REMOTE TRAJECTORY TRACKING OF RIGID BODIES IMMERSED IN A 2D PERFECT INCOMPRESSIBLE FLUID

OLIVIER GLASS, JÓZSEF J. KOLUMBÁN, AND FRANCK SUEUR

Abstract. We consider the motion of several rigid bodies immersed in a two-dimensional incompressible perfect fluid, the whole system occupying a bounded simply connected domain. The external fixed boundary is impermeable except on an open non-empty part where one controls both the normal velocity, allowing some fluid to go in and out the domain, and the entering vorticity. The motion of the rigid bodies is given by the Newton laws with forces due to the fluid pressure and the fluid motion is described by the incompressible Euler equations. We prove that it is possible to exactly achieve any non-colliding smooth motion of the rigid bodies by the remote action of a controlled normal velocity on the outer boundary which takes the form of state-feedback, with zero entering vorticity. The proof relies on a nonlinear method to solve linear perturbations of nonlinear equations associated with a quadratic operator.

Contents

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1. Presentation of the model: the "Euler+rigid bodies" system

The model that we consider in this paper describes the motion of rigid bodies immersed in a two-dimensional perfect incompressible fluid. The whole system occupies a bounded connected open subset Ω of $\mathbb{R}^2$, which to simplify we will also consider to be simply connected (though this is by no means essential to the analysis). The rigid bodies occupy at the initial time disjoint non-empty regular connected and simply connected compact sets $\mathcal{S}_{\kappa,0} \subset \Omega$, with κ in $\{1, 2, \dots, N\}$. We assume for simplicity that none of these sets is a disk, since this particular case requires a special treatment.

The rigid motion of the solid κ is described at each moment by the rotation matrix

$$R(\theta_\kappa(t)) := \begin{bmatrix} \cos\theta_\kappa(t) & -\sin\theta_\kappa(t) \\ \sin\theta_\kappa(t) & \cos\theta_\kappa(t) \end{bmatrix}, \quad \theta_\kappa(t) \in \mathbb{R},$$

and by the position $h_\kappa(t)$ in $\mathbb{R}^2$ of its center of mass. The domain of the solid κ at every time $t > 0$ is therefore

$$\mathcal{S}_\kappa(t) := R(\theta_\kappa(t))(\mathcal{S}_{\kappa,0} - h_\kappa(0)) + h_\kappa(t).$$

We will denote by $m_\kappa > 0$ and by $\mathcal{J}_\kappa > 0$ respectively the mass and the moment of inertia of the body indexed by κ. The domain occupied by the fluid is correspondingly

$$\mathcal{F}_0 := \Omega \setminus \bigcup_{\kappa\in\{1,2,\dots,N\}} \mathcal{S}_{\kappa,0}, \quad \text{at } t = 0, \quad \text{and } \mathcal{F}(t) := \Omega \setminus \bigcup_{\kappa\in\{1,2,\dots,N\}} \mathcal{S}_\kappa(t) \quad \text{at } t > 0.$$

We will denote by $u = (u_1, u_2)^t$ (the exponent t denotes the transpose of the vector) and by π the velocity and pressure fields in the fluid, respectively. Without loss of generality, the fluid is supposed to be homogeneous of density 1, to simplify the notations. The fluid dynamics is given by the incompressible Euler equations:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla\pi = 0 \quad \text{in } \mathcal{F}(t), \quad \text{for } t > 0, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } \mathcal{F}(t), \quad \text{for } t > 0. \tag{1.2}$$

The solids dynamics is given by Newton's balance law for linear and angular momenta: given κ in $\{1, 2, \dots, N\}$,

$$m_\kappa h_\kappa''(t) = \int_{\partial\mathcal{S}_\kappa(t)} \pi n \, \mathrm{d}s, \quad \text{for } t > 0, \tag{1.3}$$

$$\mathcal{J}_\kappa \theta_\kappa''(t) = \int_{\partial\mathcal{S}_\kappa(t)} (x - h_\kappa(t))^\perp \cdot \pi n \, \mathrm{d}s, \quad \text{for } t > 0. \tag{1.4}$$

When $x = (x_1, x_2)^t$ the notation $x^\perp$ stands for $x^\perp = (-x_2, x_1)^t$, n denotes the unit normal vector on $\partial\mathcal{S}_\kappa(t)$ which points outside of the fluid, so that $n = \tau^\perp$, where τ is the unit counterclockwise tangential vector on $\partial\mathcal{S}_\kappa(t)$. We assume the rigid bodies to be impermeable so that we prescribe on the interface: for every κ in $\{1, 2, \dots, N\}$,

$$u \cdot n = \left(\theta_\kappa'(\cdot - h_\kappa)^\perp + h_\kappa'\right) \cdot n \quad \text{on } \partial\mathcal{S}_\kappa(t), \quad \text{for } t > 0. \tag{1.5}$$

We will use the notations $\mathbf{q}_\kappa$ and $\mathbf{q}_\kappa'$ for vectors in $\mathbb{R}^3$ gathering both the linear and angular parts of the position and velocity:

$$\mathbf{q}_\kappa := (h_\kappa^t, \theta_\kappa)^t \quad \text{and} \quad \mathbf{q}_\kappa' := (h_\kappa'^{\,t}, \theta_\kappa')^t. \tag{1.6}$$

The vectors $\mathbf{q}_\kappa$ and $\mathbf{q}'_\kappa$ are next concatenated into vectors of length $3N$:

$$q = (\mathbf{q}_1^t, \dots, \mathbf{q}_N^t)^t \quad \text{and} \quad q' = (\mathbf{q}_1'^t, \dots, \mathbf{q}_N'^t)^t, \tag{1.7}$$

whose entries are relabeled respectively q_k and q'_k with k ranging over $\{1, \dots, 3N\}$. Hence we have also:

$$q = (q_1, q_2, \dots, q_{3N})^t \quad \text{and} \quad q' = (q'_1, q'_2, \dots, q'_{3N})^t.$$

Consequently, k in $\{1, \dots, 3N\}$ denotes the datum of both a solid number and a coordinate in $\{1, 2, 3\}$ so that q_k and q'_k denote respectively the coordinate of the position and of the velocity of a given solid. More precisely, for all k in $\{1, \dots, 3N\}$, we denote by $[\![k]\!]$ the quotient of the Euclidean division of $k-1$ by 3, $[k] = [\![k]\!] + 1$ in $\{1, \dots, N\}$ denotes the number of the solid and $(k) := k - 3[\![k]\!]$ in $\{1, 2, 3\}$ the considered coordinate.

Throughout this paper we will not consider collisions, so we introduce the set of body positions without collision:

$$\mathcal{Q} := \{q \in \mathbb{R}^{3N} \ : \ \min_{\kappa \neq \nu} \mathrm{d}(\mathcal{S}_\kappa(q), \Omega^c \cup \mathcal{S}_\nu(q)) > 0\},$$

where d is the Euclidean distance. For $\delta > 0$, we also introduce

$$\mathcal{Q}_\delta := \{q \in \mathbb{R}^{3N} \ : \ \min_{\kappa \neq \nu} \mathrm{d}(\mathcal{S}_\kappa(q), \Omega^c \cup \mathcal{S}_\nu(q)) \geqslant \delta\}.$$

The fluid domain is completely described by q in $\mathcal{Q}$ and we will therefore make use of the following abuse of notation: $\mathcal{F}(t) = \mathcal{F}(q(t))$.

2. Boundary conditions on the external boundary

Our purpose in this paper is to investigate the possibility of steering the rigid bodies according to any reasonable (smooth, non colliding) given motion by means of a boundary control acting on a part of the external boundary, while on the rest of the boundary we consider the usual impermeability condition. More precisely we consider Σ a nonempty, open part of the outer boundary $\partial\Omega$ and the following boundary conditions introduced by Yudovich in [52]. To begin with, let $\mathcal{C}$ denote the space

$$\mathcal{C} := \left\{ g \in C_0^\infty(\Sigma; \mathbb{R}) \ \text{ such that } \ \int_\Sigma g \, \mathrm{d}s = 0 \right\}. \tag{2.1}$$

For $T > 0$, we consider g in $C^\infty([0, T]; \mathcal{C})$ and the boundary condition on the normal trace of the outer boundary

$$u(t, x) \cdot n(x) = g(t, x) \text{ on } [0, T] \times \Sigma \quad \text{and} \quad u(t, x) \cdot n(x) = 0 \text{ on } [0, T] \times (\partial\Omega \setminus \Sigma). \tag{2.2}$$

Above, as for the solids boundaries, n denotes the unit normal vector pointing outside the fluid, so that $n = \tau^\perp$, where here τ denotes the unit clockwise tangential vector on $\partial\Omega$. The condition on the zero flux of g through Σ is necessary due to the incompressibility of the fluid. As noticed by Yudovich (Ibid.), this is not a sufficient boundary condition to determine the system. To complete it, we consider the set

$$\Sigma^- := \{(t, x) \in [0, T] \times \Sigma \ \text{ such that } \ g(t, x) < 0\},$$

of points of $[0, T] \times \Sigma$ where the fluid velocity field points inside Ω. Then the other part of the boundary condition consists in prescribing the entering vorticity, that is the vorticity

$$\omega := \operatorname{curl} u = \partial_1 u_2 - \partial_2 u_1 \text{ on } \Sigma^-.$$

This is natural since the fluid vorticity satisfies the transport equation:

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = 0, \quad x \in \mathcal{F}(q(t)). \tag{2.3}$$

For simplicity we will actually prescribe a null control in vorticity, that is

$$\omega(t, x) = 0 \text{ on } \Sigma^-. \tag{2.4}$$

Let us insist on the fact that the control considered here is a remote control in the sense that it is located on the external boundary, not on the moving rigid bodies. For this alternative issue of rigid or deformable bodies equipped with thrusters or locomotion devices we refer to the papers [24, 36, 37, 42].

Since the fluid occupies a multiply-connected domain, the circulations of the fluid velocity around the rigid bodies $\mathcal{S}_\kappa$:

$$\int_{\partial \mathcal{S}_\kappa(t)} u(t) \cdot \tau \, \mathrm{d}s = \gamma_\kappa, \qquad \text{for all } \ \kappa \in \{1, 2, \ldots, N\}, \tag{2.5}$$

will play an important role. Let us recall that, for each κ, the circulation γ_κ remains constant over time according to Kelvin's theorem. We will use the notation

$$\gamma := (\gamma_\kappa)_{\kappa=1,\ldots,N}.$$

To achieve our goal, we will consider a control *in feedback form*, depending on the state of the "fluid+rigid bodies" system. More precisely we will prescribe a normal velocity g on $[0, T] \times \Sigma$ of the form

$$g(t) = \mathscr{C}(q(t), q'(t), q''(t), \gamma, \omega(t, \cdot)), \tag{2.6}$$

where $\mathscr{C}$ is a Lipschitz function on

$$\cup_{q \in \mathcal{Q}_\delta} \{q\} \times \mathbb{R}^{3N} \times \mathbb{R}^{3N} \times \mathbb{R}^N \times L^\infty(\mathcal{F}(q); \mathbb{R}),$$

for any $\delta > 0$. Furthermore we will only need a finite dimensional space of controls so that $\mathscr{C}$ can be taken with values in a finite dimensional subspace of the space $\mathcal{C}$ defined in (2.1).

3. Main results: Trajectory tracking by a remote control

The problem that we raise in this paper is the trajectory tracking by means of the remote control described in the previous section. Precisely, the question is: is it possible to exactly achieve any non-colliding smooth motion of the rigid bodies by the remote action described above? The purpose is schematically described in Figure 1. Let us now explain how we positively answer to this question.

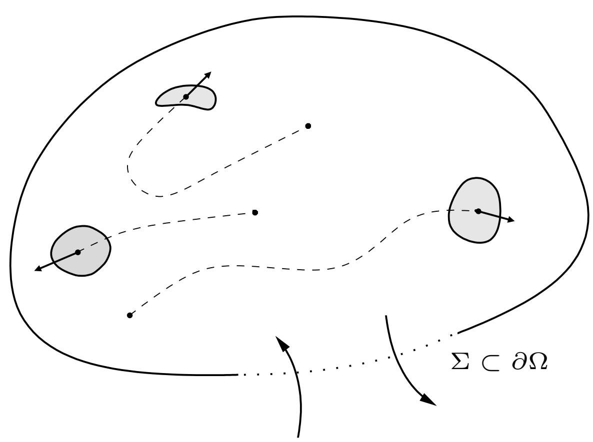

Figure 1. Controlled trajectories for solids inside Ω

To begin with, let us be more specific on the functional setting. Following Yudovich [53], we consider the case where the initial fluid vorticity is bounded. Then the natural regularity for a fluid velocity field associated with a bounded vorticity is the log-Lipschitz regularity. Precisely, for $T > 0$ and given the solids trajectories q, we will consider the space $LL(T)$ of uniformly in time log-Lipschitz in space vector fields, defined via its norm

$$\|f\|_{\mathrm{LL(T)}} := \|f\|_{L^\infty(\cup_{t\in(0,T)}\{t\}\times\mathcal{F}(q(t)))} + \sup_{t\in[0,T]}\sup_{x\neq y}\frac{|f(t,x)-f(t,y)|}{|x-y|(1+\ln^-(|x-y|))}.$$

Moreover, we will work with vorticities belonging to balls in L^∞: for any q in $\mathcal{Q}_\delta$ and $r_\omega>0$, we consider the complete metric space

$$\mathscr{B}(q,r_\omega) := \overline{B}_{L^\infty(\mathcal{F}(q))}(0,r_\omega) \text{ endowed with the } L^3(\mathcal{F}(q)) \text{ distance.} \tag{3.1}$$

Before we state the main result of this paper, let us give two words of caution.

- Below we will use the letter q as a variable for the positions of the rigid bodies, as well as a trajectory of the rigid bodies. Readers should not be confused.
- Let us also recall that, in incompressible fluid mechanics, including in the presence of moving rigid bodies, the pressure field π can be interpreted as a Lagrange multiplier associated with the divergence-free constraint; as a result it can be ignored when we speak of a solution of the problem. Consequently in the sequel we will say that (q,u) satisfies, for t in $[0,T]$, the Euler equations (1.1)-(1.2) and the Newton equations (1.3)-(1.4) for κ in $\{1,2,\dots,N\}$, without referring to the associated pressure. In the case where the vorticity is bounded, the controlled solutions which we will consider below correspond to a pressure field in $L^\infty(0,T;H^1(\mathcal{F}(t)))$ which is unique up to a function depending only on time which does not change the value of the terms involving the pressure (1.1), (1.3) and (1.4). This regularity result can be obtained as in the uncontrolled case, see [27, Corollary 2]. In particular this gives a sense to the right hand sides of the Newton equations (1.3)-(1.4) for κ in $\{1,2,\dots,N\}$.

