# A remark on the controllability of a system of conservation laws in the context of entropy solutions*

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**Abstract**

In this paper, we consider a system of conservation laws introduced by DiPerna [12], from the point of view of boundary controllability, in the context of weak entropy solutions. Bressan and Coclite [5] have shown that this system is not controllable when the solutions are of small total variation. We study the use of a large shock wave for the control.

# 1 Introduction

## 1.1 Basic question and previous results

The problems of controllability for one-dimensional systems of conservation laws and more generally quasilinear hyperbolic systems has known many progresses since the pioneering work of Cirinà [7], in particular in the framework of classical solutions of class $C^1$, see in particular Li and Rao [19] for an important work on this problem.

A general quasilinear hyperbolic system in one dimension reads as follows

$$u_t + A(u)u_x = 0 \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1.1}$$

where $u : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}^n$ is the unknown and the matrix $A(u) \in \mathcal{M}_n(\mathbb{R})$ satisfies the strict hyperbolic condition, that is, for any $u$ in the state domain $\Omega \subset \mathbb{R}^n$, one has

$$A(u) \text{ has } n \text{ real distinct eigenvalues } \lambda_1 < \cdots < \lambda_n. \tag{1.2}$$

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These eigenvalues are the characteristic speeds at which the system propagates; we associate the eigenvectors $r_i$ to them. A very important particular case of hyperbolic systems is given by the systems of conservation laws:

$$u_t + (f(u))_x = 0 \text{ for } (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1.3}$$

where the flux function $f$ is regular from $\Omega$ to $\mathbb{R}^n$. Typically, $t$ is the time and $x$ is the position.

The general problem of controllability is the following. Consider the problem posed in the interval $[0,1]$ rather than in $\mathbb{R}$. In such a case one needs of course to prescribe boundary conditions on $[0,T] \times \{0,1\}$: here boundary conditions will be considered as a *control*, that is, a way to influence the system to make it behave in a prescribed way. Let us call $u(t,\cdot)$ the *state* of the system at time $t$. The question is: given two possible states of the system, say $u_0$ and $u_1$, can we choose the control suitably, in order that the solution of the system starting from $u_0$, reaches $u_1$ at time $T$?

Let us underline that the boundary conditions for such hyperbolic systems of conservation laws are in general quite involved (in particular when the characteristic speeds can change sign). A way to overcome this difficulty is to reformulate the controllability problem in an *underdetermined* form: given $u_0$, $u_1$ and $T$, can we find a solution of (1.1) (*without boundary conditions*) satisfying

$$u_{|t=0} = u_0 \text{ and } u_{|t=T} = u_1?$$

A very general answer to this problem has been obtained by Li and Rao [19] in the case of solutions of class $C^1$ with small $C^1$ norm, when the characteristic speeds are strictly separated from zero.

**Theorem 1.1** (Li-Rao, [19], 2002)**.** *Consider the system* (1.1) *with the condition* $\lambda_1(u) < \cdots < \lambda_k(u) \leq -c < 0$ *and* $0 \leq c < \lambda_{k+1}(u) < \cdots < \lambda_n(u)$. *Then for all* $\phi, \psi \in C^1([0,1])$ *such that* $\|\phi\|_{C^1} + \|\psi\|_{C^1} < \varepsilon$, *there exists a solution* $u \in C^1([0,T] \times [0,1])$ *such that*

$$u_{|t=0} = \phi, \ \textit{and} \ u_{|t=T} = \psi.$$

In the same functional framework, a result has also been obtained in certain cases admitting vanishing characteristic speeds, see [11].

