tag{142}$$

$$\lim_{|x|\to+\infty} u_0(x) = 0,$$

there exists a unique solution (h', r, u) of (1)–(8) in $C^1(\mathbb{R}^+;\mathbb{R}^2\times\mathbb{R}) \times L^\infty(\mathbb{R}^+, \mathcal{LL}(\mathcal{F}(t))$. Moreover for all $t > 0$, $w(t) := \operatorname{curl} u(t) \in L_c^\infty(\overline{\mathcal{F}(t)})$.

We first prove the local in time existence part by Schauder's fixed point theorem. The global in time existence follows then from our a priori estimates of Section 4. Finally we follow Yudovich's approach for what concerns the uniqueness.

8.1 Proof of Theorem 3

Reformulating the problem. To begin with, we consider the equations in the body frame as in Subsection 3.1. Next, we decompose the pressure. To this purpose we recall the following result from [18] about the Leray projector on $\mathcal{F}_0$:

Lemma 9. *For any $q \in (1, \infty)$, we have*

$$L^q(\mathcal{F}_0; \mathbb{R}^2) = L^q_\sigma(\mathcal{F}_0) \oplus E^q(\mathcal{F}_0),$$

where

$$L^q_\sigma(\mathcal{F}_0) := \{u \in L^q(\mathcal{F}_0; \mathbb{R}^2) /\ \operatorname{div} u = 0 \text{ in } \mathcal{F}_0 \text{ and } u \cdot n = 0 \text{ on } \partial \mathcal{S}_0\},$$
$$E^q(\mathcal{F}_0) := \{\nabla p \ / \ p \in L^q_{loc}(\overline{\mathcal{F}_0}; \mathbb{R}) \text{ and } \nabla p \in L^q(\mathcal{F}_0; \mathbb{R}^2)\}.$$

Moreover the projection P_q from $L^q(\mathcal{F}_0; \mathbb{R}^2)$ onto $L^q_\sigma(\mathcal{F}_0)$ along $E^q(\mathcal{F}_0)$ is linear continuous.

Since, by density of smooth compactly supported vector fields in $L^q(\mathcal{F}_0; \mathbb{R}^2)$, P_q and $P_{\tilde{q}}$ coincide on $L^q(\mathcal{F}_0; \mathbb{R}^2) \cap L^{\tilde{q}}(\mathcal{F}_0; \mathbb{R}^2)$, we will simply denote P without dwelling.

This allows to reformulate the solid equation as follows. We introduce μ up to an additive constant by

$$\nabla \mu := -(Id - P)\Big((v - \ell - r x^\perp) \cdot \nabla v + r v^\perp\Big). \tag{143}$$

We will see that in the case under view this is well defined in $L^\infty_{loc}(\mathbb{R}^+; L^p(\mathcal{F}_0))$. Then, assuming that the solution has the regularity claimed in Theorem 1, we observe that

$$\partial_t v - \begin{bmatrix} \ell \\ r \end{bmatrix}' \cdot (\nabla \Phi_i)_{i=1,2,3} \in L^p_\sigma(\mathcal{F}_0).$$

We deduce that

$$\begin{bmatrix} \ell \\ r \end{bmatrix}' \cdot (\nabla \Phi_i)_{i=1,2,3} = (Id - P)\partial_t v,$$

where the functions Φ_i are defined in (37)-(39). We obtain that the pressure q can be decomposed into

$$\nabla q = \nabla \mu - \begin{bmatrix} \ell \\ r \end{bmatrix}' \cdot (\nabla \Phi_i)_{i=1,2,3}. \tag{144}$$

It is now elementary to see that the equation of the solid reads

$$\mathcal{M} \begin{pmatrix} \ell^1 \\ \ell^2 \\ r \end{pmatrix}' = \left(\int_{\mathcal{F}_0} \nabla \mu(\tau, x) \cdot \nabla \Phi_i(x)\, dx \right)_{i=1,2,3} - m r \begin{pmatrix} -\ell^2 \\ \ell^1 \\ 0 \end{pmatrix}, \tag{145}$$

where $\mathcal{M}$ is given in (64).

An operator. Let us introduce an operator, whose fixed points give local in time solutions to the system. We denote

$$\overline{\rho} := \min\{\rho > 0 \ / \ \operatorname{Supp}(w_0) \subset \overline{B}(0, \rho)\},$$

and as before we denote γ the circulation of u_0 around $\partial\mathcal{S}_0$. For $T>0$, we let