Our main result is twofold. In a first part, we prove that there exists a feedback control $\mathscr{C}$ as in (2.6) such that, for any target trajectory q and any compatible initial conditions, there exists a solution of the closed-loop system with this control $\mathscr{C}$, in which the solids follow the trajectory q exactly. The second part of our statement establishes a partial uniqueness result: any (weak) solution of the closed loop system, with the acceleration given by the targeted trajectory, does satisfy that the solids follow the trajectory q exactly.

The exact statement is as follows.

Theorem 3.1. *For any $\delta>0$, there is a finite dimensional subspace $\mathcal{E}$ of $\mathcal{C}$ such that the following holds. Let $T>0$, $r_\omega>0$ and $\mathscr{K}$ be a compact subset of $\mathbb{R}^{3N}\times\mathbb{R}^{3N}\times\mathbb{R}^N$. Then there exists a control law*

$$\mathscr{C} \in Lip(\cup_{q\in\mathcal{Q}_\delta}\{q\}\times\mathscr{K}\times\mathscr{B}(q,r_\omega);\mathcal{E}),$$

such that the two following results hold true for any given trajectory q in $C^2([0,T];\mathcal{Q}_\delta)$ and for any γ in $\mathbb{R}^N$ such that for any t in $[0,T]$, $(q'(t),q''(t),\gamma)$ belongs to $\mathscr{K}$.

1) For any initial vorticity ω_0 in $L^\infty(\mathcal{F}(q(0)))$ such that $\|\omega_0\|_{L^\infty(\mathcal{F}_0)}\leqslant r_\omega$, there exists a velocity field u in $LL(T)\cap C^0([0,T];W^{1,p}(\mathcal{F}(t)))$, for all $p\in[1,+\infty)$, with $\operatorname{curl} u(0,\cdot)=\omega_0$ and for any t in $[0,T]$, $\operatorname{curl} u(t,\cdot)$ in $\mathscr{B}(q(t),r_\omega)$, such that (q,u) satisfies for all t in $[0,T]$: the Euler equations (1.1)-(1.2), the Newton equations (1.3)-(1.4) for κ in $\{1,2,\dots,N\}$, the interface condition (1.5), the boundary condition (2.2) on the normal velocity with

$$g(t) = \mathscr{C}(q(t),q'(t),q''(t),\gamma,\operatorname{curl} u(t,\cdot)), \tag{3.2}$$

and the boundary condition (2.4) on the entering vorticity and the circulation conditions (2.5).

2) Let

$$(\tilde{q}, \tilde{u}) \in C^2([0,T]; \mathcal{Q}_\delta) \times [LL(T) \cap C^0([0,T]; W^{1,p}(\mathcal{F}(t)))], \text{ for all } p \in [1, +\infty),$$

and $\tilde{\gamma}$ in $\mathbb{R}^N$ such that for any t in $[0,T]$, $(\tilde{q}'(t), \tilde{q}''(t), \tilde{\gamma})$ belongs to $\mathscr{K}$ and $\operatorname{curl} \tilde{u}(t,\cdot)$ is in $\mathscr{B}(\tilde{q}(t), r_\omega)$. Assume that $(\tilde{q}, \tilde{u})$ satisfies: the Euler equations (1.1)-(1.2)*, the Newton equations* (1.3)-(1.4) *for κ in $\{1, 2, \ldots, N\}$, the interface condition* (1.5)*, the boundary condition* (2.2) *on the normal velocity with*

$$g(t) = \mathscr{C}(\tilde{q}(t), \tilde{q}'(t), q''(t), \tilde{\gamma}, \operatorname{curl} \tilde{u}(t,\cdot)), \tag{3.3}$$

the boundary condition (2.4) *on the entering vorticity, the circulation conditions* (2.5) *(with $\tilde{\gamma}$ in place of γ) and the initial conditions $\tilde{q}(0) = q(0)$ and $\tilde{q}'(0) = q'(0)$ on the initial positions and velocities of the rigid bodies. Then $\tilde{q} = q$ on $[0,T]$.*

We note that there is a slight abuse of notation in writing $C^0([0,T]; W^{1,p}(\mathcal{F}(t)))$, for space of functions defined for each t in the fluid domain $\mathcal{F}(t)$. More precisely and more generally, for a functional space X of functions depending on the variable x, the notation $C^0([0,T]; X(\mathcal{F}(t)))$ stands for the space of functions defined for each t in the fluid domain $\mathcal{F}(t)$, and which can be extended to functions in $C^0([0,T]; X(\mathbb{R}^2))$. Furthermore, we require this regularity of the velocity field to insure that the trace is well-defined at $t = 0$.

A few further comments are in order.

Comparison with the controllability result in [21]**.** Theorem 3.1 extends the result in [21] where the exact controllability of a single rigid body immersed in a 2D irrotational perfect incompressible fluid from an initial position and velocity to a final position and velocity was investigated. There the control was already set on a non-empty open part of the external boundary and was obtained as a regularization of some time impulses. On the opposite Theorem 3.1 proves that it is possible to drive some rigid bodies along a given admissible trajectory by a control which is active all the time, while, in terms of the space variable, this control is also supported on a non-empty open part of the external boundary. Moreover the control in Theorem 3.1 has the convenience to be achieved as a feedback law, depending only on the instantaneous state of the fluid-rigid bodies system. Thus Theorem 3.1 provides a positive answer to the open problem mentioned in [21], in the wider setting where several rigid bodies and irrotational flows are considered. Of course Theorem 3.1 also implies controllability of the positions and velocities of the rigid bodies at final time, by considering a targeted trajectory with the desired final positions and velocities. For instance, in view of practical applications, one may think at a regrouping of the rigid bodies in a given subregion of the domain, with enough volume to contain them all with positive distances. In the opposite direction one may think at a spreading of the rigid bodies in the fluid domain, thinking at a medical treatment which requires dispersion of some medicinal particles.

Hence this provides an extension to the main result of [21], but on the other hand, since the control can be active all the time in the result of Theorem 3.1, it is not possible to guarantee a small total flux condition as we did in [21, Remark 1] by a simple rescaling in time.

Uniqueness part (second part) of Theorem 3.1. In the case of an L^∞ vorticity, a uniqueness result for the fluid-solid system has been obtained in [27] in the case without control, that is, of impermeable boundary condition (vanishing normal component) on the whole external boundary, rather than the permeable boundary conditions (2.2) and (2.4). For the latter, uniqueness in the setting of bounded vorticity is a delicate issue, already in the case of a fluid alone. Indeed, in contrast to his celebrated result in the impermeable case [53], Yudovich only succeeded to obtain uniqueness for solutions which are much more regular in [52]. Recently Weigant and Papin obtained in [51] the uniqueness of the solutions with bounded vorticity with a proof in the case of a rectangle with the flow entering on a lateral side and exiting on the opposite side. The extension of such a uniqueness result to the case of the fluid-rigid bodies system seems challenging, as it involves a free boundary problem and a more involved geometry. Hence this leaves the following open problem.

Open problem 3.2. *For any $T > 0$, for any $\delta > 0$, for any q_0 in $\mathcal{Q}_\delta$ and q_1 in $\mathbb{R}^{3N}$, for any initial vorticity ω_0 in $L^\infty(\mathcal{F}(q(0)))$ for any g in $C^\infty([0,T];\mathcal{C})$, for any γ in $\mathbb{R}^N$, does there exist at most one velocity field u in $LL(T) \cap C^0([0,T]; W^{1,p}(\mathcal{F}(t)))$, for all $p \in [1, +\infty)$, with $\operatorname{curl} u(0,\cdot) = \omega_0$ and for any t in $[0,T]$, $\operatorname{curl} u(t,\cdot)$ in $L^\infty(\mathcal{F}(q))$, and does there exist at most one q in $C^2([0,T];\mathcal{Q}_\delta)$ with the initial conditions $q(0) = q_0$ and $q'(0) = q_1$ on the initial positions and velocities of the rigid bodies, such that (q,u) satisfies for all t in $[0,T]$: the Euler equations* (1.1)-(1.2)*, the Newton equations* (1.3)-(1.4) *for κ in $\{1,2,\dots,N\}$, the interface condition* (1.5)*, the boundary condition* (2.2) *on the normal velocity, the null boundary condition* (2.4) *on the entering vorticity and the circulation conditions* (2.5) *?*

However the second part of Theorem 3.1 claims that, for a given bounded initial vorticity, should there be several solutions, the control would drive the rigid bodies of all these solutions along the targeted motion. On the other hand the fluid motions are not guaranteed to coincide, except in the very particular case where the initial vorticity ω_0 is identically null. Indeed, in this case, the regularity of the velocity field and the condition (2.4) entail that the vorticity is identically null for all times. It then follows from Section 5 that the corresponding velocity u is uniquely determined.

Let us emphasize that the control in (3.3) involves the acceleration $q''(t)$ of the targeted motion, rather than the acceleration $\tilde{q}''$ of the solution itself as in (3.2) for the part of the statement regarding the existence of a motion associated with the targeted trajectory q.

The result in the second part of Theorem 3.1 covers the case of solutions corresponding to distinct initial vorticities and velocity circulations. On the other hand if the initial positions and velocities of the rigid bodies do not match, that is if $(\tilde{q}(0), \tilde{q}'(0)) \neq (q(0), q'(0))$, the control law (3.2) will not guarantee any decay of the initial condition errors. Still the control law can be adapted to provide a stability result in the case where the sole assumption on the initial conditions is that $\tilde{q}(0)$ and $q(0)$ are sufficiently close and that the boundary condition (2.2) on the normal velocity is satisfied with

$$g(t) = \mathscr{C}(\tilde{q}(t), \tilde{q}'(t), q''(t) + K_P(q(t) - \tilde{q}(t)) + K_D(q'(t) - \tilde{q}'(t)), \tilde{\gamma}, \operatorname{curl} \tilde{u}(t,\cdot)), \tag{3.4}$$

where K_P and K_D are positive definite, symmetric $3N \times 3N$ matrices. Then the error $q(t) - \tilde{q}(t)$ exponentially decays to 0 as the time t goes to $+\infty$, with a rate which can be made arbitrarily fast by appropriate choices of K_P and K_D. The indexes P and D in the notations K_P and K_D respectively refer to "proportional" and "derivative", according to the usual terminology in robotics, see [44, Chapter 11], [47, Section 4.5] and [46, Section 8.5]. This stability result will be proved by a little modification of the proof of the second part of Theorem 3.1. The interest of such a result is that in practice the positions and velocities of the rigid bodies cannot be determined exactly, see [6, 7, 43].

Regularity issues. A natural issue is whether it is possible to preserve the regularity of the vorticity when time proceeds while achieving the targeted motion. In that case it could be possible to adjust the boundary condition on the entering vorticity, by substituting to (2.4) an appropriate inhomogeneous condition. Such a construction was performed in [9] in the case of the controllability of a fluid alone by means of an open-loop control.

On the opposite direction, one may wonder whether it is possible to extend the existence part of Theorem 3.1 to the case where the initial vorticity is only in L^p with $p \geqslant 1$. Let us recall in that direction that, on the one hand, some existence results, in the case of impermeable boundary condition, rather than the conditions (2.2) and (2.4), have been obtained in [25, 26, 22, 49, 50] in the case of systems coupling one single rigid body and a two-dimensional perfect incompressible flow without any external boundary; and, on the other hand, some existence results have been obtained in [4, 41, 40] in the case of boundary conditions such as (2.2) and (2.4), but for a fluid alone.

One may also wonder whether it is possible to reach some targeted trajectories with lower regularity in time, by the means of controls g which are also of lower regularity in time than continuous. Indeed this is very much related to the strategy of [21] where impulsive controls were considered.

Energy saving. A natural question is whether it is possible to turn off the control when the targeted motion is, at some time, solution of the uncontrolled equation, that is to guarantee that the mapping $\mathscr{C}$ vanishes on (q, q', q'', ω) satisfying (1.1)-(2.3), (1.3)-(1.4) for κ in $\{1, 2, \dots, N\}$, (1.5), (2.2) with $g = 0$ and (2.4). Some extra comments on this issue are given in Section 10, after the proof of Theorem 3.1; this postponing allows us to be more precise regarding some technical aspects of the question.

Three dimensional case. Another natural extension of the results of Theorem 3.1 is the case where the system is set in three space dimensions. In the impermeable/uncontrolled case a reformulation of the Newton equations as a second-order ODE for the solid positions is tackled in [28]. However the design of the control below relies on the possibility to use complex analysis in two dimensions. Therefore several arguments of the proof of Theorem 3.1 regarding the construction of the control law would need to be adapted.

Controlled collisions. A challenging question is whether it is to possible to provoke some controlled collisions. Let us recall that collisions can occur even without control, see [32, 33, 13]. However one may imagine to be able, given a couple of rigid bodies, some collision positions —and perhaps also some collision velocities— to prove the existence of a control such that for the corresponding controlled solution of (1.1)-(2.3), (1.3)-(1.4) for κ in $\{1, 2, \dots, N\}$, (1.5), this collision occurs. In this direction let us mention that in the proof of Theorem 3.1 we will use certain arguments of complex analysis, see also [21], which involve approximations by rational functions of harmonic functions which are first defined in some respective neighborhoods of the rigid bodies. When two rigid bodies become close one could face some difficulty in the fusion of such local approximations, see [17].

Case of the Navier-Stokes equations. One could also be interested in extending the results of Theorem 3.1 to the case of the Navier-Stokes equations rather than the Euler equations as a model for the fluid part of the system. Then the boundary conditions have to be modified and several choices can be made. Beside the classical no-slip conditions, there is a condition which models slip and friction at the boundaries referred to here as Navier slip-with-friction boundary conditions. The latter case is closer to the case of the Euler equation where slip, that is discrepancy of the tangential velocity at the boundary, is allowed. Regarding the Cauchy problem, in the uncontrolled case, the existence of weak Leray-type solutions to the Navier-Stokes system in presence of a rigid body when the Navier slip-with-friction conditions are considered at the boundaries has been proved in [45] when the system occupies the whole space and in [29] when the system occupies a bounded domain. These two results tackle the three-dimensional case but the latter result has been adapted in [3] to the two-dimensional case with some extra properties. For strong solutions, existence and uniqueness of solutions in some Hilbert spaces have been proved in [54].