But the situation is far less well understood in the context of *entropy solutions* of systems of conservation laws (1.3). The origin of this theory stems from the fact that in general the solutions of these equations develop singularities in finite time. It is hence natural to consider discontinuous (weak) solutions. As well-known, such weak solutions are no longer unique, and it is natural to consider weak solutions which satisfy *entropy conditions* aimed at singling out the physically relevant solution. These entropy conditions are the following:

**Definition 1.2.** We define an entropy/entropy flux couple as a couple of functions $(\eta, q)$ such that

$$\forall u \in \mathbb{R}_+^* \times \mathbb{R}, \quad D\eta(u).Df(u) = Dq(u).$$

Then entropy solutions are defined as weak solutions of the system,

$$u_t + (f(u))_x = 0,$$

which moreover satisfy that, for all $(\eta, q)$ entropy couple with $\eta$ convex, stands, in the sense of distributions:

$$\eta(u)_t + q(u)_x \leq 0.$$

An important difference between the theory of entropy solutions and the one of classical solutions is that in the context of entropy solutions, the system is no longer reversible. This is of course, of great significance for the study of these equations, and particularly for what concerns controllability problems. Of course, the $C^1$ solutions of the system are in particular entropy solutions.

To be more precise, in this paper, we will consider solutions *à la* Glimm [15], that is, entropy solutions in the sense above, of small total variation in $x$ for all times. Note that the meaning of the boundary value in this context is intricate, especially when the characteristic speeds are not separated from zero, see in particular the reference of Dubois and LeFloch [13]. Hence the underdetermined version of the problem is particularly well suited here.

There are very few studies concerning the controllability problem for hyperbolic systems of conservation laws in the context of entropy solutions. Ancona and Marson [2] described the attainable set on a half line for convex scalar ($n = 1$) conservation laws. In the case of the Burgers equation, Horsin [16] considered the case of a bounded interval, when the initial data is not necessarily zero. His method relies on J.-M. Coron's so-called *return method*, on which we shall come back later. For what concerns systems of conservation laws ($n \geq 2$), Ancona and Coclite described the attainable set for the particular case of Temple systems [1]. Bressan and Coclite [5] showed that for a hyperbolic system of conservation laws with fields either linearly degenerate or genuinely nonlinear in the sense of Lax [17], with characteristic speeds strictly separated from zero, one can *asymptotically* converge toward any constant state. But for what concerns the finite time controllability, Bressan and Coclite showed the following very surprising result.

**Theorem 1.3** (Bressan-Coclite, [5], 2002). *For a class of systems containing DiPerna's system [12]:*

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t v + \partial_x \left( \frac{v^2}{2} + \frac{K^2}{\gamma - 1} \rho^{\gamma-1} \right) = 0, \end{cases} \tag{1.4}$$

*there are initial conditions $\varphi \in BV([0,1])$ of arbitrary small total variation such that any entropy solution $u$ remaining of small total variation for all times satisfies:*

$$\textit{for any } t, \ u(t,\cdot) := (\rho, v) \ \textit{is not constant.}$$

We see that the situation is strikingly different from the case of $C^1$ solutions. As we will see, DiPerna's system is strictly hyperbolic, has genuinely nonlinear characteristic fields, and there are large zones in which the two characteristic speeds are away from zero. Hence the Li-Rao theorem applies, and in the context of $C^1$ solutions, one can reach constant states in finite time, at least if one stays away from the critical states when one of the characteristic speeds vanishes. Hence Bressan and Coclite's result describes a particular phenomenon due to discontinuities. To describe very roughly their counterexample, the initial state that they consider is constituted with a dense distribution of shock waves in $[0,1]$; a particular feature of DiPerna's system is that when two shocks of the same characteristic family interact, they merge into a larger shock and create an additional shock in the other characteristic family. This involves that there is a permanent creation of shocks in the domain, hence the solution cannot be driven to a constant state.