$$\mathcal{C} := \Big\{(\omega,\ell,r) \in L^\infty(0,T;L^\infty_c(\mathcal{F}_0)) \times W^{1,1}([0,T];\mathbb{R}^3) \ / \\ i.\ \ \|\omega\|_{L^\infty(0,T;L^1(\mathcal{F}_0))} \leqslant \|w_0\|_{L^1(\mathcal{F}_0)},\ \|\omega\|_{L^\infty(0,T;L^\infty(\mathcal{F}_0))} \leqslant \|w_0\|_{L^\infty(\mathcal{F}_0)}, \\ \text{and} \int_{\mathcal{F}_0} w(t,x)\,dx = \int_{\mathcal{F}_0} w_0(x)\,dx,\ \forall t\in[0,T], \\ ii.\ \ \mathrm{Supp}\,(\omega(t)) \subset \overline{B}(0,\overline{\rho}+1), \\ iii.\ \ \partial_t\omega \in L^1(0,T;W^{-1,3}(\mathcal{F}_0)) \ \text{ and } \ \|\partial_t\omega\|_{L^1(0,T;W^{-1,3}(\mathcal{F}_0))} \leqslant 1, \\ iv.\ \ \|\ell-\ell_0\|_{W^{1,1}(0,T)} \leqslant 1,\ \|r-r_0\|_{W^{1,1}(0,T)} \leqslant 1\Big\}.$$

Now we define the operator $\mathcal{V}:\mathcal{C}\to\mathcal{C}$, mapping (ω,ℓ,r) to $(\tilde\omega,\tilde\ell,\tilde r)$ as follows.

We first introduce v as the solution of (44) associated to (ω,ℓ,r) and γ. As is classical (see Proposition 1), the resulting v is bounded and log-Lipschitz uniformly in time with

$$\|v\|_{L^\infty(0,T;\mathcal{LL}(\mathcal{F}_0))} \leqslant C(\mathcal{S}_0)(\|\omega\|_{L^\infty(0,T;L^1(\mathcal{F}_0))} + \|\omega\|_{L^\infty(0,T;L^\infty(\mathcal{F}_0))} + \|\ell\|_{L^\infty(0,T)} + \|r\|_{L^\infty(0,T)} + |\gamma|), \tag{146}$$

and it also satisfies (see e.g. [8]) for $p\in(1,+\infty)$:

$$\|\nabla v\|_{L^\infty(0,T;L^p(\mathcal{F}_0))} \leqslant C(\mathcal{S}_0)\,\frac{p^2}{p-1}(\|\omega\|_{L^\infty(0,T;L^1(\mathcal{F}_0))} + \|\omega\|_{L^\infty(0,T;L^\infty(\mathcal{F}_0))} + \|\ell\|_{L^\infty(0,T)} + \|r\|_{L^\infty(0,T)} + |\gamma|). \tag{147}$$

As a consequence of (146), we have also a uniform log-Lipschitz estimate on $v-\ell-rx^\perp$, so one can define a unique flow associated to $v-\ell-rx^\perp$, that is Φ such that

$$\partial_t\Phi(t,s,x) = (v-\ell-rx^\perp)(t,\Phi(t,s,x)) \ \text{ and } \ \Phi(s,s,x)=x \ \text{ for } (t,s,x)\in[0,T]^2\times\mathcal{F}_0.$$

Then the ω-part of $\mathcal{V}$ is defined as

$$\tilde\omega(t,x) := w_0(\Phi(0,t,x)).$$

It satisfies

$$\partial_t\tilde\omega + \mathrm{div}((v-\ell-rx^\perp)\tilde\omega) = 0. \tag{148}$$

Next we introduce μ as

$$\nabla\mu := -(Id-P)\Big((v-\ell-rx^\perp)\cdot\nabla v + rv^\perp\Big). \tag{149}$$

Let us justify that $\nabla\mu$ is well defined in $L^\infty(0,T;L^p(\mathcal{F}_0))$, $p>2$. Due to Lemma 9 one has only to check that $(v-\ell-rx^\perp)\cdot\nabla v + rv^\perp \in L^\infty(0,T;L^p(\mathcal{F}_0))$. For what concerns the part $(v-\ell)\cdot\nabla v$, this comes directly from (146) and (147). For what concerns $r(x^\perp\cdot\nabla)v$ and $rv^\perp$ we use the fact that uniformly in t,

$$v = \mathcal{O}\left(\frac{1}{|x|}\right) \ \text{ and } \nabla v = \mathcal{O}\left(\frac{1}{|x|^2}\right) \ \text{ as } x\to+\infty. \tag{150}$$

To prove (150), we recall that v is harmonic for $|x|$ large and tends to zero at infinity, and use the following estimate for its (convergent) Laurent series development: for $R>0$ large enough, one has

$$\Big\|z\sum_{k\geqslant1}\frac{a_k}{z^k}\Big\|_{L^\infty(\mathbb{C}\setminus B(0,R))} \leqslant \max\left(|a_1|, \Big\|z\sum_{k\geqslant1}\frac{a_k}{z^k}\Big\|_{L^\infty(S(0,R))}\right) \leqslant C\Big\|\sum_{k\geqslant1}\frac{a_k}{z^k}\Big\|_{L^\infty(S(0,R))},$$

as follows by the maximum principle and Cauchy's Residue Theorem. The conclusion on v follows then from the $L^\infty(\mathcal{F}_0)$ part of the estimate (146). We proceed analogously on ∇v, adding the classical interior elliptic regularity estimate:

$$\Big\|\sum_{k\geqslant1}\frac{b_k}{z^k}\Big\|_{L^\infty(S(0,R))} \leqslant C\Big\|\sum_{k\geqslant1}\frac{b_k}{z^k}\Big\|_{L^p(B(0,R+1)\setminus B(0,R-1))},$$

and using (147). The estimate (150) follows.