On the other hand the result in [21], on the exact controllability of a single rigid body immersed in a 2D irrotational perfect incompressible fluid mentioned above was extended in [35] to the Navier-Stokes equations in the case where the Navier slip-with-friction boundary conditions are prescribed on the interface between the fluid and the body, see also [14, 15] for complementary results. One key ingredient was a rescaling in time which allows to reduce the problem to the case where the viscosity is small (first introduced by Coron in [10]), to use an asymptotic expansion and the inviscid result of [21] for the leading order. This strategy works for the Navier slip-with-friction boundary conditions because the corresponding boundary layers have a small amplitude. Unfortunately, for the problem of trajectory tracking considered in this paper, one is not allowed to effectuate such a time-rescaling.

However it is possible that the method used in the proof of Theorem 3.1 could be adapted to the case the Navier-Stokes equations, with the Navier slip-with-friction boundary conditions. The case of the Navier-Stokes equations, with the no-slip boundary conditions is clearly more challenging.

A nonlinear method reminiscent of Coron's return method. To prove Theorem 3.1 we will make use of a nonlinear method which is reminiscent of Coron's return method, cf. [12, Chapter 6], in the sense that it takes advantage of the nonlinearity of the problem out of equilibria. However our method rather considers time as a parameter and allows us to prove a trajectory tracking result rather than a controllability result. It uses the homogeneity of the nonlinear part of a nonlinear equation and the existence of a single non-trivial zero at which the differential of this term is right-invertible to solve the equation for general data, see Section 7.1. One could also compare to Coron's Phantom tracking method from [11], which takes advantage of the nonlinearity in a similar fashion in order to establish a stabilization result.

Practical use. An attempt to put in practice the theoretical result in Theorem 3.1 would face the drawback of the feedback laws (3.3) and (3.2) depend on the full state-function $\omega(t,\cdot)$, rather than on only some norms, moments or any finite dimensional information extracted from it. However the Lipschitz dependence leads to the hope that a bad identification of the vorticity of the fluid by the operator in charge to apply the control at the boundary may not affect the resulting controlled trajectory too drastically. Another difficulty is linked to the design of this control law by itself. Indeed, in this direction, for a quite important part of the analysis performed below in the proof of Theorem 3.1 the observations done in [31] for a slightly different problem are also relevant. There the authors discuss an alternative method to the complex-analytic one which is developed here. This method is more application-friendly. However this alternative method relies on linear techniques which seem difficult to adapt here.

4. Organisation of the rest of the paper.

The rest of the paper is organised as follows. In Section 5 we recall the decomposition of the fluid velocity into elementary velocities according to the vorticity, the circulations, the external boundary control and the velocities of the rigid bodies. Then we reformulate in Section 6 the solid equations as an equation with the control as the unknown, and the solid motion and the vorticity as data. In Section 7 we design the feedback control. Section 8 is devoted to the end of the proof of the first part of Theorem 3.1 regarding the existence of a controlled solution with the targeted motion of the rigid bodies. Then, Section 9 is devoted to the end of the proof of the second part of Theorem 3.1 regarding the uniqueness of the motion of the rigid bodies for the hybrid control law (3.3). Finally in Section 10 we give some extra comments on the issue of energy saving discussed above.

5. Decomposition of the fluid velocity according to the solids motions, the vorticity, the circulation and the external control

Let q in $\mathcal{Q}$. For any q' in $\mathbb{R}^{3N}$, for any ω bounded over $\mathcal{F}(q)$, for any $\gamma := (\gamma_\kappa)_{\kappa=1,\dots,N}$ in $\mathbb{R}^N$, for any g in $\mathcal{C}$, classically there exists a unique log-Lipschitz vector field u such that

$$\operatorname{div} u = 0 \text{ in } \mathcal{F}(q), \quad \operatorname{curl} u = \omega \text{ in } \mathcal{F}(q), \quad u \cdot n = g \text{ on } \partial\Omega, \tag{5.1a}$$

(5.1b)

$$u \cdot n = \big(\theta'_\kappa(\cdot - h_\kappa)^\perp + h'_\kappa\big) \cdot n \text{ on } \partial\mathcal{S}_\kappa(t) \text{ and } \int_{\partial\mathcal{S}_\kappa(t)} u(t)\cdot\tau \,\mathrm{d}s = \gamma_\kappa, \quad \text{for all } \kappa \in \{1,2,\dots,N\}.$$

The circulations conditions above are important to guarantee the uniqueness of the system (5.1); this is related to the Hodge-De Rham theory. See for example Kato [34].

We now decompose the vector field u in several elementary contributions which convey the influence of the vorticity, of the circulations, of the external boundary control and of the velocities of the rigid bodies.

5.1. **Kirchhoff potentials.** Consider for any κ in $\{1, 2, \dots, N\}$ the functions $\xi_{\kappa,j}(q,\cdot) = \xi_k(q,\cdot)$, for $j = 1, 2, 3$ and $k = 3(\kappa - 1) + j$, defined by $\xi_{\kappa,j}(q,x) := 0$ on $\partial\mathcal{F}(q) \setminus \partial\mathcal{S}_\kappa$ and by $\xi_{\kappa,j}(q,x) := e_j$, for $j = 1, 2$, and $\xi_{\kappa,3}(q,x) := (x - h_\kappa)^\perp$ on $\partial\mathcal{S}_\kappa$. Above e_1 and e_2 are the unit vectors of the canonical basis.

We denote by $K_{\kappa,j}(q,\cdot) = K_k(q,\cdot)$ the normal trace of $\xi_{\kappa,j}$ on $\partial\mathcal{F}(q)$, that is: $K_{\kappa,j}(q,\cdot) := n \cdot \xi_{\kappa,j}(q,\cdot)$ on $\partial\mathcal{F}(q)$, where as before n denotes the unit normal vector pointing outside $\mathcal{F}(q)$.

We introduce the Kirchhoff potentials $\varphi_{\kappa,j}(q,\cdot) = \varphi_k(q,\cdot)$, for $j = 1, 2, 3$ and $k = 3(\kappa - 1) + j$, as the unique (up to an additive constant) solutions in $\mathcal{F}(q)$ of the following Neumann problem:

$$\Delta\varphi_{\kappa,j} = 0 \qquad \text{in } \mathcal{F}(q), \tag{5.2a}$$

$$\frac{\partial\varphi_{\kappa,j}}{\partial n}(q,\cdot) = K_{\kappa,j}(q,\cdot) \qquad \text{on } \partial\mathcal{F}(q). \tag{5.2b}$$

We also denote

$$\boldsymbol{K}_\kappa(q,\cdot) := (K_{\kappa,1}(q,\cdot), K_{\kappa,2}(q,\cdot), K_{\kappa,3}(q,\cdot))^t \text{ and } \boldsymbol{\varphi}_\kappa(q,\cdot) := (\varphi_{\kappa,1}(q,\cdot), \varphi_{\kappa,2}(q,\cdot), \varphi_{\kappa,3}(q,\cdot))^t. \tag{5.3}$$

Following the same rules of notation as for q, see (1.6), we define the function $\varphi(q,\cdot)$ by concatenating into a vector of length $3N$ the functions $\boldsymbol{\varphi}_\kappa(q,\cdot)$, namely:

$$\varphi(q,\cdot) := (\boldsymbol{\varphi}_1(q,\cdot)^t, \dots, \boldsymbol{\varphi}_N(q,\cdot)^t)^t.$$

Let us also recall that the Kirchhoff potentials are involved in the so-called added inertia of the rigid bodies, through the formula

$$\mathcal{M}^a_{\kappa,\nu}(q) := \int_{\partial\mathcal{S}_\kappa(q)} \boldsymbol{\varphi}_\nu(q,\cdot) \otimes \frac{\partial\boldsymbol{\varphi}_\kappa}{\partial n}(q,\cdot)\,\mathrm{d}s. \tag{5.4}$$

It follows from integrations by parts that these matrices are symmetric and non negative, since

$$\mathcal{M}^a_{\kappa,\nu}(q) = \int_{\mathcal{F}(q)} \nabla\boldsymbol{\varphi}_\nu(q,\cdot) \otimes \nabla\boldsymbol{\varphi}_\kappa(q,\cdot)\,\mathrm{d}s. \tag{5.5}$$

The total added inertia matrix of the system reads:

$$\mathcal{M}^a(q) := \left(\mathcal{M}^a_{\kappa,\nu}(q)\right)_{1\leqslant\kappa,\nu\leqslant N}. \tag{5.6}$$

Notice that this matrix is in the set $S^+_{3N}(\mathbb{R})$ of the real symmetric positive-semidefinite $3N \times 3N$ matrices. These matrices depend on the shape of the fluid domain, therefore on the shape of the external boundaries and of the shape and position of the rigid bodies.

We also define for any $q \in \mathcal{Q}$, for any $p \in \mathbb{R}^{3N}$, the Christoffel symbols of the first kind

$$\langle\Gamma(q), p, p\rangle := \Big(\sum_{1\leqslant\kappa,\nu\leqslant 3N} (\Gamma(q))^\lambda_{\kappa,\nu}\, p_\kappa p_\nu\Big)_{1\leqslant\lambda\leqslant 3N} \in \mathbb{R}^{3N}, \tag{5.7}$$

with for every $i, k, j \in \{1, \dots, 3N\}$,

$$(\Gamma(q))^i_{k,j} := \frac{1}{2}\Big(\frac{\partial(\mathcal{M}^a(q))_{k,i}}{\partial q_j} + \frac{\partial(\mathcal{M}^a(q))_{j,i}}{\partial q_k} - \frac{\partial(\mathcal{M}^a(q))_{k,j}}{\partial q_i}\Big).$$

The smoothness of the mapping $(q,p,p) \mapsto \langle\Gamma(q), p, p\rangle$ follows from classical results on the shape derivatives of the Laplace problem for which we refer for example to [30].

5.2. **Stream functions for the circulation.** To account for the velocity circulations around the solids, we introduce for each κ in $\{1,\dots,N\}$ the stream function $\psi_\kappa = \psi_\kappa(q,\cdot)$ defined on $\mathcal{F}(q)$ as the harmonic vector field which has circulation $\delta_{\kappa,\nu}$ around $\partial\mathcal{S}_\nu(q)$, for each ν in $\{1,\dots,N\}$. More precisely, for every q, one can show that there exists a unique family $(C_{\kappa,\nu}(q))_{\nu\in\{1,2,\dots,N\}}$ in $\mathbb{R}^N$ such that the unique solution $\psi_\kappa(q,\cdot)$ of the Dirichlet problem:

$$\Delta\psi_\kappa(q,\cdot) = 0 \quad \text{in } \mathcal{F}(q) \tag{5.8a}$$

$$\psi_\kappa(q,\cdot) = C_{\kappa,\nu}(q) \quad \text{on } \partial\mathcal{S}_\nu(q), \text{ for } \nu \in \{1,2,\dots,N\}, \tag{5.8b}$$

$$\psi_\kappa(q,\cdot) = 0 \quad \text{on } \partial\Omega, \tag{5.8c}$$

satisfies

$$\int_{\partial\mathcal{S}_\nu(q)} \frac{\partial\psi_\kappa}{\partial n}(q,\cdot)\mathrm{d}s = -\delta_{\kappa,\nu}, \text{ for } \nu \in \{1,2,\dots,N\}, \tag{5.8d}$$

where $\delta_{\nu,\kappa}$ is the Kronecker symbol. As before, we define the concatenation into a vector of length N:

$$\psi(q,\cdot) := (\psi_1(q,\cdot),\dots,\psi_N(q,\cdot))^t.$$

5.3. **Hydrodynamic stream function.** For every bounded scalar function ω over $\mathcal{F}(q)$, there exists a unique family $(C_{\omega,\nu}(q))_{\nu\in\{1,2,\dots,N\}} \in \mathbb{R}^N$ such that the unique solution $\psi_\omega(q,\cdot)$ in $H^1(\mathcal{F}(q))$ of:

$$\Delta\psi_\omega(q,\cdot) = \omega \quad \text{in } \mathcal{F}(q) \tag{5.9a}$$

$$\psi_\omega(q,\cdot) = C_{\omega,\nu}(q) \quad \text{on } \partial\mathcal{S}_\nu(q), \text{ for } \nu \in \{1,2,\dots,N\}, \tag{5.9b}$$

$$\psi_\omega(q,\cdot) = 0 \quad \text{on } \partial\Omega, \tag{5.9c}$$

satisfies

$$\int_{\partial\mathcal{S}_\nu(q)} \frac{\partial\psi_\omega}{\partial n}(q,\cdot)\mathrm{d}s = 0, \text{ for } \nu \in \{1,2,\dots,N\}. \tag{5.9d}$$

It is classical that $\nabla^\perp\psi_\omega$ has log-Lipschitz regularity (see again [34] for instance).

We gather the stream functions due to the fluid vorticity and to the circulations by setting

$$\psi_{\omega,\gamma}(q,\cdot) := \psi_\omega(q,\cdot) + \psi(q,\cdot)\cdot\gamma. \tag{5.10}$$

5.4. **Potential due to the external control.** With any q in $\mathcal{Q}$ and g in $\mathcal{C}$ we associate

$$\alpha := \mathcal{A}[q,g] \in C^\infty(\overline{\mathcal{F}(q)};\mathbb{R}), \tag{5.11}$$

the unique solution to the following Neumann problem:

$$\Delta\alpha = 0 \text{ in } \mathcal{F}(q) \quad \text{and} \quad \partial_n\,\alpha = g\mathbb{1}_\Sigma \text{ on } \partial\mathcal{F}(q), \tag{5.12}$$

with zero mean on $\mathcal{F}(q)$ (recall (2.1)), where $\mathbb{1}_\Sigma$ is the indicator function of the set Σ. This zero mean condition allows to determine a unique solution to the Neumann problem but plays no role in the sequel.