Now the introduction of system (1.4) was motivated by isentropic fluid dynamics, which is described by a system very close to (1.4):

$$\begin{cases} \partial_t \rho + \partial_x m = 0, \\ \partial_t m + \partial_x (\frac{m^2}{\rho} + \kappa \rho^\gamma) = 0. \end{cases} \tag{1.5}$$

In the above equation, $\rho = \rho(t,x) \geq 0$ is the density of the fluid, $m(t,x)$ is the momentum ($v(t,x) = \frac{m(t,x)}{\rho(t,x)}$ is the velocity of the fluid), the pressure law is $p(\rho) = \kappa \rho^\gamma$, $\gamma \in (1,3]$. Equation (1.5) is formulated in Eulerian coordinates. The problem of one-dimensional isentropic gas dynamics is also frequently studied in Lagrangian coordinates:

$$\begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x (\kappa \tau^{-\gamma}) = 0, \end{cases} \tag{1.6}$$

in which case the system is referred to as the $p$-system; here $\tau = 1/\rho$ is the specific volume. What we have shown in [14] is that the particular behavior of system (1.4) does not occur in the case of equations (1.5) and (1.6):

**Theorem 1.4** (G., [14], 2007)**.** *Consider two constant states* $\overline{u}_0 := (\rho_0, m_0)$ *and* $\overline{u}_1 := (\rho_1, m_1)$ *in* $\mathbb{R}_+^* \times \mathbb{R}$*. There exist* $\varepsilon > 0$ *and* $T > 0$*, such that, for any* $u_0 \in BV([0,1])$ *satisfying:*

$$\|u_0 - \overline{u}_0\|_{L^1} \leq \varepsilon \text{ and } TV(u_0) \leq \varepsilon,$$

*there is an entropy solution* $u$ *of* (1.5) *in* $[0,T] \times [0,1]$ *such that*

$$u_{|t=0} = u_0 \quad \text{and} \quad u_{|t=T} = \overline{u}_1.$$

*The same result applies for equation* (1.6)*.*

*Remark* 1.5. Actually, the result of [14] describes a broader set of final states that can be reached via suitable boundary controls. Typically, this set contains all small $C^1$ states, and also states containing shocks, which fulfill a so-called Oleinik-type inequality. Also, one can see that in the case (1.5), no condition of separation of the characteristics speeds from zero is imposed, despite the fact that these speed can actually vanish.

The proof of Theorem 1.4 given in [14] relies in fact on two different methods for the case (1.5) and the case (1.6) and give in fact slightly different results. Actually, the method that we give for (1.6) applies also for equation (1.5) (see [14] for more details), and allows to get the following property:

$$\text{if } u_0 - \overline{u}_1 \text{ is small in total variation, then the solution of the control problem can be chosen small as well.} \tag{1.7}$$

This does not mean that necessarily we will have for all times that $u(t,\cdot)$ is of total variation of order $TV(u_0 - \overline{u}_1)$; actually this is more like $[TV(u_0 - \overline{u}_1)]^{1/3}$, but this is typically a behavior which is excluded for system (1.4). One of the main points is that, for systems (1.5) and (1.6), when two shocks of the same characteristic family interact, they merge into a larger shock and create a rarefaction wave in the other characteristic family. The other method which we present in [14] for (1.5) does not yield property (1.7), and does not apply to system (1.6). But what we are going to see in this paper is that it applies to system (1.4).

## 1.2 The result

What we show is the following.

**Theorem 1.6.** *Given* $\overline{u}_0 := (\rho_0, v_0), \overline{u}_1 := (\rho_1, v_1)$ *in* $\mathbb{R}_+^* \times \mathbb{R}$*, there exist* $\varepsilon > 0$ *and* $T > 0$*, such that, for any* $u_0 \in BV([0,1]; \mathbb{R}_+^* \times \mathbb{R})$ *satisfying:*

$$\|u_0 - \overline{u}_0\|_{L^1} \leq \varepsilon \text{ and } TV(u_0) \leq \varepsilon,$$

*there is an entropy solution $u$ of* (1.4) *in* $[0,T]\times[0,1]$ *such that*

$$u_{|t=0}=u_0, \text{ and } u_{|t=T}=\overline{u}_1.$$

But of course, the solution which we obtain does not stay of small total variation for all times!

### 1.3 Structure of the proof

As in [14], the proof of Theorem 1.6 consists in proving these two consecutive propositions.