Now we can define $(\tilde{\ell},\tilde{r})$ as follows:

$$\begin{pmatrix}\tilde{\ell}^1\\ \tilde{\ell}^2\\ \tilde{r}\end{pmatrix}(t)=\begin{pmatrix}\ell_0^1\\ \ell_0^2\\ r_0\end{pmatrix}+\mathcal{M}^{-1}\int_0^t\left[\left(\int_{\mathcal{F}_0}\nabla\mu(\tau,x)\cdot\nabla\Phi_i(x)\,dx\right)_{i=1,2,3}-mr\begin{pmatrix}-\ell^2\\ \ell^1\\ 0\end{pmatrix}\right]d\tau. \tag{151}$$

Existence of a solution via a fixed point. Let us now prove that for $T>0$ suitably small, $\mathcal{V}$ admits a fixed point in $\mathcal{C}$. We endow $\mathcal{C}$ with the $L^\infty(0,T;L^p(\mathcal{F}_0))\times C^0([0,T];\mathbb{R}^3)$ topology (for some $p\in(2,+\infty)$) and use Schauder's fixed point theorem. It is clear that $\mathcal{C}$ is a closed convex subset of $L^\infty(0,T;L^p(\mathcal{F}_0))\times C^0([0,T];\mathbb{R}^3)$. Hence it remains to prove that $\mathcal{V}(\mathcal{C})\subset\mathcal{C}$, that $\mathcal{V}(\mathcal{C})$ is relatively compact and that $\mathcal{V}$ is continuous.

• Let $(\omega,\ell,r)\in\mathcal{C}$. That $\tilde{\omega}$ satisfies point *i.* in the definition of $\mathcal{C}$ is immediate. Due to (146), we see that

$$\|v-\ell\|_\infty\leqslant C(\mathcal{S}_0)(\|w_0\|_1+\|w_0\|_\infty+|\ell_0|+|r_0|+|\gamma|+1),$$

so that $\operatorname{Supp}(\tilde{\omega}(t))\subset\overline{B}(0,\overline{\rho}+1)$ is granted for T small. For what concerns point *iii.* in the definition of $\mathcal{C}$, we simply use (148) and the estimates on v, ℓ, r and $\operatorname{Supp}(\tilde{\omega})$ to see that it is satisfied by $\tilde{\omega}$ for T suitably small.

Due to (146), (147), (149) and Lemma 9, we deduce that

$$\|\nabla\mu\|_{L^p(\mathcal{F}_0)}\leqslant C(\mathcal{S}_0)(\|w_0\|_1+\|w_0\|_\infty+|\ell_0|+|r_0|+|\gamma|+1)^2. \tag{152}$$

Hence with (151) we obtain easily that for T suitably small, $(\tilde{\ell},\tilde{r})$ satisfies the estimates in the definition of $\mathcal{C}$. Hence we have $\mathcal{V}(\mathcal{C})\subset\mathcal{C}$ for T small.

• Let us now prove that $\mathcal{V}(\mathcal{C})$ is relatively compact. Let us consider $(\tilde{\omega}_n,\tilde{\ell}_n,\tilde{r}_n)$ a sequence in $\mathcal{V}(\mathcal{C})$, let us say $(\tilde{\omega}_n,\tilde{\ell}_n,\tilde{r}_n)=\mathcal{V}(\omega_n,\ell_n,r_n)$. Call v_n the velocity field associated to (ω_n,ℓ_n,r_n) by (44), and Φ_n the corresponding flow. Using the definition of $\mathcal{C}$, (147) and Aubin-Lions' lemma, one deduces that (v_n) is relatively compact in $L^\infty_{loc}([0,T]\times\overline{\mathcal{F}_0})$. Let us say that $v_{\varphi(n)}$ converges uniformly to v. Due to the uniform log-Lipschitz estimates on v, we infer that $\Phi_{\varphi(n)}$ converges uniformly towards the flow Φ associated to v on $[0,T]^2\times\overline{B}(0,\overline{\rho}+1)$. This involves that

$$\tilde{\omega}_n\longrightarrow\tilde{\omega}(t,x):=w_0(\Phi(0,t,x))\ \text{ in } L^\infty(0,T;L^p(\mathcal{F}_0)).$$

(This convergence can for instance be established by using the density of $C_0^\infty(\mathcal{F}_0)$ in $L^p(\mathcal{F}_0)$.) The compactness for $(\tilde{\ell},\tilde{r})$ is straightforward.