5.5. **Decomposition of the velocity.** Now, by the linearity of System (5.1), we see that the unique solution u to (5.1) can be decomposed into

$$u = u_f + u_c, \text{ with } u_f := \sum_{\kappa=1}^N \nabla(\boldsymbol{\varphi}_\kappa(q,\cdot)\cdot\boldsymbol{q}'_\kappa) + \nabla^\perp\psi_{\omega,\gamma}(q,\cdot) \text{ and } u_c := \nabla\alpha. \tag{5.13}$$

Above the index "f" stands for free and the index 'c" for controlled.

6. Reformulation of the Newton equations as a quadratic equation for the control

This section is devoted to the reformulation of the solid equations in terms of the control, of the solid variables and of the vorticity.

To obtain this reformulation, we introduce test functions as follows. For each integer κ between 1 and N, q in $\mathcal{Q}$, ℓ^*_κ in $\mathbb{R}^2$ and r^*_k in $\mathbb{R}$, we consider the following potential vector field $\mathcal{F}(q)$:

$$u^*_\kappa := \nabla(\boldsymbol{\varphi}_\kappa(q,\cdot)\cdot p^*_\kappa), \quad \text{where} \quad p^*_\kappa := (\ell^{*t}_\kappa, r^*_\kappa)^t.$$

By (5.2a), we have that

$$u^*_\kappa \cdot n = \delta_{\kappa,\nu}(\ell^*_\kappa + r^*_\kappa(\cdot - h_\kappa)^\perp)\cdot n \ \text{ on } \ \partial\mathcal{S}_\nu(q) \quad \text{and} \quad u^*_\kappa\cdot n = 0 \ \text{ on } \ \partial\Omega.$$

By (1.3), (1.4), (5.2b), the fact that u^*_κ is divergence-free in $\mathcal{F}(q)$, an integration by parts and (1.1), we have

$$m_\kappa h''_\kappa \cdot \ell^*_\kappa + \mathcal{J}_\kappa \theta''_\kappa r^*_\kappa = \int_{\mathcal{F}(q)} \nabla\pi\cdot u^*_\kappa \,\mathrm{d}x = -\int_{\mathcal{F}(q)} \left(\frac{\partial u}{\partial t} + \frac{1}{2}\nabla|u|^2 + \omega u^\perp\right)\cdot u^*_\kappa \,\mathrm{d}x. \tag{6.1}$$

We introduce the global test function

$$u^* := \sum_{1\leqslant\kappa\leqslant N} u^*_\kappa \quad \text{for} \quad p^* := (p^*_\kappa)_{1\leqslant\kappa\leqslant N} \in \mathbb{R}^{3N},$$

and the genuine mass matrix $\mathcal{M}^g$ defined as the positive definite diagonal $3N\times 3N$ matrix

$$\mathcal{M}^g := \operatorname{diag}(\mathcal{M}^g_1,\dots,\mathcal{M}^g_N) \quad \text{with} \quad \mathcal{M}^g_\kappa := \operatorname{diag}(m_\kappa, m_\kappa, \mathcal{J}_\kappa).$$

Summing (6.1) over all indices κ and using the decomposition (5.13), we therefore obtain:

$$\begin{aligned}\int_{\mathcal{F}(q)} \frac{\partial u_c}{\partial t}\cdot u^*\,\mathrm{d}x + \int_{\mathcal{F}(q)} \left(\frac{1}{2}\nabla|u_c|^2\right)\cdot u^*\,\mathrm{d}x + \int_{\mathcal{F}(q)} \nabla(u_f\cdot u_c)\cdot u^*\,\mathrm{d}x + \int_{\mathcal{F}(q)} \omega u_c^\perp\cdot u^*\,\mathrm{d}x \\ = -\mathcal{M}^g q''\cdot p^* - \int_{\mathcal{F}(q)}\left(\frac{\partial u_f}{\partial t} + \frac{1}{2}\nabla|u_f|^2\right)\cdot u^*\,\mathrm{d}x - \int_{\mathcal{F}(q)} \omega u_f^\perp \cdot u^*\,\mathrm{d}x.\end{aligned} \tag{6.2}$$

We now reformulate each term starting with the right-hand side, and then handle the terms in the left-hand side of (6.2), where the terms involving u_c are regrouped.

- We first turn to the right-hand side of (6.2), for which the analysis of [23] can be applied. Proposition 6.1 below gathers the properties which will be useful in the sequel. It involves the matrix $\mathfrak{A}$ defined by

$$\mathfrak{A}(q,\omega,\gamma) := \Big(\mathfrak{A}_{\kappa,\nu}(q,\omega,\gamma)\Big)_{1\leqslant\kappa,\nu\leqslant N},$$

with

$$\begin{aligned}\mathfrak{A}_{\kappa,\nu}(q,\omega,\gamma) := &\int_{\partial\mathcal{S}_\kappa(q)} \frac{\partial\psi_{\omega,\gamma}}{\partial n}(q,\cdot)\left(\frac{\partial\boldsymbol{\varphi}_\kappa}{\partial n}\otimes\frac{\partial\boldsymbol{\varphi}_\nu}{\partial\tau}\right)(q,\cdot)\,\mathrm{d}s \\ &- \int_{\partial\mathcal{S}_\nu(q)} \frac{\partial\psi_{\omega,\gamma}}{\partial n}(q,\cdot)\left(\frac{\partial\boldsymbol{\varphi}_\kappa}{\partial\tau}\otimes\frac{\partial\boldsymbol{\varphi}_\nu}{\partial n}\right)(q,\cdot)\,\mathrm{d}s,\end{aligned}$$

where we recall the notations (5.10) for the stream function $\psi_{\omega,\gamma}$ and (5.2)-(5.3), for the Kirchhoff potentials. We observe that the matrix $\mathfrak{A}$ is skew-symmetric and linear with respect to ω and γ.

We also define:

$$\mathfrak{E}(q,\omega,\gamma) := (\mathfrak{E}_1(q,\omega,\gamma),\dots,\mathfrak{E}_N(q,\omega,\gamma))^t,$$

where

$$\mathfrak{E}_\kappa(q,\omega,\gamma) := -\frac{1}{2}\int_{\partial\mathcal{S}_\kappa(q)} \left(\left|\frac{\partial\psi_{\omega,\gamma}}{\partial n}\right|^2 \frac{\partial\boldsymbol{\varphi}_\kappa}{\partial n}\right)(q,\cdot)\,\mathrm{d}s,$$

and

$$\mathfrak{D}(q,q',\omega,\gamma) := (\mathfrak{D}_1(q,q',\omega,\gamma),\dots,\mathfrak{D}_{3N}(q,q',\omega,\gamma))^t,$$

where

$$\mathfrak{D}_k(q,q',\omega,\gamma) := \int_{\mathcal{F}(q)} \omega u_f^\perp(q,q',\omega,\gamma,\cdot)\cdot\nabla\varphi_k(q,\cdot)\,\mathrm{d}x,$$

recalling the definition of u_f in (5.13). Finally we set

$$\mathfrak{F}(q,q',q'',\gamma,\omega) := -\big(\mathcal{M}^g + \mathcal{M}^a(q)\big)q'' - \langle\Gamma(q),q',q'\rangle + \mathfrak{E}(q,\omega,\gamma) + \mathfrak{A}(q,\omega,\gamma)q' + \mathfrak{D}(q,q',\omega,\gamma), \tag{6.3}$$

where we recall (5.4) and (5.7).

Proposition 6.1. *The mapping $\mathfrak{F}$ is Lipschitz on $\mathcal{Q}\times\mathbb{R}^{3N}\times\mathbb{R}^{3N}\times\mathbb{R}^N\times\mathscr{B}(q,r_\omega)$, for any $r_\omega>0$ (recall that $\mathscr{B}(q,r_\omega)$ is defined in* (3.1)*), and*

$$\mathcal{M}^g q''\cdot p^* + \int_{\mathcal{F}(q)}\left(\frac{\partial u_f}{\partial t} + \frac{1}{2}\nabla|u_f|^2\right)\cdot u^*\,\mathrm{d}x - \int_{\mathcal{F}(q)} \omega u_f^\perp\cdot u^*\,\mathrm{d}x = -\mathfrak{F}(q,q',q'',\gamma,\omega)\cdot p^*. \tag{6.4}$$

- Let us now consider the terms in the left-hand side of (6.2). First, by Leibniz's formula and Reynolds' transport formula, observing that the fluid domain is transported by the vector field u_f, we have

$$\begin{aligned}\int_{\mathcal{F}(q)}\frac{\partial u_c}{\partial t}\cdot u^*\,\mathrm{d}x &= \int_{\mathcal{F}(q)}\frac{\partial(u_c\cdot u^*)}{\partial t}\,\mathrm{d}x - \int_{\mathcal{F}(q)} u_c\cdot\frac{\partial u^*}{\partial t}\,\mathrm{d}x\\ &= \frac{d}{dt}\left(\int_{\mathcal{F}(q)} u_c\cdot u^*\,\mathrm{d}x\right) - \int_{\mathcal{F}(q)} u_f\cdot\nabla\left(u_c\cdot u^*\right)\,\mathrm{d}x - \int_{\mathcal{F}(q)} u_c\cdot\frac{\partial u^*}{\partial t}\,\mathrm{d}x.\end{aligned}$$

Then we integrate by parts the first two first integrals in the right hand side above and we compute the last one by using the shape derivatives of the Kirchhoff potentials. We obtain

$$\begin{aligned}\int_{\mathcal{F}(q)}\frac{\partial u_c}{\partial t}\cdot u^*\,\mathrm{d}x = \frac{d}{dt}&\left(\left(\int_{\partial\mathcal{S}_\kappa(q)}\alpha\,\partial_n\boldsymbol{\varphi}_\kappa(q,\cdot)\mathrm{d}s\right)_{1\leqslant\kappa\leqslant N}\cdot p^*\right)\\ &- \left(\left(\int_{\partial\mathcal{S}_\kappa(q)}\nabla\alpha\cdot\nabla\boldsymbol{\varphi}_\nu\,\partial_n\boldsymbol{\varphi}_\kappa(q,\cdot)\mathrm{d}s\right)_{1\leqslant\kappa\leqslant N}\cdot q'\right)_{1\leqslant\nu\leqslant N}\cdot p^*\\ &\qquad - \left(\left(\int_{\mathcal{F}(q)}\nabla\alpha\cdot\partial_{\boldsymbol{q}_\kappa}\nabla\boldsymbol{\varphi}_\nu(q,\cdot)\mathrm{d}x\right)_{1\leqslant\kappa\leqslant N}\cdot q'\right)_{1\leqslant\nu\leqslant N}\cdot p^*.\end{aligned} \tag{6.5}$$

Let us highlight right here that the first term in the right hand side above is discarded below by an additional assumption on the control.

We now consider the second and third terms in the left-hand side of (6.2). By integrations by parts, we obtain

$$\int_{\mathcal{F}(q)} \left(\frac{1}{2}\nabla|u_c|^2\right) \cdot u^* \,\mathrm{d}x = \frac{1}{2}\left(\int_{\partial\mathcal{S}_\kappa(q)} |\nabla\alpha|^2\, \partial_n \boldsymbol{\varphi}_\kappa(q,\cdot)\,\mathrm{d}s\right)_{1\leqslant\kappa\leqslant N} \cdot p^*, \tag{6.6}$$

$$\int_{\mathcal{F}(q)} \nabla(u_f \cdot u_c) \cdot u^* \,\mathrm{d}x = \left(\int_{\partial\mathcal{S}_\kappa(q)} (\nabla\alpha \cdot u_f)\, \partial_n \boldsymbol{\varphi}_\kappa(q,\cdot)\, ds\right)_{1\leqslant\kappa\leqslant N} \cdot p^*. \tag{6.7}$$

Concerning the last term in the left-hand side of (6.2), we simply decompose:

$$\int_{\mathcal{F}(q)} \omega u_c^\perp \cdot u^* \,\mathrm{d}x = \left(\int_{\mathcal{F}(q)} \omega \nabla^\perp\alpha \cdot \nabla\boldsymbol{\varphi}_k(q,\cdot)\,\mathrm{d}x\right)_{1\leqslant\kappa\leqslant N} \cdot p^*. \tag{6.8}$$

Accordingly to the computations above we introduce the following notations. For q in $\mathcal{Q}$ and g in $\mathcal{C}$, we set

$$\mathfrak{Q}(q)[g] := \frac{1}{2}\left(\int_{\partial\mathcal{S}_\kappa(q)} |\nabla\alpha|^2\, \partial_n \boldsymbol{\varphi}_\kappa(q,\cdot)\,\mathrm{d}s\right)_{1\leqslant\kappa\leqslant N}, \tag{6.9}$$

and recalling the notation $\alpha = \mathcal{A}[q,g]$ of Section 5.4, we set

$$\begin{aligned}\mathfrak{L}(q,q',\gamma,\omega)[g] := &\left(\int_{\mathcal{F}(q)} \omega\nabla^\perp\alpha \cdot \nabla\boldsymbol{\varphi}_\kappa(q,\cdot)\,\mathrm{d}x\right)_{1\leqslant\kappa\leqslant N} - \left(\int_{\partial\mathcal{S}_\kappa(q)} (\nabla\alpha\cdot u_f)\,\partial_n\boldsymbol{\varphi}_\kappa(q,\cdot)\,\mathrm{d}s\right)_{1\leqslant\kappa\leqslant N} \\ &+ \left(\left(\int_{\partial\mathcal{S}_\kappa(q)} \nabla\alpha\cdot\nabla\boldsymbol{\varphi}_\kappa\,\partial_n\boldsymbol{\varphi}_\nu(q,\cdot)\mathrm{d}s\right)_{1\leqslant\nu\leqslant N} \cdot q'\right)_{1\leqslant\kappa\leqslant N} \\ &+ \left(\left(\int_{\mathcal{F}(q)} \nabla\alpha\cdot\partial_{\boldsymbol{q}_\nu}\nabla\boldsymbol{\varphi}_\kappa(q,\cdot)\mathrm{d}x\right)_{1\leqslant\nu\leqslant N} \cdot q'\right)_{1\leqslant\kappa\leqslant N}.\end{aligned} \tag{6.10}$$