**Proposition 1.7.** *Let $u_0 \in BV([0,1];\mathbb{R}_+^* \times \mathbb{R})$ as in Theorem 1.6. Then there exist $T_1>0$, a constant state $\omega_1 \in \Omega$, and an entropy solution $u:[0,T_1]\times[0,1]\rightarrow\Omega$ of (1.4) such that*

$$u_{|t=0}=u_0 \tag{1.8}$$

$$u_{|t=T_1}=\omega_1. \tag{1.9}$$

This part is the part where we show that the fact that the solution stays of small total variation is central in Theorem 1.3. This is connected to Coron's return method, which was introduced in [9]; see [10] for more details on it. Basically, this method advocates that in many situations, one has better controllability properties when the system goes far from the base point and returns to it. In the context here the Bressan-Coclite theorem shows that it is more or less necessary.

The second proposition (which can be seen as finite-dimensional control result) is the following.

**Proposition 1.8.** *For any $(\omega,\omega')\in(\mathbb{R}_+^*\times\mathbb{R})^2$, there is some $T_2>0$ and an entropy solution $u$ of* (1.4) *in* $[0,T]\times[0,1]$*, such that:*

$$u_{|t=0}=\omega \tag{1.10}$$

$$u_{|t=T_2}=\omega'. \tag{1.11}$$

We show this two propositions in Sections 3 and 4, which establishes Theorem 1.6.

## 2 Characteristics of DiPerna's system

Let us briefly describe the main characteristics of system (1.4). The Jacobian matrix $A=df$ associated to (1.4) is the following

$$A(\rho,v)=\begin{pmatrix} v & \rho \\ K^2\rho^{\gamma-2} & v \end{pmatrix} \tag{2.1}$$

Hence it is easily seen that this system is strictly hyperbolic for $(\rho, v) \in \Omega := \mathbb{R}_+^* \times \mathbb{R}$, with eigenvalues

$$\lambda_1 = v - K\rho^\beta \quad \text{and} \quad \lambda_2 = v + K\rho^\beta, \tag{2.2}$$

and eigenvectors

$$r_1 = \begin{pmatrix} -\rho \\ K\rho^\beta \end{pmatrix} \text{ and } r_2 = \begin{pmatrix} \rho \\ K\rho^\beta \end{pmatrix} \tag{2.3}$$

where

$$\beta := \frac{\gamma - 1}{2} \in (0, 1).$$

It is straightforward to check that the system is genuinely nonlinear in the sense of Lax [17]

$$r_i.\nabla\lambda_i > 0 \text{ in } \Omega.$$

Let us finally describe the wave curves associated to this system. The wave curves, that is, shock curves and rarefaction curves, is the set of states in $\Omega$ which can be connected to a given fixed state on the left $u_l := (\rho_l, v_l)$ via a shock wave or a rarefaction wave. Shock waves (associated to each characteristic family) are discontinuities satisfying Rankine-Hugoniot (in order to be a weak solution of the equation) relations

$$f(u_r) - f(u_l) = s[u_r - u_l], \tag{2.4}$$

and Lax's inequalities (in order to be entropic): for the $i$-th family of shocks,

$$\lambda_i(u_r) < s < \lambda_i(u_l) \tag{2.5}$$

$$\lambda_{i-1}(u_l) < s < \lambda_{i+1}(u_r). \tag{2.6}$$

where $s$ is the speed of the shock, which gives the particular solution

$$u(t, x) = \begin{cases} u_l \text{ for } x/t < s, \\ u_r \text{ for } x/t > s. \end{cases}$$

Rarefaction waves are defined by introducing integral curves of $r_i$, and are a discontinuity-free solutions:

$$\begin{cases} \frac{d}{d\sigma} W_i(\sigma) = r_i(W_i(\sigma)), \\ W_i(0) = u_l, \\ \sigma \geq 0, \end{cases}$$