• We finally prove the continuity of $\mathcal{V}$. Suppose that (ω_n,ℓ_n,r_n) converges to (ω,ℓ,r) in $L^\infty(0,T;L^p(\mathcal{F}_0))\times C^0([0,T];\mathbb{R}^3)$. Associate to them v_n, Φ_n, etc. and v, Φ, etc. Then as previously one deduces that v_n converges to v uniformly and that ∇v_n converges to ∇v in $L^\infty(0,T;L^p(\mathcal{F}_0))$. We deduce consequently that Φ_n converges to Φ uniformly, so in the same way as above, $\tilde{\omega}_n$ converges to $\tilde{\omega}$ in $L^\infty(0,T;L^p(\mathcal{F}_0))$. Also, using the fact that the estimates (150) are uniform in t and n, we see that $\nabla\mu_n$ converges to $\nabla\mu$ in $L^\infty(0,T;L^p(\mathcal{F}_0))$. We deduce that (ℓ_n,r_n) converges uniformly to (ℓ,r).

• Hence for T small we obtain a fixed point. One deduces that this yields a solution to the original system (1)-(8), going back to the original frame. Then the fact that a maximal solution is global comes from the a priori estimates on ω, and the estimates of Section 4 for what concerns $\operatorname{Supp}(\omega(t))$, ℓ and r.

Uniqueness. This relies on Yudovich's idea [19]. Suppose that we have two solutions (ℓ_1,r_1,v_1) and (ℓ_2,r_2,v_2) with the same initial data. (In this part of the proof the indices do not stand for the components.) In particular, they share the same circulation γ and initial vorticity w_0. As a consequence, despite the fact that v_1 and v_2 are not necessarily in $L^2(\mathcal{F}_0)$, their difference v_1-v_2 does belong to $L^\infty(0,T;L^2(\mathcal{F}_0))$ with

$$v_1-v_2=\mathcal{O}\left(\frac{1}{|x|^2}\right)\ \text{and}\ \nabla(v_1-v_2)=\mathcal{O}\left(\frac{1}{|x|^3}\right)\quad\text{as } |x|\to+\infty. \tag{153}$$

(Recall that both v_1 and v_2 are harmonic for $|x|$ large enough and converge to 0 at infinity).

Let us also remark that due to (152) and (145), ℓ_1, ℓ_2, r_1, r_2 belong to $W^{1,\infty}(0,T)$. As a consequence, using (144), we see that ∇q_1 and ∇q_2 belong to $L^\infty(0,T;L^2(\mathcal{F}_0))$.

Then defining $\breve{\ell} := \ell_1 - \ell_2$, $\breve{r} := r_1 - r_2$, $\breve{v} := v_1 - v_2$ and $\breve{q} = q_1 - q_2$, we deduce from (50) that

$$\frac{\partial \breve{v}}{\partial t} + \big[(v_1 - \ell_1 - r_1 x^\perp)\cdot\nabla\big]\,\breve{v} + \Big[(\breve{v} - \breve{\ell} - \breve{r}x^\perp)\cdot\nabla\Big]\,v_2 + r_1\breve{v}^\perp + \breve{r}v_2^\perp + \nabla\breve{q} = 0.$$

We multiply by $\breve{v}$, integrate over $\mathcal{F}_0$ and integrate by parts (which is permitted by (153) and by the regularity of the pressure), and deduce:

$$\frac{1}{2}\frac{d}{dt}\|\breve{v}\|_{L^2}^2 + \int_{\mathcal{F}_0} \breve{v}\cdot\Big[(\breve{v} - \breve{\ell} - \breve{r}x^\perp)\cdot\nabla v_2\Big]\,dx + \breve{r}\int_{\mathcal{F}_0}\breve{v}\cdot v_2^\perp\,dx + \int_{\partial\mathcal{F}_0}\breve{q}\breve{v}\cdot n = 0.$$

For what concerns the last term,

$$\begin{aligned}
\int_{\partial\mathcal{F}_0}\breve{q}\breve{v}\cdot n &= \breve{\ell}\cdot\int_{\partial\mathcal{F}_0}\breve{q}n + \breve{r}\int_{\partial\mathcal{F}_0}\breve{q}x^\perp\cdot n \\
&= m\breve{\ell}\cdot\big(\breve{\ell}' + \breve{r}\ell_1^\perp + r_2\breve{\ell}^\perp\big) + \mathcal{J}\breve{r}\breve{r}' \\
&= m\breve{r}\,\breve{\ell}\cdot\ell_1^\perp + m\breve{\ell}\cdot\breve{\ell}' + \mathcal{J}\breve{r}\breve{r}'.
\end{aligned}$$