Additional assumption on the control. In the sequel we will make the following additional assumption on the controls that we consider, in order to eliminate the first term in the right hand side of (6.5). Let us first recall that the space $\mathcal{C}$ is defined in (2.1). Now given q in $\mathcal{Q}_\delta$, we define the set

$$\mathcal{C}_b(q) := \left\{ g\in\mathcal{C},\ \int_{\mathcal{S}_\kappa(q)} \mathcal{A}[q,g]\,\partial_n\boldsymbol{\varphi}_\kappa(q,\cdot)\mathrm{d}s = 0 \ \text{ for } \ 1\leqslant\kappa\leqslant N\right\}. \tag{6.11}$$

To obtain Theorem 3.1, we will consider controls g in this set $\mathcal{C}_b$. In this case when g is in $\mathcal{C}_b(q)$, by (6.5), (6.6), (6.7), (6.8) and (6.4), the equation (6.2) now reads

$$\mathfrak{Q}(q)[g] + \mathfrak{L}(q,q',\gamma,\omega)[g] = \mathfrak{F}(q,q',q'',\gamma,\omega). \tag{6.12}$$

Conclusion. Therefore, under the assumption $g\in\mathcal{C}_b(q)$, the Euler equations (1.1)-(1.2), the Newton equations (1.3)-(1.4) for κ in $\{1,2,\dots,N\}$, the interface condition (1.5), and the boundary conditions (2.2) with $g(t)$ in $\mathcal{C}_b(q(t))$ for every t in $[0,T]$, are equivalent to the problem

$$\begin{cases} \mathfrak{Q}(q)[g] + \mathfrak{L}(q,q',\gamma,\omega)[g] = \mathfrak{F}(q,q',q'',\gamma,\omega), \\ \partial_t\omega + \left(\sum_{\kappa=1}^N \nabla(\boldsymbol{\varphi}_\kappa(q,\cdot)\cdot\boldsymbol{q}'_\kappa) + \nabla^\perp\psi_{\omega,\gamma}(q,\cdot) + \nabla\mathcal{A}[q,g]\right)\cdot\nabla\omega = 0 \text{ in } \mathcal{F}(q).\end{cases} \tag{6.13}$$

Remark 6.2. *Three comments are in order.*

- *Above it is understood that, in the converse way, the fluid velocity is recovered by the equation* (5.13). *That it satisfies the Euler equations* (1.1) *for a pressure field in* $L^\infty(0,T;H^1(\mathcal{F}(t)))$, *which is unique up to a function depending only on time, follows from the second equation of* (6.13) *and the property of the* curl *operator. On the other hand it follows from* (5.13) *that it satisfies the divergence free condition* (1.2), *the interface condition* (1.5), *and the boundary conditions* (2.2) *with* $g(t)$ *in* $\mathcal{C}_b(q(t))$ *for every* t *in* $[0,T]$.

- *The system* (6.13) *will be useful in the proof of the existence part of Theorem 3.1 in Section 8, while* (6.12) *alone will be used in the proof of the uniqueness part of Theorem 3.1 in Section 9.*

- *The unknowns of the problem* (6.13) *are* g *and* ω, *and one may observe that this system is completely coupled in the sense that* g *and* ω *are involved in both equations. Still we will tackle these two equations separately. In Section 7 we will start by proving the existence of a solution of the first equation of* (6.13) *for the unknown* g *in terms of* ω *considered as a parameter. Then in Section 8 we will solve the second equation of* (6.13) *for* ω *with* g *given by the solution of the first equation that is identified in Section 7.*

7. Design of a feedback control law

This section is devoted to the design of a control g (for the trace of normal velocity on the exterior boundary) on $[0,T]\times\Sigma$ of the form $g=\mathscr{C}(q,q',q'',\gamma,\omega)$, where $\mathscr{C}$ is a Lipschitz function on

$$\cup_{q\in\mathcal{Q}_\delta}\{q\}\times\mathscr{K}\times\mathscr{B}(q,r_\omega),$$

with values in $\mathcal{C}_b$ (defined in (6.11)), and aimed at fulfilling the first equation of (6.13). The second equation of (6.13) will be tackled in the next section.

Recall that $\mathscr{K}$ is a compact subset of $\mathbb{R}^{3N}\times\mathbb{R}^{3N}\times\mathbb{R}^N$ and $\mathscr{B}(q,r_\omega)$ is defined in (3.1). We further define the topological space

$$\mathscr{B}^*(q,r_\omega) := \overline{B}_{L^\infty(\mathcal{F}(q))}(0,r_\omega) \text{ endowed with the weak-}L^3(\mathcal{F}(q)) \text{ topology}.$$

Precisely in this section we show the following.

Proposition 7.1. *Let* $\delta>0$. *There exists a finite dimensional subspace* $\mathcal{E}\subset\mathcal{C}$ *and for any* $r_\omega>0$ *and any compact subset* $\tilde{\mathscr{K}}$ *of* $\mathbb{R}^{3N}\times\mathbb{R}^N$, *there exists a locally Lipschitz mapping*

$$\mathfrak{R}:\cup_{q\in\mathcal{Q}_\delta}\{q\}\times\tilde{\mathscr{K}}\times\mathscr{B}(q,r_\omega)\times\mathbb{R}^{3N}\longrightarrow\mathcal{E},\quad (q,q',\gamma,\omega,p^*)\longmapsto\mathfrak{R}(q,q',\gamma,\omega,p^*)\in\mathcal{E}\cap\mathcal{C}_b(q),$$

such that for any p^* *in* $\mathbb{R}^{3N}$,

$$\mathfrak{Q}(q)[\mathfrak{R}(q,q',\gamma,\omega,p^*)]+\mathfrak{L}(q,q',\gamma,\omega)[\mathfrak{R}(q,q',\gamma,\omega,p^*)]=p^*.$$

Furthermore, the map

$$\mathfrak{R}:\cup_{q\in\mathcal{Q}_\delta}\{q\}\times\tilde{\mathscr{K}}\times\mathscr{B}^*(q,r_\omega)\times\mathbb{R}^{3N}\longrightarrow\mathcal{E}$$

is continuous.

This proposition being granted, we will be able to design the control as follows. We set

$$\tilde{\mathscr{K}} := \{(q',\gamma),\ (q',q'',\gamma)\in\mathscr{K} \text{ for some } q''\}. \tag{7.1}$$

and we define $\mathscr{C}\in\mathrm{Lip}(\cup_{q\in\mathcal{Q}_\delta}\{q\}\times\mathscr{K}\times\mathscr{B}(q,r_\omega);\mathcal{E})$ by

$$\mathscr{C}(q,q',q'',\gamma,\omega) := \mathfrak{R}(q,q',\gamma,\omega,\mathfrak{F}(q,q',q'',\gamma,\omega)). \tag{7.2}$$

The Lipschitz regularity of $\mathscr{C}$ follows from the boundedness and regularity of $\mathfrak{F}$ mentioned in Proposition 6.1 and from the regularity of $\mathfrak{R}$ given by Proposition 7.1. We note that the last statement regarding the continuity with respect to the weak-L^3 topology for ω will be useful in proving the compactness of an appropriate operator in Section 8.6. It relies on the following observation on $\mathfrak{F}$.

Lemma 7.2. *The mapping $\mathfrak{F}$ is continuous with respect to the weak-L^3 topology for ω.*

Proof of Lemma 7.2. Referring to the definition (6.3) we observe that merely three terms depend on ω: $\mathfrak{A}$, $\mathfrak{E}$ and $\mathfrak{D}$. Suppose that the sequence (ω_n) of $\mathcal{B}(0, r_\omega)$ converges to $\omega^\star$ in the L^3-weak topology. Then, due to standard elliptic estimates, the corresponding $\psi_{\omega,\gamma}$ and u_f converge in the $W^{2,3}$-weak and $W^{1,3}$-weak topologies. This involves directly the continuity of the term $\mathfrak{A}$, that is linear. By a weak-strong convergence argument, this also gives the continuity of $\mathfrak{D}$, due to the compact embedding of $W^{1,3}(\mathcal{F}(q))$ in $L^{3/2}(\mathcal{F}(q))$.

Concerning the quadratic term $\mathfrak{E}$, we see that the terms $\frac{\partial \psi_{\omega_n,\gamma}}{\partial n}$ converge weakly in $W^{2/3,3}(\partial\mathcal{F}(q))$. This is compactly embedded in $L^2(\partial\mathcal{F}(q))$, which gives the continuity of $\mathfrak{E}$. □

The rest of the section is devoted to the proof of Proposition 7.1.

7.1. **A nonlinear method to solve linear perturbations of nonlinear equations.** Our strategy relies on a nonlinear method: we prove the existence of solutions to nonlinear equations of the type

$$Q(X) + L(X) = Y \tag{7.3}$$

where L is linear continuous and Q is a quadratic operator admitting a non trivial zero at which the differential is right-invertible. The main point is using the difference of homogeneity between Q and L. Some other superlinear homogeneous terms could be considered in place of Q, the key point being to deal with the linear part of the equation as a perturbation of the nonlinear part by means of a scaling argument. This type of strategy where one takes advantage of the nonlinearity is reminiscent of Coron's phantom tracking method and return method, see [11], respectively [12]. Here we will only be interested in the existence of a solution to an equation of the form (7.3), which holds as a prototype for Equation (6.12) where the unknown is the control, so that we do not expect any uniqueness properties. Moreover we need to consider some parameterized version of (7.3), as the operators Q and L above depend on the parameters q, q', γ, ω, and the feedback control law that we are looking for has to be robust and to depend on these parameters in a sufficiently regular manner. The precise result which we prove in this subsection is the following.

Proposition 7.3. *(a) Let $d \geqslant 1$, let $(E, \|\cdot\|)$ be a finite-dimensional normed linear space of dimension larger than d, F a bounded metric space, $Q : F \times E \to \mathbb{R}^d$ a Lipschitz map which to each $\mathfrak{p}$ in F associates a quadratic operator $Q_\mathfrak{p}$ from E to $\mathbb{R}^d$, and $L : F \times E \to \mathbb{R}^d$ a Lipschitz map which with each $\mathfrak{p}$ in F associates a linear operator $L_\mathfrak{p}$ from E to $\mathbb{R}^d$. Furthermore, assume that there exists a Lipschitz map $\mathfrak{p} \in F \mapsto \overline{X}_\mathfrak{p}$ in E satisfying for any p in F,*

$$\|\overline{X}_\mathfrak{p}\| = 1, \quad Q_\mathfrak{p}(\overline{X}_\mathfrak{p}) = 0,$$

and such that the family of linear operators $\big(DQ_\mathfrak{p}(\overline{X}_\mathfrak{p})\big)_{\mathfrak{p}\in F}$ admits a family of right inverses depending on $\mathfrak{p} \in F$ in a Lipschitz way. Then there exists a locally Lipschitz mapping $R : F \times \mathbb{R}^d \to E$ such that

$$(Q_\mathfrak{p} + L_\mathfrak{p}) \circ R(\mathfrak{p}, \cdot) = Id_{\mathbb{R}^d}.$$

(b) Let τ be another topology on F. Suppose furthermore that the maps $Q : F \times E \to \mathbb{R}^d$, $L : F \times E \to \mathbb{R}^d$ and $\mathfrak{p} \in F \mapsto \overline{X}_\mathfrak{p}$ are continuous when F is endowed with τ. Then there exists a map R which satisfies the conclusions given in (a) and which is also continuous when F is endowed with τ.

To prove Proposition 7.3 we will make use of the following version of the inverse function theorem where the size of the neighborhood is precised with respect to a parameter. In the proof of Proposition 7.3 we will need to add a scalar parameter to $\mathfrak{p}$, hence we introduce the notation $\tilde{\mathfrak{p}}$ and $\tilde{F}$. Furthermore, the space E that we refer to in the Lemma below will not be quite the same as the one in Proposition 7.3. Hence we will rather use the notation $\tilde{E}$. Despite the fact that we will actually use it on a finite dimensional space, we state the result in the slightly more general setting of Banach spaces.

Lemma 7.4. *(a) Let $\tilde{E}$ be a Banach space, $\tilde{F}$ a metric space, for any $\tilde{\mathfrak{p}}$ in $\tilde{F}$, $f_{\tilde{\mathfrak{p}}} : \tilde{E} \to \tilde{E}$ a mapping which is C^1 in a neighborhood of 0, such that the following are satisfied:*

(i) *for any $\tilde{\mathfrak{p}}$ in $\tilde{F}$, the linear map $Df_{\tilde{\mathfrak{p}}}(0)$ is one-to-one on $\tilde{E}$, the maps $(\tilde{\mathfrak{p}}, x) \in \tilde{F} \times \tilde{E} \mapsto f_{\tilde{\mathfrak{p}}}(x)$ and $\tilde{\mathfrak{p}} \in \tilde{F} \mapsto Df_{\tilde{\mathfrak{p}}}(0)^{-1}$ are Lipschitz;*

(ii) *there exist $r > 0$ and $M > 0$ such that for all x_1, x_2 in $B_{\tilde{E}}(0, r)$ and $\tilde{\mathfrak{p}}$ in $\tilde{F}$,*

$$\|Df_{\tilde{\mathfrak{p}}}(0)^{-1}\|_{\mathcal{L}(\tilde{E};\tilde{E})} \leqslant M \quad \text{and} \quad \|Df_{\tilde{\mathfrak{p}}}(x_1) - Df_{\tilde{\mathfrak{p}}}(x_2)\|_{\mathcal{L}(\tilde{E};\tilde{E})} \leqslant \frac{1}{2M}.$$

Then there exists a unique Lipschitz map

$$\tilde{\mathscr{R}} : \cup_{\tilde{\mathfrak{p}} \in \tilde{F}} \left(\{\tilde{\mathfrak{p}}\} \times B_{\tilde{E}} \left(f_{\tilde{\mathfrak{p}}}(0), \frac{r}{2M} \right) \right) \longrightarrow B_{\tilde{E}}(0, r),$$

such that for any $\tilde{\mathfrak{p}}$ in $\tilde{F}$, for any y in $B_{\tilde{E}}\left(f_{\tilde{\mathfrak{p}}}(0), \frac{r}{2M}\right)$,

$$f_{\tilde{\mathfrak{p}}}(\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}, y)) = y.$$

(b) Let $\tilde{\tau}$ be another topology on $\tilde{F}$. Suppose that the maps $(\tilde{\mathfrak{p}}, x) \in \tilde{F} \times \tilde{E} \mapsto f_{\tilde{\mathfrak{p}}}(x)$ and $\tilde{\mathfrak{p}} \in \tilde{F} \mapsto Df_{\tilde{\mathfrak{p}}}(0)^{-1}$ are continuous when $\tilde{F}$ is equipped with $\tilde{\tau}$. Then the map $\tilde{\mathscr{R}}$ given in (a) is also continuous when $\tilde{F}$ is equipped with $\tilde{\tau}$.