The standard (right) shock curves at the point $(\rho_0, v_0) \in \Omega$ are given by the following

$$\begin{cases} v = v_0 - \frac{K}{\sqrt{\beta}} \sqrt{(\rho^{2\beta} - \rho_0^{2\beta}) \frac{\rho - \rho_0}{\rho + \rho_0}} \text{ with } \rho \geq \rho_0, \quad \text{along } R_1(\rho_0, v_0), \\ v = v_0 + \frac{K}{\sqrt{\beta}} \sqrt{(\rho^{2\beta} - \rho_0^{2\beta}) \frac{\rho - \rho_0}{\rho + \rho_0}} \text{ with } 0 < \rho \leq \rho_0, \quad \text{along } R_2(\rho_0, v_0). \end{cases} \tag{2.7}$$

The left shock curves at the point $(\rho_0, v_0) \in \Omega$ (when we fix the right state and look for the right one) are given by the following

$$\begin{cases} v = v_0 - \frac{K}{\sqrt{\beta}}\sqrt{(\rho^{2\beta} + \rho_0^{2\beta})\frac{\rho-\rho_0}{\rho+\rho_0}} \text{ with } 0 < \rho \leq \rho_0, \quad \text{along } L_1(\rho_0, v_0), \\ v = v_0 + \frac{K}{\sqrt{\beta}}\sqrt{(\rho^{2\beta} - \rho_0^{2\beta})\frac{\rho-\rho_0}{\rho+\rho_0}} \text{ with } \rho \geq \rho_0, \quad \text{along } L_2(\rho_0, v_0). \end{cases} \tag{2.8}$$

Instead of describing of the rarefaction curves in the plane $(\rho, v)$, we introduce the Riemann invariants associated to system (1.4). Precisely define

$$z = v - \frac{K}{\beta}\rho^{\beta} \text{ and } w = v + \frac{K}{\beta}\rho^{\beta}, \tag{2.9}$$

so that

$$r_1.\nabla w = r_2.\nabla z = 0 \text{ and } r_1.\nabla z > 0, \ r_2.\nabla w > 0.$$

In the $(w, z)$-plane, rarefaction curves are horizontal and vertical half-lines.

# 3 Proof of Proposition 1.7

The proof of Proposition 1.7 relies on large shocks for system (1.4). We will be able to treat them thanks to the next lemma.

**Lemma 3.1.** *All shocks $(u_-, u_+)$ are Majda-stable in the sense that*

**i.** *$s$ is not an eigenvalue of $A(u^{\pm})$,*
**ii.** $\{r_j(u^+) \ / \ \lambda_j(u^+) > s\} \cup \{u^+ - u^-\} \cup \{r_j(u^-) \ / \ \lambda_j(u^-) < s\}$ *is a basis of $\mathbb{R}^2$ (for a $j$-shock).*

(3.1)

*Proof.* Taking into account the fact that Lax's inequalities are globally satisfied along the shock curves (see [12]) this means that 1-shocks (resp. 2-shocks) $(u_l, u_r)$ satisfy

$$(r_1(u^-), u_+ - u_-) \text{ (resp. } (u_+ - u_-, r_2(u^+))) \text{ is a basis of } \mathbb{R}^2. \tag{3.2}$$

One of the properties of the system (1.4) as shown by DiPerna [12] is that its shock curves have special behavior in the plane given by the Riemann invariants. One can express all the wave curves in terms of $\eta := (2K/\beta)\rho^{\beta}$ and check that

$$\frac{\partial R_1}{\partial \eta}, \ \frac{\partial L_1}{\partial \eta} \leq 0 \text{ and } \frac{\partial R_2}{\partial \eta}, \ \frac{\partial L_2}{\partial \eta} \geq 0,$$

which involves that expressed in terms of $w$, the curves satisfy

$$-\infty \leq \frac{\partial R_1}{\partial w}, \ \frac{\partial L_1}{\partial w} \leq -1 \text{ and } -1 \leq \frac{\partial R_2}{\partial w}, \ \frac{\partial L_2}{\partial w} \leq 0.$$

(This is referred to as property $A_2$ in [12].) Hence the shock curves are in confined in cones which involves that (3.2) is satisfied since in the $(w, z)$ plane, $r_1$ and $r_2$ are vertical and horizontal respectively. □

The other ingredient which appeared in [14] was the following.