Using (83), we deduce

$$(x^\perp\cdot\nabla)v_2 = \nabla(x^\perp\cdot v_2) - v_2^\perp - x^\perp\omega_2,$$

so that after integration by parts

$$\int_{\mathcal{F}_0}\breve{v}\cdot[(x^\perp\cdot\nabla)v_2]\,dx = \int_{\mathcal{S}_0}(x^\perp\cdot v_2)[(\breve{\ell}+\breve{r}x^\perp)\cdot n]\,ds + \int_{\mathcal{F}_0}\breve{v}\cdot(-v_2^\perp - x^\perp\omega_2)\,dx.$$

Hence using the boundedness of v_2 and ω_2 in $L^\infty(0,T;L^\infty(\mathcal{F}_0))$, the boundedness of ℓ^1 and the one of $\mathrm{Supp}\,(\omega_2)$, we arrive to

$$\frac{d}{dt}\big(\|\breve{v}\|_{L^2}^2 + \|\breve{\ell}\|^2 + \|\breve{r}\|^2\big) \leqslant C\Big(\|\breve{v}\|_{L^2}^2 + \|\breve{\ell}\|^2 + \|\breve{r}\|^2 + \|\nabla v_2\|_{L^p}\|\breve{v}^2\|_{L^{p'}}\Big),$$

for $p>2$. (Here, the various constants C may depend on $\mathcal{S}_0$ and on the solutions (ℓ_1,r_1,v_1) and (ℓ_2,r_2,v_2), but not on p.) Hence using (147), we obtain that for p large,

$$\begin{aligned}
\frac{d}{dt}\big(\|\breve{v}\|_{L^2}^2 + \|\breve{\ell}\|^2 + \|\breve{r}\|^2\big) &\leqslant C\Big(\|\breve{v}\|_{L^2}^2 + \|\breve{\ell}\|^2 + \|\breve{r}\|^2\Big) + \tilde{C}p\|\breve{v}^2\|_{L^{p'}} \\
&\leqslant C\Big(\|\breve{v}\|_{L^2}^2 + \|\breve{\ell}\|^2 + \|\breve{r}\|^2\Big) + \tilde{C}p\|\breve{v}\|_{L^2}^{\frac{2}{p'}}\|\breve{v}^2\|_{L^\infty}^{\frac{1}{p}}.
\end{aligned}$$

For some constant $K>0$, we have on $[0,T]$:

$$\|\breve{v}\|_{L^2}^2 + \|\breve{\ell}\|^2 + \|\breve{r}\|^2 \leqslant K,$$

so for some $C>0$ one has in particular

$$\frac{d}{dt}\big(\|\breve{v}\|_{L^2}^2 + \|\breve{\ell}\|^2 + \|\breve{r}\|^2\big) \leqslant Cp\Big(\|\breve{v}\|_{L^2}^2 + \|\breve{\ell}\|^2 + \|\breve{r}\|^2\Big)^{1/p'}.$$

Now the unique solution of $y' = Ny^\delta$ and $y(0) = \varepsilon > 0$ for $\delta\in(0,1)$ and $N>0$ is given by

$$y(t) = \Big[(1-\delta)Nt + \varepsilon^{1-\delta}\Big]^{\frac{1}{1-\delta}}.$$

Hence a comparison argument proves that

$$\|\breve{v}\|_{L^2}^2 + \|\breve{\ell}\|^2 + \|\breve{r}\|^2 \leqslant (Ct)^p.$$

We conclude that $\breve{v} = 0$ for $t < 1/C$ by letting $p\to+\infty$.

8.2 Proof of Theorem 1 (Existence)

We now briefly explain how to deduce Theorem 1 from Theorem 3, by a quite straightforward adaptation of the methods of [15].

Consider an initial data for the vorticity $w_0 \in L^p_c(\overline{\mathcal{F}_0})$ with $p \in (2, +\infty)$. Let us introduce a sequence $(w_0^n)_{n\in\mathbb{N}} \in (L^\infty_c(\overline{\mathcal{F}_0}))^{\mathbb{N}}$ converging to w_0 in $L^p(\mathcal{F}_0)$. Let us consider (ℓ_n, r_n, v_n) the corresponding solutions given by Theorem 3 in the body frame.

We use the conservation of the energy given in Proposition 4, of the L^p norm of the vorticity and of the circulation around the body an proceed as in Section 4 to deduce that for any $T > 0$:

$$\|\rho_{\omega_n}\|_{L^\infty(0,T)} + \|\ell_n\|_{L^\infty(0,T)} + \|r_n\|_{L^\infty(0,T)} + \|v_n\|_{L^\infty(0,T;W^{1,p}(\mathcal{F}_0))} \leqslant C, \tag{154}$$

where we used the notation (66). (As a matter of fact, the proof could be simpler, since we have a classical flow associated to v_n.)