Proof of Lemma 7.4. For $\tilde{\mathfrak{p}}$ in $\tilde{F}$, y in $B_{\tilde{E}}\left(f_{\tilde{\mathfrak{p}}}(0), \frac{r}{2M}\right)$ and x in $B_{\tilde{E}}(0, r)$, we set

$$\begin{aligned} g_{\tilde{\mathfrak{p}},y}(x) &:= x + Df_{\tilde{\mathfrak{p}}}(0)^{-1}\left(y - f_{\tilde{\mathfrak{p}}}(x)\right) \\ &= Df_{\tilde{\mathfrak{p}}}(0)^{-1}\left(y - f_{\tilde{\mathfrak{p}}}(0)\right) - Df_{\tilde{\mathfrak{p}}}(0)^{-1}\left(f_{\tilde{\mathfrak{p}}}(x) - f_{\tilde{\mathfrak{p}}}(0) - Df_{\tilde{\mathfrak{p}}}(0)x\right). \end{aligned} \tag{7.4}$$

Using (7.4), the triangle inequality, (ii) and

$$f_{\tilde{\mathfrak{p}}}(x) - f_{\tilde{\mathfrak{p}}}(0) = \left(\int_0^1 Df_{\tilde{\mathfrak{p}}}(tx)\, dt \right) x, \tag{7.5}$$

we observe that

$$g_{\tilde{\mathfrak{p}},y}(B_{\tilde{E}}(0, r)) \subset B_{\tilde{E}}(0, r).$$

Similarly, for any x_1, x_2 in $B_{\tilde{E}}(0, r)$, using again (7.4) , (ii) and (7.5), we obtain:

$$\|g_{\tilde{\mathfrak{p}},y}(x_1) - g_{\tilde{\mathfrak{p}},y}(x_2)\| \leqslant \frac{1}{2}\|x_1 - x_2\|. \tag{7.6}$$

Therefore, $g_{\tilde{\mathfrak{p}},y}$ is a contraction, and from the Banach fixed point theorem it follows that the mapping $g_{\tilde{\mathfrak{p}},y}$ has a unique fixed point in $B_{\tilde{E}}(0, r)$, which is also the unique solution of $f_{\tilde{\mathfrak{p}}}(\cdot) = y$. We consequently define $\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}, y)$ as this fixed point.

Now let $\tilde{\mathfrak{p}}_1$ and $\tilde{\mathfrak{p}}_2$ in $\tilde{F}$,

$$y_1 \in B_{\tilde{E}}\left(f_{\tilde{\mathfrak{p}}_1}(0), \frac{r}{2M}\right) \quad \text{and} \quad y_2 \in B_{\tilde{E}}\left(f_{\tilde{\mathfrak{p}}_2}(0), \frac{r}{2M}\right).$$

By the triangle inequality, using (7.6) and

$$\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_2, y_2) = g_{\tilde{\mathfrak{p}}_2,y_2}(\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_2, y_2)) \quad \text{and} \quad \tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_1, y_1) = g_{\tilde{\mathfrak{p}}_1,y_1}(\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_1, y_1)),$$

we obtain

$$\|g_{\tilde{\mathfrak{p}}_1,y_1}(\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_1,y_1))-g_{\tilde{\mathfrak{p}}_2,y_2}(\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_1,y_1))\|\geqslant\|\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_2,y_2)-\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_1,y_1)\|-\|g_{\tilde{\mathfrak{p}}_2,y_2}(\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_2,y_2))-g_{\tilde{\mathfrak{p}}_2,y_2}(\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_1,y_1))\| \\ \geqslant \frac{1}{2}\|\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_2,y_2)-\tilde{\mathscr{R}}(\tilde{\mathfrak{p}}_1,y_1)\|. \tag{7.7}$$

Now since the mapping $(\tilde{\mathfrak{p}},y,x)\mapsto g_{\tilde{\mathfrak{p}},y}(x)$ is Lipschitz due to (i), we deduce that $\tilde{\mathscr{R}}$ is Lipschitz.

The continuity property in (b) follows similarly, using once more (7.4) and (7.7). This concludes the proof of Lemma 7.4. □

We can now start the proof of Proposition 7.3.

Proof of Proposition 7.3. We begin by observing that for any $\mathfrak{p}$ in F, there exist some linear isomorphisms $\varphi_{\mathfrak{p}}$ from $\mathbb{R}^d$ to the orthogonal of the kernel of $DQ_{\mathfrak{p}}(\overline{X}_{\mathfrak{p}})$, such that the maps $F\ni\mathfrak{p}\mapsto\varphi_{\mathfrak{p}}$ and $F\ni\mathfrak{p}\mapsto\varphi_{\mathfrak{p}}^{-1}$ are Lipschitz and bounded (since F is bounded). Furthermore, if the assumptions of part (b) are satisfied, then these maps are continuous with respect to the topology τ on F.

The proof is of Proposition 7.3 is then based on a scaling argument. We introduce $\varepsilon_0>0$ such that for any ε in $[0,\varepsilon_0]$, for any $\mathfrak{p}$ in F:

(1) the linear operator $DQ_{\mathfrak{p}}(\overline{X}_{\mathfrak{p}})+\varepsilon L_{\mathfrak{p}}:E\longrightarrow E$ is right invertible, with some right inverses which are uniformly bounded as $(\varepsilon,\mathfrak{p})$ runs over $[0,\varepsilon_0]\times F$,
(2) the linear isomorphism $\varphi_{\mathfrak{p}}$ also allows to select a right inverse of $DQ_{\mathfrak{p}}(\overline{X}_{\mathfrak{p}})+\varepsilon L_{\mathfrak{p}}$.

We will further denote $\tilde{\mathfrak{p}}:=(\varepsilon,\mathfrak{p})$ and $\tilde{F}:=[0,\varepsilon_0]\times F$. Then we consider the mapping $f_{\tilde{\mathfrak{p}}}$ from $\mathbb{R}^d$ to $\mathbb{R}^d$ which maps x in $\mathbb{R}^d$ to

$$f_{\tilde{\mathfrak{p}}}(x):=(Q_{\mathfrak{p}}+\varepsilon L_{\mathfrak{p}})(\overline{X}_{\mathfrak{p}}+\varphi_{\mathfrak{p}}(x))\in\mathbb{R}^d.$$

Our goal is to apply Lemma 7.4 to the mapping $f_{\tilde{\mathfrak{p}}}$ and to the space $\tilde{E}=\mathbb{R}^d$.

First, we see that the map $(\tilde{\mathfrak{p}},x)$ in $\tilde{F}\times\mathbb{R}^d\mapsto f_{\tilde{\mathfrak{p}}}(x)$ is Lipschitz and for any $\tilde{\mathfrak{p}}$ in $\tilde{F}$, the map x in $\mathbb{R}^d\mapsto f_{\tilde{\mathfrak{p}}}(x)$ is C^1 in a neighborhood $\mathcal{V}$ of 0 in $\mathbb{R}^d$ and for any x in $\mathcal{V}$,

$$Df_{\tilde{\mathfrak{p}}}(x)=(DQ_{\mathfrak{p}}+\varepsilon L_{\mathfrak{p}})(\overline{X}_{\mathfrak{p}}+\varphi_{\mathfrak{p}}(x))\circ\varphi_{\mathfrak{p}}.$$

In particular, since $\varphi_{\mathfrak{p}}(0)=0$,

$$Df_{\tilde{\mathfrak{p}}}(0)=(DQ_{\mathfrak{p}}+\varepsilon L_{\mathfrak{p}})(\overline{X}_{\mathfrak{p}})\circ\varphi_{\mathfrak{p}},$$

is one to one and, using (2) from the above choice of $\varepsilon_0>0$, one can see that its inverse is given by

$$Df_{\tilde{\mathfrak{p}}}(0)^{-1}=\varphi_{\mathfrak{p}}^{-1}\circ((DQ_{\mathfrak{p}}+\varepsilon L_{\mathfrak{p}})(\overline{X}_{\mathfrak{p}}))^{-1}.$$

Thus the map $\tilde{\mathfrak{p}}\mapsto Df_{\tilde{\mathfrak{p}}}(0)^{-1}$ is Lipschitz and bounded as the composition of bounded Lipschitz maps. Therefore the assumption (i) of Lemma 7.4 is satisfied.

Moreover for all x_1,x_2 in $\mathcal{V}$ and $\tilde{\mathfrak{p}}\in\tilde{F}$, we have

$$Df_{\tilde{\mathfrak{p}}}(x_1)-Df_{\tilde{\mathfrak{p}}}(x_2)=\Big((DQ_{\mathfrak{p}}+\varepsilon L_{\mathfrak{p}})(\overline{X}_{\mathfrak{p}}+\varphi_{\mathfrak{p}}(x_1))-(DQ_{\mathfrak{p}}+\varepsilon L_{\mathfrak{p}})(\overline{X}_{\mathfrak{p}}+\varphi_{\mathfrak{p}}(x_2))\Big)\circ\varphi_{\mathfrak{p}}.$$

Using that the mapping $F\times E\ni(\mathfrak{p},x)\longmapsto Q_{\mathfrak{p}}(x)\in\mathbb{R}^d$ is Lipschitz and that $\varphi_{\mathfrak{p}}$ is linear continuous, we deduce that that the assumption (ii) of Lemma 7.4 is satisfied.

Hence we can apply Lemma 7.4 to $f_{\tilde{\mathfrak{p}}}$ and obtain the map $\tilde{\mathscr{R}}$. Since $Q_{\mathfrak{p}}(\overline{X}_{\mathfrak{p}})=0$ and $\varphi_{\mathfrak{p}}(0)=0$ (by linearity), we have $f_{\tilde{\mathfrak{p}}}(0)=\varepsilon L_{\mathfrak{p}}(\overline{X}_{\mathfrak{p}})$. Therefore $f_{\tilde{\mathfrak{p}}}(0)$ converges to 0 as ε converges to 0, uniformly in $\tilde{\mathfrak{p}}\in\tilde{F}$. Then reducing $\varepsilon_0>0$ again if necessary, there exists $r>0$ such that the Lipschitz mapping

$$\mathscr{R}:[0,\varepsilon_0]\times F\times B_{\mathbb{R}^d}(0,r)\longrightarrow E,\qquad(\varepsilon,\mathfrak{p},y)\longmapsto\overline{X}_{\mathfrak{p}}+\tilde{\mathscr{R}}(\varepsilon,\mathfrak{p},y),$$

satisfies for any ε in $[0, \varepsilon_0]$, for any $\mathfrak{p}$ in F, for any y in $\mathbb{R}^d$ with $|y| < r$,

$$(Q_{\mathfrak{p}} + \varepsilon L_{\mathfrak{p}})(\mathscr{R}(\varepsilon, \mathfrak{p}, y)) = y. \tag{7.8}$$

Now we define $R : F \times \mathbb{R}^d \to E$ by setting, for any $\mathfrak{p}$ in F and any y in $\mathbb{R}^d$,

$$R(\mathfrak{p}, y) := \frac{1}{\varepsilon(y)} \mathscr{R}(\varepsilon(y), \mathfrak{p}, \varepsilon(y)^2 y) \quad \text{where} \quad \varepsilon(y) := \min \left\{ \varepsilon_0, \frac{\sqrt{r}}{(1 + |y|^2)^{1/4}} \right\}.$$

Observe that this definition makes sense since, for any y in $\mathbb{R}^d$, $\varepsilon(y)$ is in $(0, \varepsilon_0]$ and $|\varepsilon(y)^2 y| < r$. Moreover R is a locally Lipschitz mapping as composition of locally Lipschitz mappings. Finally, using that the mapping $Q_{\mathfrak{p}}$ is quadratic, that the mapping $L_{\mathfrak{p}}$ is linear, and (7.8), we obtain that for any $\mathfrak{p}$ in F, for any y in $\mathbb{R}^d$,

$$(Q_{\mathfrak{p}} + L_{\mathfrak{p}})(R(\mathfrak{p}, y)) = \frac{1}{\varepsilon(y)^2} (Q_{\mathfrak{p}} + \varepsilon(y) L_{\mathfrak{p}})(\mathscr{R}(\varepsilon(y), \mathfrak{p}, \varepsilon(y)^2 y)) = y.$$

This concludes the proof of part (a) of Proposition 7.3.

Concerning Part (b) of the statement, we rely on the observation at the beginning of the proof concerning the continuity with respect to τ. This allows us to deduce that the function $f_{\tilde{\mathfrak{p}}}$ satisfies the conditions from part (b) of Lemma 7.4, defining the topology $\tilde{\tau}$ as the product of the topology of $\mathbb{R}$ and τ. □

7.2. **Restriction of the quadratic mapping $\mathfrak{Q}(q)$ and determination of a particular non-trivial zero point.** We go back to the framework of Proposition 7.1. We first recall that $\mathfrak{Q}(q)[g]$ for q in $\mathcal{Q}$ and g in $\mathcal{C}_b(q)$ was defined in (6.9) with $\alpha = \mathcal{A}[q, g]$ introduced in (5.11). Accordingly we have

$$\mathfrak{Q}(q)[g] = \frac{1}{2} \left(\int_{\partial \mathcal{S}_\kappa(q)} |\nabla \mathcal{A}[q, g]|^2 \, \partial_n \boldsymbol{\varphi}_\kappa(q, \cdot) \, \mathrm{d}s \right)_{\kappa = 1, \ldots, N} \in \mathbb{R}^{3N}. \tag{7.9}$$

The goal of this section is to associate with the operators $\mathfrak{Q}(q)$ a finite-dimensional subspace $\mathcal{E} \subset \mathcal{C}_b$ and, for each q, a point in $\mathcal{E}$ which is a non trivial zero of $\mathfrak{Q}(q)_{|\mathcal{E}}$ at which the derivative is right-invertible. This will allow us to apply Proposition 7.3. Precisely, we show the following.