**Lemma 3.2.** *For any* $(\overline{\rho}_0, \overline{v}_0) \in \Omega$*, there exists* $(\rho, v) \in L_2(\omega)$*, such that*

$$\lambda_2((\rho, v)) > \lambda_1((\rho, v)) \geq 3, \tag{3.3}$$

$$s((\overline{\rho}_0, \overline{v}_0), (\rho, v)) \geq 3. \tag{3.4}$$

Here $s$ is the shock speed given by the Rankine-Hugoniot relation (2.4).

*Proof.* Consider $(\rho, v) \in L_2(\overline{\rho}_0, \overline{v}_0)$, with $\rho \to +\infty$. From the Rankine-Hugoniot relations, one easily computes

$$s((\overline{\rho}_0, \overline{v}_0), (\rho, v)) = \overline{v}_0 + \frac{K\rho}{\sqrt{\beta}\sqrt{\rho + \overline{\rho}_0}} \sqrt{\frac{\rho^{2\beta} - \overline{\rho}_0^{2\beta}}{\rho - \overline{\rho}_0}}$$

Hence clearly $s((\overline{\rho}_0, \overline{v}_0), (\rho, v)) \to +\infty$ as $\rho \to +\infty$. Next one sees that

$$\lambda_1((\rho, v)) = \overline{v}_0 + \frac{K}{\sqrt{\beta}} \sqrt{\frac{(\rho^{2\beta} - \overline{\rho}_0^{2\beta})(\rho - \overline{\rho}_0)}{\rho + \overline{\rho}_0}} - K\rho^{\beta}.$$

But since $\beta \in (0, 1)$, one has

$$\frac{K}{\sqrt{\beta}} > K.$$

Hence one deduces that as well $\lambda_1((\rho, v)) \to +\infty$ as $\rho \to +\infty$. With the global strict hyperbolicity this concludes the proof. □

Now given $\overline{u}_0 := (\overline{\rho}_0, \overline{v}_0)$ and $u_0$ as in Theorem 1.6, we introduce $\overline{u} = (\rho, v)$ as in Lemma 3.2. We introduce the following function $U_0 \in BV_{loc}(\mathbb{R}; \mathbb{R}_+^* \times \mathbb{R})$:

$$U_0(x) = \begin{cases} \overline{u} \text{ for } x < 0, \\ u_0(x) \text{ for } 0 \leq x \leq 1, \\ \overline{u}_0 \text{ for } x > 1. \end{cases} \tag{3.5}$$

Exactly as in [14], we can prove the following proposition.

**Proposition 3.3.** *If* $u_0$ *as small enough total variation, there is a global-in-time entropy solution* $U$ *of* (1.4) *in* $[0, +\infty) \times \mathbb{R}$ *satisfying*

$$U(0, \cdot) = U_0 \text{ in } \mathbb{R}. \tag{3.6}$$

*Moreover it satisfies:*

$$U_{|\{1\} \times [0,1]} \text{ is constant.} \tag{3.7}$$

By simply taking the restriction of $U$ to $[0,1] \times [0,1]$, we obtain a solution of the problem consisting in driving $u_0$ to a constant. The proof of Proposition 3.3 is exactly the same as in [14] (see also studies [6, 8, 18, 20, 21] for related problems). It relies only on the Majda-stability of the large shock and on the positivity of the propagation speeds on its left. Basically we show that the above initial condition is propagated for all times as a large shock plus small waves on both sides of it. However, due to condition (3.3), all these waves travel at positive speed and eventually leave the domain. The basic ingredient to prove this is the use of a front-tracking algorithm (see [4] for more details on this particular construction of solutions of systems of conservation laws). We refer to the above articles for a complete proof.