Using the definition (143) of μ, Lemma 9, (150) and (154), we deduce that

$$\|\nabla\mu_n\|_{L^\infty(0,T;L^p(\mathcal{F}_0))} \leqslant C.$$

Going back to (145) and then to (144), we deduce that

$$\|\ell_n\|_{W^{1,\infty}(0,T)} + \|r_n\|_{W^{1,\infty}(0,T)} + \|\nabla q_n\|_{L^\infty(0,T;L^p(\mathcal{F}_0))} \leqslant C.$$

Now, using (50) and (150), we deduce that

$$\|\partial_t v_n\|_{L^\infty(0,T;L^p(\mathcal{F}_0))} \leqslant C.$$

Moreover, (50) can be written as follows:

$$-\partial_t\hat{v}_n = \nabla q_n + (v_n - \ell_n)\cdot\nabla v_n - r_n x^\perp\cdot\nabla\hat{v}_n + r_n\hat{v}_n^\perp + (\alpha+\gamma)r_n[H^\perp - (x^\perp\cdot\nabla)H],$$

where $\hat{v}_n := v_n - (\gamma+\alpha)H$, with α given by (46). Using (36), we infer that

$$\partial_t v_n = \partial_t\hat{v}_n = \mathcal{O}\left(\frac{1}{|x|^2}\right), \tag{155}$$

uniformly in n and in t, so that $\partial_t v_n$ is bounded in $L^\infty(0,T;L^q(\mathcal{F}_0))$ for any $q \in (1,p]$.

Hence by a straightforward compactness argument, we extract a sequence, that we still index by n such that

$$\ell_n \overset{w*}{\rightharpoonup} \ell,\ r_n \overset{w*}{\rightharpoonup} r \text{ in } W^{1,\infty}_{loc}(\mathbb{R}^+),\ \nabla q_n \overset{w*}{\rightharpoonup} \nabla q \text{ in } L^\infty_{loc}(\mathbb{R}^+;L^p(\mathcal{F}_0)), \text{ and } v_n \overset{w*}{\rightharpoonup} v \text{ in } L^\infty_{loc}(\mathbb{R}^+;W^{1,p}(\mathcal{F}_0)),$$

and, using Aubin-Lions' lemma, such that

$$v_n \longrightarrow v \text{ in } L^\infty_{loc}(\mathbb{R}^+\times\mathcal{F}_0).$$

This is enough to pass to the limit into the equation, in the sense of distributions.

Going back to the original variables, we get that

$$u \in L^\infty(0,T;W^{1,p}(\mathcal{F}(t))),\ \nabla p \in L^\infty(0,T;L^p(\mathcal{F}(t))),$$

satisfy (1). Using $(u\cdot\nabla)u = \operatorname{div}(u\otimes u)$, we deduce that $\partial_t u \in L^\infty(0,T;L^p(\mathcal{F}(t)))$, and, considering its behaviour at infinity, $\partial_t u \in L^\infty(0,T;L^q(\mathcal{F}(t)))$ for all $q \in (1,p]$. We deduce finally that $\nabla p \in L^\infty(0,T;L^q(\mathcal{F}(t)))$ for all $q \in (1,p]$.

8.3 Proof of Lemma 3 and of Theorem 1 (Additional properties)

We begin by proving Lemma 3. This will in particular establish the assertion in Theorem 1 concerning the compactness of the support of $w(t,\cdot)$.

Proof of Lemma 3. In the case $p = +\infty$, we have a unique flow $\Phi(t, s, x)$ associated to $v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp$, since this vector field is log-Lipschitz uniformly in time, and the conclusion follows easily, because

$$\omega(t, x) = w_0(\Phi(0, t, x)).$$

Let us discuss the main case, that is $p < +\infty$. We will use the renormalization theory for which we refer to [6, 15, 2]. In particular it follows from [2, Theorem 3.2], that ω^ε is a renormalized solution of (58), in the sense that for any $\beta \in \mathrm{Lip}(\mathbb{R}; \mathbb{R})$,

$$\partial_t \beta(\omega^\varepsilon) + \mathrm{div}\left(\beta(\omega^\varepsilon)(v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp)\right) = 0,$$

in $(0, T) \times \mathcal{F}_0^\varepsilon$ in the sense of distributions. We deduce that for any $\psi \in C_c^\infty([0, T] \times \mathcal{F}_0^\varepsilon)$,

$$\int_{\mathcal{F}_0^\varepsilon} \beta(\omega^\varepsilon(t, \cdot))\psi(t, \cdot) = \int_{\mathcal{F}_0^\varepsilon} \beta(w_0)\psi(0, \cdot) + \int_{[0,t]\times\mathcal{F}_0^\varepsilon} (\partial_t \psi + (v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp) \cdot \nabla_x \psi)\beta(\omega^\varepsilon). \tag{156}$$