Proposition 7.5. *Let $\delta > 0$. There exists a finite dimensional subspace $\mathcal{E} \subset \mathcal{C}$ and Lipschitz mappings*

$$q \in \mathcal{Q}_\delta \longmapsto g_i(q, \cdot) \in \mathcal{C}_b(q) \cap \mathcal{E}, \quad \text{for } 1 \leqslant i \leqslant (3N+1)^2,$$

such that the following holds. Define $Q_q : \mathbb{R}^{(3N+1)^2} \to \mathbb{R}^{3N}$ the quadratic operator which maps $X := (X_i)_{1 \leqslant i \leqslant (3N+1)^2}$ to

$$Q_q(X) := \mathfrak{Q}(q) \left[\sum_{i=1}^{(3N+1)^2} X_i \, g_i(q, \cdot) \right]. \tag{7.10}$$

Then there exists a Lipschitz map $q \in \mathcal{Q}_\delta \mapsto \overline{X}_q$ in $\mathbb{R}^{(3N+1)^2}$ satisfying

$$\|\overline{X}_q\| = 1 \quad \text{and} \quad Q_q(\overline{X}_q) = 0,$$

and such that $DQ_q(\overline{X}_q)$, the derivative with respect to the second argument, is right-invertible with right inverses which depend on q in a bounded Lipschitz way.

To prove Proposition 7.5, we extend the analysis performed in [21] for a single solid to the case of several solids. In particular we will use some arguments of complex analysis and convexity which are similar to the ones already used in [21]. We recall that the conical hull of $A \subset \mathbb{R}^d$ is defined as

$$\operatorname{coni}(A) := \left\{ \sum_{i=1}^{k} \lambda_i a_i,\ k \in \mathbb{N}^*,\ \lambda_i \geqslant 0,\ a_i \in A \right\}.$$

Proof of Proposition 7.5. First, we recall, see [21, Lemma 14], that if $\mathcal{S}_0 \subset \Omega$ is a bounded, closed, simply connected domain of $\mathbb{R}^2$ with smooth boundary, which is not a disk, then

$$\operatorname{coni}\{(n(x), (x-h_0)^\perp \cdot n(x)),\ x \in \partial\mathcal{S}_0\} = \mathbb{R}^3,$$

for any h_0 in $\mathbb{R}^2$, where $n(x)$ denotes the unit normal vector to $\mathcal{S}_0$. Therefore, taking into account the boundary conditions of the Kirchhoff potentials, see Subsection 5.1, we deduce that for any q_0 in $\mathcal{Q}_\delta$,

$$\operatorname{coni}\left\{ (\partial_n \boldsymbol{\varphi}_\kappa(q_0, x^\kappa))_{\kappa=1,\dots,N},\ \ (x^1, \dots, x^N) \in \partial\mathcal{S}_1(q_0) \times \dots \times \partial\mathcal{S}_N(q_0) \right\} = \mathbb{R}^{3N}. \tag{7.11}$$

This allows to establish the following lemma.

Lemma 7.6. *Fix q_0 in $\mathcal{Q}_\delta$. For $1 \leqslant i \leqslant (3N+1)^2$ and $\kappa = 1, \dots, N$, there exists x_i^κ in $\partial\mathcal{S}_\kappa(q_0)$ and positive smooth mapping $\tilde{\mu}_i : \mathbb{R}^{3N} \to \mathbb{R}$, such that*

$$\sum_{i=1}^{(3N+1)^2} \tilde{\mu}_i(v)\, (\partial_n \boldsymbol{\varphi}_\kappa(q_0, x_i^\kappa))_{\kappa=1,\dots,N} = v \quad \textit{for all } v \in \mathbb{R}^{3N}. \tag{7.12}$$

Proof of Lemma 7.6. We introduce the following notations: let $\{\mathbf{b}_1, \dots, \mathbf{b}_{3N}\}$ the canonical orthonormal basis of $\mathbb{R}^{3N}$, and $\mathbf{b}_{3N+1} := -(1, \dots, 1)$; let $\lambda_\ell(v) := v_\ell + \sqrt{1+|v|^2}$ for $\ell = 1, \dots, 3N$, and $\lambda_{3N+1}(v) := \sqrt{1+|v|^2}$. For any v in $\mathbb{R}^{3N}$, we have

$$v = \sum_{\ell=1}^{3N+1} \lambda_\ell(v)\, \mathbf{b}_\ell. \tag{7.13}$$

Now, thanks to (7.11), we see that for some radius $r > 0$ the sphere $S(0,r)$ of $\mathbb{R}^{3N}$ is contained in the interior of the convex hull of

$$\left\{ (\partial_n \boldsymbol{\varphi}_\kappa(q_0, x^\kappa))_{\kappa=1,\dots,N},\ (x^1, \dots, x^N) \in \partial\mathcal{S}_1(q_0) \times \dots \times \partial\mathcal{S}_N(q_0) \right\}.$$

For any $\ell \in \{1, \dots, 3N+1\}$, by Carathéodory's theorem there exist points $x^\kappa_{(\ell-1)(3N+1)+j}$ in $\partial\mathcal{S}_\kappa(q_0)$ for $\kappa \in \{1, \dots, N\}$ and j in $\{1, \dots, 3N+1\}$ and scalars $\tilde{\lambda}_{(\ell-1)(3N+1)+j} \in [0,1)$ for j in $\{1, \dots, 3N+1\}$ such that

$$r\mathbf{b}_\ell = \sum_{j=1}^{3N+1} \tilde{\lambda}_{(\ell-1)(3N+1)+j} \left(\partial_n \boldsymbol{\varphi}_\kappa(q_0, x^\kappa_{(\ell-1)(3N+1)+j}) \right)_{\kappa=1,\dots,N}. \tag{7.14}$$

We may exclude the possibility that some $\tilde{\lambda}_i$ is 0 as follows: if for some i, $\tilde{\lambda}_i = 0$, then we move the corresponding points x_i^κ on another x_k^κ for which $\tilde{\lambda}_k \neq 0$; then we split the value $\tilde{\lambda}_k$ between $\tilde{\lambda}_k$ and $\tilde{\lambda}_i$ so that no coefficient $\tilde{\lambda}_i$ vanishes. We consider this to be the case from now on.

Combining (7.13) with (7.14), we arrive at (7.12) with for any v in $\mathbb{R}^{3N}$ and for $i = 1, \dots, (3N+1)^2$,

$$\tilde{\mu}_i(v) := \frac{1}{r} \tilde{\lambda}_i \lambda_{1+i/(3N+1)}(v),$$

where $i/(3N+1)$ denotes the quotient of the Euclidean division of i by $3N+1$. It follows that the mappings $\tilde{\mu}_i$ are smooth with positive values on $\mathbb{R}^{3N}$. This ends the proof of Lemma 7.6. ∎

Now given the points x_i^κ of Lemma 7.6, let us denote

$$x_i^\kappa(q) = R(\theta_\kappa)(x_i^\kappa - h_{\kappa,0}) + h_\kappa \in \partial\mathcal{S}_\kappa(q).$$

We consider the 3×3 and $3N \times 3N$ rotation matrices

$$\mathcal{R}_\kappa(q) = \begin{pmatrix} R(\vartheta_\kappa) & 0 \\ 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3} \quad \text{for } \kappa = 1, \dots, N,$$

$$\mathcal{R}(q) = \begin{pmatrix} \mathcal{R}_1(q) & 0 & \dots & 0 \\ 0 & \mathcal{R}_2(q) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathcal{R}_N(q) \end{pmatrix} \in \mathbb{R}^{3N\times 3N}.$$

Recalling the definition of $\partial_n \boldsymbol{\varphi}_\kappa(q,\cdot)$, we find

$$\sum_{i=1}^{(3N+1)^2} \tilde{\mu}_i(\mathcal{R}(q)^{-1}v)\,(\partial_n \boldsymbol{\varphi}_\kappa(q, x_i^\kappa(q)))_{\kappa=1,\dots,N} = v, \text{ for all } v \in \mathbb{R}^{3N},\ q \in \mathcal{Q}_\delta. \tag{7.15}$$

For $1 \leqslant i \leqslant (3N+1)^2$ and q in $\mathcal{Q}_\delta$, we set

$$e_i(q) := \big(\partial_n \boldsymbol{\varphi}_1(q, x_i^1(q)), \dots, \partial_n \boldsymbol{\varphi}_N(q, x_i^N(q))\big) \quad \text{and} \quad \mu_i(q,v) := \tilde{\mu}_i(\mathcal{R}(q)^{-1}v), \tag{7.16}$$

so that, for any q in $\mathcal{Q}_\delta$ and v in $\mathbb{R}^{3N}$,

$$\sum_{i=1}^{(3N+1)^2} \mu_i(q,v) e_i(q) = v. \tag{7.17}$$

Now the following lemma is the adaptation of [21, Lemma 10] to the case of several rigid bodies.

Lemma 7.7. *Let κ in $\{1,\dots,N\}$. Given q in $\mathcal{Q}_\delta$, there exists a family of functions*

$$(\tilde{\alpha}_{\kappa,\varepsilon}^{i,j}(q,\cdot))_{\varepsilon\in(0,1)} \textit{ for } 1 \leqslant i \leqslant (3N+1)^2,\ 1 \leqslant j \leqslant 3N+1 \textit{ and } \varepsilon \in (0,1),$$

which are defined and harmonic in a closed neighbourhood $\mathcal{V}_{\kappa,\varepsilon}^{i,j}$ of $\partial\mathcal{S}_\kappa(q)$, satisfy $\partial_n \tilde{\alpha}_{\kappa,\varepsilon}^{i,j}(q,\cdot) = 0$ on $\partial\mathcal{S}_\kappa(q)$, and moreover, for any $1 \leqslant i,k \leqslant (3N+1)^2$, for any $1 \leqslant j,l \leqslant 3N+1$,

$$\left| \int_{\partial\mathcal{S}_\kappa(q)} \nabla\tilde{\alpha}_{\kappa,\varepsilon}^{i,j}(q,\cdot)\cdot\nabla\tilde{\alpha}_{\kappa,\varepsilon}^{k,l}(q,\cdot)\partial_n\boldsymbol{\varphi}_k(q,\cdot)\mathrm{d}s - \delta_{(i,j),(k,l)}\,\partial_n\boldsymbol{\varphi}_\kappa(q,x_i^\kappa(q)) \right| \leqslant \varepsilon.$$

Proof of Lemma 7.7. We deduce from the Riemann mapping theorem the existence of a conformal map $\Psi_\kappa : \overline{\mathbb{C}} \setminus B(0,1) \to \overline{\mathbb{C}} \setminus \mathcal{S}_\kappa(q)$, where $\overline{\mathbb{C}}$ is the Riemann sphere. Classically, this conformal map is smooth up to the boundary thanks to the regularity of $\partial\mathcal{S}_\kappa(q)$. For any smooth function $\alpha : \partial\mathcal{S}_\kappa(q) \to \mathbb{R}$, the Cauchy-Riemann relations imply that for any x in $\partial B(0,1)$,

$$\partial_n \alpha(\Psi_\kappa(x)) = \frac{1}{\sqrt{|\det(D\Psi_\kappa(x))|}} \partial_{n_B}(\alpha\circ\Psi_\kappa)(x),$$

$$\int_{\partial\mathcal{S}_\kappa(q)} |\nabla\alpha(x)|^2\,\partial_n\boldsymbol{\varphi}_\kappa(q,x)\,\mathrm{d}s = \int_{\partial B(0,1)} |\nabla\alpha(\Psi_\kappa(x))|^2\,\partial_{n_B}\boldsymbol{\varphi}_\kappa(q,\Psi_\kappa(x))\,\frac{1}{\sqrt{|\det(D\Psi_\kappa(x))|}}\,\mathrm{d}s,$$

where n and n_B respectively denote the normal vectors on $\partial\mathcal{S}_\kappa(q)$ and $\partial B(0,1)$. Since Ψ_κ is invertible, we have $|\det(D\Psi_\kappa(x))| > 0$, for any x in $\partial B(0,1)$.

We consider the parameterizations

$$\{c(s) = (\cos(s), \sin(s)),\ s \in [0,2\pi]\} \text{ of } \partial B(0,1) \quad \text{and} \quad \{\Psi_\kappa(c(s)),\ s \in [0,2\pi]\} \text{ of } \partial\mathcal{S}_\kappa(q),$$

and the corresponding values s_i such that $x_i^\kappa(q) = \Psi_\kappa(c(s_i))$, for $1 \leqslant i \leqslant (3N+1)^2$. We introduce a family of smooth functions $\beta_\rho^{i,j} : [0, 2\pi] \to \mathbb{R}$ defined for $\rho > 0$, $1 \leqslant i \leqslant (3N+1)^2$ and j in $\{1, \ldots, 3N+1\}$, satisfying:

$$\operatorname{supp}\, \beta_\rho^{i,j} \cap \operatorname{supp}\, \beta_\rho^{k,l} = \emptyset \text{ for } (i,j) \neq (k,l), \quad \int_0^{2\pi} \beta_\rho^{i,j}(s)\, ds = 0,$$

and such that, as $\rho \to 0^+$, $\operatorname{diam}\left(\operatorname{supp}\, \beta_\rho^{i,j}\right) \to 0$ and

$$\int_0^{2\pi} |\beta_\rho^{i,j}(s)|^2\, \partial_n \boldsymbol{\varphi}_\kappa(q, c(s)) \frac{1}{\sqrt{|\det(D\Psi_\kappa(c(s)))|}}\, ds \longrightarrow \frac{1}{\sqrt{|\det(D\Psi_\kappa(c(s_i)))|}} \partial_n \boldsymbol{\varphi}_\kappa(q, \Psi_\kappa(c(s_i))).$$

We denote $\hat{a}_{k,\rho}^{i,j}$ and $\hat{b}_{k,\rho}^{i,j}$ the k-th Fourier coefficients of the function $\beta_\rho^{i,j}$.

Now for suitable ρ and K, one then defines $\tilde{\alpha}_{\kappa,\varepsilon}^{i,j}(q, \cdot)$ as the truncated Laurent series:

$$\frac{1}{2} \sum_{0<k\leqslant K} \frac{1}{k} \left(r^k + \frac{1}{r^k}\right) (-\hat{b}_{k,\rho}^{i,j} \cos(k\theta) + \hat{a}_{k,\rho}^{i,j} \sin(k\theta)),$$

composed with Ψ_κ^{-1}. Choosing first $\rho > 0$ small enough, and then $K \in \mathbb{N}$ large enough, it is easy to check that this family satisfies the required properties. This ends the proof of Lemma 7.7. ☐

The following result of approximation of the functions $\tilde{\alpha}_{\kappa,\varepsilon}^{i,j}(q, \cdot)$ given by Lemma 7.7 is close to [16, p. 147-149] and [19]. Recall the definition of $\mathcal{C}$ in (2.1).