# 4 Proof of Proposition 1.8

This is almost exactly the same as in [14]. There are three zones in $\Omega$ with respect to the signs of the characteristic speeds: the zone $\Omega_- := \{(\rho,u) \ / \ u < -K\rho^\beta\}$ where both characteristic speeds are negative, the zone $\Omega_+ := \{(\rho,u) \ / \ u > K\rho^\beta\}$ where both characteristic speeds are positive and the zone $\Omega_\pm := \{(\rho,u) \ / \ -K\rho^\beta < u < K\rho^\beta\}$ where $\lambda_1$ is negative and $\lambda_2$ is positive. These three zones are separated by the two critical curves $\mathcal{C}_- := \{(\rho,u) \ / \ u = -K\rho^\beta\}$ and $\mathcal{C}_+ := \{(\rho,u) \ / \ u = K\rho^\beta\}$ .

Now to prove Proposition 1.8, it suffices to prove that

1. Given $\omega$ and $\omega'$ in the same zone ($\Omega_-$, $\Omega_+$ or $\Omega_\pm$), one can find a solution from $\omega$ to $\omega'$,
2. One can always find a solution from a given zone to another,
3. One can always go out a critical curve or reach it from one of the above zones.

**1.** To prove the first point, let us limit ourselves to the case where $\omega$ and $\omega'$ are sufficiently close one to another. Then since the zones are path-connected and since a path is compact, one easily deduces the general case.

Now what one does depends on the zone the states are into. Given $\omega$ and $\omega'$ in $\Omega_+$ and sufficiently close one to another, we solve the Riemann problem $(\omega',\omega)$ (see e.g. [4, 17]). If the two states are sufficiently close one to another, the intermediate state is in $\Omega_+$ as well, hence the two waves obtained in this Riemann problem are of fixed sign speed. Hence the Riemann solution of this problem answers the question: if one waits long enough, the solution of this problem with $\omega'$ on $\mathbb{R}_-$ and $\omega$ on $\mathbb{R}_+$, will reach $\omega'$ in $[0,1]$.

If both states are in $\Omega_-$, the idea is the same, but one has to let the waves enter from the right boundary, that is, one solves the Riemann problem $(\omega, \omega')$, where the separation between the states occurs at $x = 1$. Wait long enough, and $\omega'$ enters $[0, 1]$.

If both states are in $\Omega_\pm$, again, we manage in other that the intermediate state $\omega_m$ in the resolution of the Riemann problem $(\omega', \omega)$ is in $\Omega_\pm$. Now we use the solution of the Riemann problem $(\omega, \omega_m)$ with the states separated at $x = 1$ and the solution of the Riemann problem $(\omega', \omega_m)$ with the states separated at $x = 0$ to join $\omega'$ from $\omega$.

**2.** To prove the second point, we use roughly the same remark as for Lemma 3.2. If the state that you consider is in $\Omega_-$ or $\Omega_\pm$, then by a large 2-shock on the left of the domain, you can reach $\Omega_+$. In the same way, if the state that you consider is $\Omega_-$, then you can reach $\Omega_\pm$: reach the point on the second left shock curve for which $v = 0$ and observe that its speed is necessarily positive. The same (with 1-shocks on the right) can be done to go from $\Omega_+$ to $\Omega_-$ or $\Omega_\pm$. Also, by the same method, one can leave a critical curve.

**3.** It remains to explain how to reach a critical curve. It is not difficult to see that one can arrive to the critical curve $\mathcal{C}_-$ by a small 1-shock that one lets enter by $x = 1$ (with the critical state on $x > 1$, and a non-critical state for $x < 1$), and that one can reach the critical curve $\mathcal{C}_+$ by a small 2-shock that one lets enter by $x = 0$ (with the critical state on $x < 0$, and a non-critical state for $x > 0$). It suffices to check that $r_1$ is transverse to $\mathcal{C}_-$ and $r_2$ is transverse to $\mathcal{C}_+$. This is easily established by noticing that $\beta < 1$.

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