Now let us prove that (156) holds also for any $\psi \in C_c^\infty([0, T] \times \overline{\mathcal{F}_0^\varepsilon})$ and any bounded $\beta \in \mathrm{Lip}(\mathbb{R}; \mathbb{R})$. Let $\eta \in C^\infty(\mathbb{R}^+; \mathbb{R})$ such that $\eta = 1$ in $[0, 1]$ and $\eta = 0$ in $[2, +\infty)$. Given $\phi \in C_c^\infty([0, T] \times \overline{\mathcal{F}_0^\varepsilon})$ and a bounded $\beta \in \mathrm{Lip}(\mathbb{R}; \mathbb{R})$, we apply (156) to $\psi(t, x) := \phi(t, x)\eta(d(x, \partial\mathcal{S}_0^\varepsilon)/\delta)$ for $\delta > 0$ small. Using $\nabla d(x, \partial\mathcal{S}_0^\varepsilon) \cdot [v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp] = 0$ on $\partial\mathcal{S}_0^\varepsilon$, the uniform continuity of $\nabla d(x, \mathcal{S}_0^\varepsilon) \cdot [v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp]$ on some compact neighborhood of $[0, T] \times \mathcal{S}_0^\varepsilon$ and letting $\delta \to 0^+$, we deduce the claim.

Using Lebesgue's dominated convergence theorem, we deduce that (156) is still valid when $\beta(t) := |t|^p$ and when ψ is replaced by some $\phi \in C^\infty([0, t] \times \overline{\mathcal{F}_0^\varepsilon})$, bounded as well as its first derivatives. As a consequence we obtain that for all t,

$$\int_{\mathcal{F}_0^\varepsilon} |\omega^\varepsilon(t, \cdot))|^p\phi(t, \cdot) = \int_{\mathcal{F}_0^\varepsilon} |w_0|^p\phi(0, \cdot) + \int_{[0,t]\times\mathcal{F}_0^\varepsilon} (\partial_t \phi + (v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp) \cdot \nabla_x \phi)|\omega^\varepsilon|^p. \tag{157}$$

Now, as in [14], let ϕ_0 denote a smooth nondecreasing function from $\mathbb{R}$ to $\mathbb{R}$ such that $\phi_0(s) = 0$ for $s \leqslant 1$ and $\phi_0(s) = 1$ for $s \geqslant 2$. We also define $\phi(s, x) := \phi_0(|x|/R(s))$, where

$$R(t) := \rho^\varepsilon(0) + \int_0^t \|v^\varepsilon - \ell^\varepsilon\|_{L^\infty(\mathbb{R}^2\setminus B(0,1))}.$$

With this test function (157) now reads

$$\int_{\mathcal{F}_0^\varepsilon} |\omega^\varepsilon(t, \cdot)|^p\phi_0\left(\frac{|x|}{R}\right) = \int_{[0,t]\times\mathcal{F}_0^\varepsilon} \left((v^\varepsilon - \ell^\varepsilon - r^\varepsilon x^\perp) \cdot \frac{x}{|x|} - \frac{R'}{R}|x|\right) \frac{\phi_0'\left(\frac{|x|}{R}\right)}{R} |\omega^\varepsilon|^p. \tag{158}$$

Since ϕ_0' has non trivial values only when $\frac{|x|}{R} \geqslant 1$, we get that

$$\int_{\mathcal{F}_0^\varepsilon} |\omega^\varepsilon(t, \cdot)|^p\phi_0\left(\frac{|x|}{R}\right) \leqslant \int_{[0,t]\times\mathcal{F}_0^\varepsilon} \left(\|v^\varepsilon - \ell^\varepsilon\|_{L^\infty(\mathbb{R}^2\setminus B(0,1))} - R'\right) \frac{\phi_0'\left(\frac{|x|}{R}\right)}{R} |\omega^\varepsilon|^p = 0, \tag{159}$$

which proves the lemma. □

Let us finally establish the last properties of the solutions announced in Theorem 1.

End of the proof of Theorem 1. That the quantities mentioned in the statement are preserved over time can be seen as follows.

- Concerning the conservation of energy, this was proven in Proposition 4.
- Concerning the conservation of $\|w(t)\|_q$ for $q \in [1, p]$, we use again (156). By using Lebesgue's dominated convergence theorem, it is easy to see that one can apply it to $\psi = 1$ and $\beta(t) = |t|^q$, which yields the result.

- The conservation of $\int_{\mathcal{F}_0} w(t,x)\,dx$ can be seen likewise (this is just the trivial case $\beta(t)=t$.)
- Finally, the conservation of $\int_{\partial\mathcal{S}_0} u\cdot\tau\,ds$ can be seen as follows. For $R>0$ large enough, due to Lemma 3, one can see that $v(t,\cdot)$ is harmonic (and therefore smooth) outside $B(0,R)$. It follows then from the usual Kelvin's circulation theorem that the circulation on large circles is conserved along the flow. Combined with Green's formula and the conservation of $\int_{\mathcal{F}_0} w(t,x)\,dx$, this proves the claim.

Finally, we can deduce the continuity in time of v with values in $W^{1,p}(\mathcal{F}_0)$ (for $p<+\infty$). Indeed, we already have $v\in L^\infty_{loc}(\mathbb{R}^+;W^{1,p}(\mathcal{F}_0))$ and $v\in C^0(\mathbb{R}^+;L^p(\mathcal{F}_0))$; consequently one has $v\in C^0(\mathbb{R}^+;W^{1,p}(\mathcal{F}_0)-w)$. It follows that

$$w(s,\cdot)\overset{w}{\rightharpoonup} w(t,\cdot) \text{ in } L^p(\mathcal{F}_0)-w \text{ as } s\to t.$$

Due to the conservation of $\|w(t)\|_p$, this convergence is strong, and the claim follows. □

9 Hamiltonian structure of the limit system

First we endow the manifold $\mathcal{P}$ of the triplets $(w,h,\xi):=(w,h,mh')$ with a Poisson structure (see [1]). That is to say, we endow $\mathcal{P}$ with a bracket $\{\cdot,\cdot\}$ acting on C^∞ functionals $F:\mathcal{P}\to\mathbb{R}$, bilinear and skew-symmetric, satisfying the Jacobi and the Leibniz identities. Here this is obtained by setting, for any smooth functionals F^1,F^2 on $\mathcal{P}$,

$$\{F^1,F^2\}:=\gamma F^1_\xi\cdot(F^2_\xi)^\perp-(F^1_\xi\cdot F^2_h-F^1_h\cdot F^2_\xi)-\int_{\mathbb{R}^2} w\;\nabla_x F^1_w\cdot\nabla_x^\perp F^2_w,$$

where F_w, F_h and F_ξ denote the derivatives with respect to w,h and ξ of a functional F. The properties cited above are clear from this definition.

Moreover, $\mathcal{H}$ (given by (23)) can also be seen as a functional on the manifold $\mathcal{P}$, and this endows the system with a Hamiltonian structure in the following sense. The following computations are valid either when both F and the solution are smooth, or in the framework considered in this paper when $F=\mathcal{H}$.

Proposition 11. *When (w,h,ξ) solves the equations (18)–(21) we have for any smooth functional F, the ordinary differential equation*

$$\frac{d}{dt}F=\{F,\mathcal{H}\},\tag{160}$$

where F and $\mathcal{H}$ in (160) stand respectively for $F(w,h,\xi)$ and $\mathcal{H}(w,h,\xi)$.

Proof of Proposition 11. According to the chain rule, we have

$$\frac{d}{dt}F(w,h,\xi)=\int_{\mathbb{R}^2} F_w\frac{\partial w}{\partial t}+F_h\cdot h'+F_\xi\cdot mh''.$$

Using the equations (18)–(21) we arrive to

$$\frac{d}{dt}F(w,h,\xi)=-\int_{\mathbb{R}^2} F_w\Big(\tilde u+\frac{\gamma}{2\pi}\frac{(x-h(t))^\perp}{|x-h(t)|^2}\Big)\cdot\nabla_x w+F_h\cdot h'+\gamma F_\xi\cdot\Big(h'(t)-\tilde u(t,h(t))\Big)^\perp=:I_1+I_2+I_3.$$

On the other side the derivatives of $\mathcal{H}$ are

$$\mathcal{H}_w=-\int_{\mathbb{R}^2} G(\cdot-y)\,w(y)\,dy-\gamma G(\cdot-h(t)),\quad \mathcal{H}_h=\gamma\tilde u(t,h(t))^\perp,\quad \mathcal{H}_\xi=\frac{\xi}{m}.$$

Then, integrating by parts, we find

$$I_1=\int_{\mathbb{R}^2} w\nabla^\perp\left(\int_{\mathbb{R}^2} G(\cdot-y)\,w(y)\,dy+\gamma G(\cdot-h(t))\right)\cdot\nabla F_w=-\int_{\mathbb{R}^2} w\nabla F_w\cdot\nabla^\perp\mathcal{H}_w.$$

On the other side, we have

$$I_2=h'\cdot F_h=\mathcal{H}_\xi\cdot F_h,$$
$$I_3=\frac{\gamma}{m}\xi^\perp\cdot F_\xi-\gamma\tilde u(t,h(t))^\perp\cdot F_\xi=\gamma\mathcal{H}_\xi^\perp\cdot F_\xi-\mathcal{H}_h\cdot F_\xi,$$

which yields (160). □

As a simple corollary of (160) and of the skew-symmetry of the bracket we get that $\mathcal{H}$ is conserved by the solutions of (18)–(20).

Acknowledgements. The first and third authors were partially supported by the Agence Nationale de la Recherche, Project CISIFS, grant ANR-09-BLAN-0213-02. The second author is partially supported by the Agence Nationale de la Recherche, Project MathOcéan, grant ANR-08-BLAN-0301-01.

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