Lemma 7.8. *For fixed k in $\mathbb{N}$, $\varepsilon > 0$, and for any κ in $\{1, \ldots, N\}$, q in $\mathcal{Q}_\delta$, there exists a family of functions*

$$(g_{\kappa,\eta}^{i,j}(q, \cdot))_{\eta \in (0,1)} \in \mathcal{C}, \ \text{ for } 1 \leqslant i \leqslant (3N+1)^2 \ \text{ and } 1 \leqslant j \leqslant 3N+1,$$

with for any $\bar{\kappa}$ in $\{1, \ldots, N\}$

$$\left\| \mathcal{A}[q, g_{\kappa,\eta}^{i,j}(q, \cdot)] - \delta_{\kappa,\bar{\kappa}} \tilde{\alpha}_{\bar{\kappa},\varepsilon}^{i,j}(q, \cdot) \right\|_{C^k(\mathcal{V}_{\bar{\kappa},\varepsilon}^{i,j} \cap \overline{\mathcal{F}(q)})} \leqslant \eta. \tag{7.18}$$

Proof of Lemma 7.8. Let k in $\mathbb{N}$, $\varepsilon > 0$, κ in $\{1, \ldots, N\}$, $1 \leqslant i \leqslant (3N+1)^2$, $1 \leqslant j \leqslant 3N+1$ and q in $\mathcal{Q}_\delta$. We approximate $\tilde{\alpha}_{\kappa,\varepsilon}^{i,j}(q, \cdot)$ by a function defined on $\mathcal{F}(q)$ using Runge's theorem. Namely, we first introduce a neighbourhood V of $\partial\Omega \setminus \Sigma$, disjoint from $\mathcal{V}_{\bar{\kappa},\varepsilon}^{i,j}(q)$ for any $\bar{\kappa}$ in $\{1, \ldots, N\}$. Next we define the holomorphic function $f^{i,j}$ on the set

$$V \cup \left(\bigcup_{\bar{\kappa}=1}^{N} \mathcal{V}_{\bar{\kappa},\varepsilon}^{i,j} \right),$$

by

$$f^{i,j} = \partial_{x_1} \tilde{\alpha}_{\kappa,\varepsilon}^{i,j}(q, \cdot) - i \partial_{x_2} \tilde{\alpha}_{\kappa,\varepsilon}^{i,j}(q, \cdot) \text{ on } \mathcal{V}_{\kappa,\varepsilon}^{i,j}, \ \text{ and } \ f^{i,j} = 0 \text{ on } V \cup \left(\bigcup_{\bar{\kappa} \neq \kappa} \mathcal{V}_{\bar{\kappa},\varepsilon}^{i,j} \right).$$

For each $\eta > 0$, there exists a rational function $r_{\kappa,\eta}^{i,j}$ with one pole in each $\mathcal{S}_{\bar{\kappa}}(q)$ and another outside of $\overline{\Omega}$ such that

$$\| r_{\kappa,\eta}^{i,j} - f^{i,j} \|_{C^k(\mathcal{V}_{\bar{\kappa},\varepsilon}^{i,j}(q) \cup V)} \leqslant \eta, \tag{7.19}$$

for any $\bar{\kappa}$ in $\{1, \ldots, N\}$. The fact that we may take the C^k norm comes from the interior regularity of harmonic functions, enlarging a bit the neighbourhoods.

The function $(\operatorname{Re}(r_{\kappa,\eta}^{i,j}), -\operatorname{Im}(r_{\kappa,\eta}^{i,j}))$ is curl-free in $\mathcal{F}(q)$, however since $\mathcal{F}(q)$ is not a simply-connected domain, we can not directly conclude that it is a gradient, which would require it to have vanishing circulations around the solids. However, since $(\mathrm{Re}(f^{i,j}), -\mathrm{Im}(f^{i,j}))$ is a gradient in $\mathcal{V}^{i,j}_{\kappa,\varepsilon}$ and vanishes in the neighbourhood of the other solids, we may conclude from (7.19) that $(\mathrm{Re}(r^{i,j}_{\kappa,\eta}), -\mathrm{Im}(r^{i,j}_{\kappa,\eta}))$ has circulations of size $\mathcal{O}(\eta)$ around each solid. Therefore, up to subtracting to $(\mathrm{Re}(r^{i,j}_{\kappa,\eta}), -\mathrm{Im}(r^{i,j}_{\kappa,\eta}))$ harmonic fields corresponding to these circulations, we obtain a gradient field and consequently we can define (up to a constant) a function $\bar{\alpha}^{i,j}_{\kappa,\eta}$ which is harmonic on $\mathcal{F}(q)$ such that

$$\|\nabla\bar{\alpha}^{i,j}_{\kappa,\eta} - \nabla\tilde{\alpha}^{i,j}_{\kappa,\varepsilon}(q,\cdot)\|_{C^k(V\cup\bigcup_\kappa \mathcal{V}^{i,j}_{\kappa,\varepsilon}(q))} \leqslant C\eta.$$

Now, by using a continuous extension operator, we may define $g(q,\cdot)$ as a function on $\partial\mathcal{F}(q)$ such that

$$g(q,\cdot) := \partial_n\bar{\alpha}^{i,j}_{\kappa,\eta} \text{ on } \partial\mathcal{F}(q)\setminus\Sigma, \quad \int_{\partial\mathcal{F}(q)} g = 0 \text{ and } \|g\|_{C^k(\Sigma)} = \mathcal{O}(\|g\|_{C^k(\partial\mathcal{F}(q)\setminus\Sigma)}).$$

Then we introduce φ as the solution of the Neumann problem $\Delta\varphi = 0$ in $\mathcal{F}(q)$ and $\partial_n\varphi = g$ on $\partial\mathcal{F}(q)$. Using elliptic regularity we deduce that

$$\|\varphi\|_{C^{k,1/2}(\mathcal{F}(q))} \leqslant C\eta,$$

for some $C>0$ independent of η. Therefore setting

$$g^{i,j}_{\kappa,\eta}(q,\cdot) := \partial_n\bar{\alpha}^{i,j}_{\kappa,\eta} - g \text{ on } \partial\mathcal{F}(q),$$

we obtain

$$\mathcal{A}[q, g^{i,j}_{\kappa,\eta}(q,\cdot)] := \bar{\alpha}^{i,j}_{\kappa,\eta} - \varphi,$$

and this allows us to obtain (7.18) with $C\eta$ in the right-hand side, for some constant $C>0$, instead of η. Then to conclude, we just reparameterize the family $(\bar{\alpha}^{i,j}_{\kappa,\eta})$ with respect to η. This ends the proof of Lemma 7.8. ☐

Now one proceeds as in the proof of [21, Lemma 12], using a partition of unity argument, to make the above construction Lipschitz continuous with respect to q. At the same time we reduce the control space to a finite dimensional subspace of $\mathcal{C}$. More precisely, we have the following result.

Lemma 7.9. *Let $\delta>0$ be fixed, there exists a finite dimensional subspace $\mathcal{E}\subset\mathcal{C}$ such that for any $\nu>0$, there exist Lipschitz mappings*

$$q\in\mathcal{Q}_\delta \longmapsto \overline{g}^{i,j}_{\kappa}(q,\cdot)\in\mathcal{C}(q)\cap\mathcal{E}, \quad \text{for } 1\leqslant i\leqslant(3N+1)^2,\ 1\leqslant j\leqslant 3N+1,\ 1\leqslant\kappa\leqslant N,$$

such that for any q in $\mathcal{Q}_\delta$, $i,k\in\{1,\dots,(3N+1)^2\}$, $j,\ell\in\{1,\dots,3N+1\}$,

$$\left|\int_{\partial\mathcal{S}_\kappa(q)} \nabla\mathcal{A}[q,\overline{g}^{i,j}_{\bar\kappa}(q,\cdot)]\cdot\nabla\mathcal{A}[q,\overline{g}^{k,\ell}_{\hat\kappa}(q,\cdot)]\,\partial_n\boldsymbol{\varphi}_\kappa(q,\cdot)\,\mathrm{d}s - \delta_{\kappa,\bar\kappa,\hat\kappa}\delta_{(i,j),(k,\ell)}\,\partial_n\boldsymbol{\varphi}_\kappa(q,x^\kappa_i(q))\right| \leqslant \nu, \tag{7.20}$$

where $\delta_{\kappa,\bar\kappa,\hat\kappa}$ in $\{0,1\}$ is zero unless $\kappa=\bar\kappa=\hat\kappa$.

Proof of Lemma 7.9. Consider q in Q_δ and $\nu>0$. Choosing first $\varepsilon>0$ small enough in Lemma 7.7 and then $\eta=\eta(\varepsilon)>0$ small enough in Lemma 7.8, we may find for this q in Q_δ functions $g^{i,j}_{\kappa,\eta} = g^{i,j}_{\kappa,\eta}[q]$ satisfying the properties above, and in particular such that (7.20) is valid.

Note that for any q in $\mathcal{Q}_\delta$, the unique solution $\hat{\alpha}^{i,j}_{\kappa,\eta}(\tilde{q},q,\cdot)$, up to an additive constant, to the Neumann problem

$$\Delta_x\hat{\alpha}^{i,j}_{\kappa,\eta}(\tilde{q},q,x) = 0 \text{ in } \mathcal{F}(\tilde{q}),\ \partial_n\hat{\alpha}^{i,j}_{\kappa,\eta}(\tilde{q},q,x) = 0 \text{ on } \partial\mathcal{F}(\tilde{q})\setminus\Sigma,\ \partial_n\hat{\alpha}^{i,j}_{\kappa,\eta}(\tilde{q},q,x) = g^{i,j}_{\kappa,\eta}(q,x) \text{ on } \Sigma,$$

is Lipschitz with respect to $\tilde{q}$ in $\mathcal{Q}_\delta$ (for a detailed proof using shape derivatives, see e.g. [5, 30, 39]). Therefore, if a family of functions $g^{i,j}_{\kappa,\eta}$ satisfies (7.20) at some point q in Q_δ, it also satisfies (7.20) (with say 2ν in the right hand side) in some neighborhood of q. Due to the compactness of Q_δ, since it can be covered with such neighborhoods, one can extract a finite subcover by balls $\{B(q_\ell,r_\ell)\}_{\ell=1\dots N_\delta}$.

We introduce a partition of unity $\varrho_1, \dots, \varrho_{N_\delta}$ (according to the variable q) adapted to this subcover. Defining

$$\overline{g}_\kappa^{i,j}(q,\cdot) := \sum_{\ell=1}^{N_\delta} \varrho_\ell(q)\, g_{\kappa,\eta}^{i,j}[q_\ell](\cdot),$$

we can deduce an estimate like (7.20) with $C\nu$ on the right hand side, for some positive constant C independent of ν. It remains then to reparameterize with respect to ν to obtain (7.20) exactly.

Finally, the finite dimensional subspace $\mathcal{E}$ is then generated by $\{g_{\kappa,\eta}^{i,j}(q_i,\cdot)\}_{i=1}^{N_\delta}$ and its dimension N_δ only depends on δ. This ends the proof of Lemma 7.9. $\square$

Now we would like to enforce the additional condition on the control introduced in Section 6, that is, that the control belongs to $\mathcal{C}_b$ (defined in (6.11)). The starting point is as follows: for any q in $\mathcal{Q}_\delta$, $1 \leqslant i \leqslant (3N+1)^2$, the vectors

$$\left(\int_{\partial \mathcal{S}_\kappa(q)} \mathcal{A}\left[q, \sum_{\bar\kappa=1}^{N} \overline{g}_{\bar\kappa}^{i,j}(q,\cdot) \right] \partial_n \boldsymbol{\varphi}_\kappa(q,\cdot)\, ds \right)_{\kappa=1,\dots,N},$$

for $j = 1, \dots, 3N+1$, are linearly dependent in $\mathbb{R}^{3N}$. Therefore, there exist $\lambda^{i,j}(q)$ in $\mathbb{R}$ such that

$$\sum_{j=1}^{3N+1} \lambda^{i,j}(q) \left(\int_{\partial \mathcal{S}_\kappa(q)} \mathcal{A}[q, \overline{g}_{\bar\kappa}^{i,j}(q,\cdot)]\, \partial_n \boldsymbol{\varphi}_\kappa(q,\cdot)\, ds \right)_{\kappa=1,\dots,N} = 0 \quad \text{and} \quad \sum_{j=1}^{3N+1} \lambda^{i,j}(q)^2 = 1.$$

Moreover, relying on Cramer's formula, we can manage in order that these coefficients are Lipschitz with respect to q.

Now for any $q \in \mathcal{Q}_\delta$, for any $1 \leqslant i \leqslant (3N+1)^2$, we set

$$g_i(q,\cdot) := \sum_{j=1}^{3N+1} \lambda^{i,j}(q) \sum_{\kappa=1}^{N} \overline{g}_\kappa^{i,j}(q,\cdot).$$

Using (7.20), up to further reducing $\nu > 0$, we obtain

$$\left| \left(\int_{\partial \mathcal{S}_\kappa(q)} \nabla \mathcal{A}[q, g_i(q,\cdot)] \cdot \nabla \mathcal{A}[q, g_j(q,\cdot)] \partial_n \boldsymbol{\varphi}_\kappa(q,\cdot)\, ds \right)_{\kappa\in\{1,\dots,N\}} - \delta_{i,j}\, e_i(q) \right| \leqslant \nu, \tag{7.21}$$

$$\int_{\partial \mathcal{S}_\kappa(q)} \mathcal{A}[q, g_i(q,\cdot)] \partial_n \boldsymbol{\varphi}_\kappa(q,\cdot)\, ds = 0,$$

where e_i is defined in (7.16) for $i = 1, \dots, (3N+1)^2$. Hence, $g_i(q,\cdot) \in \mathcal{C}_b(q) \cap \mathcal{E}$. With this family of elementary controls g_i, we are finally in position to prove Proposition 7.5.

For $\delta > 0$, $\nu > 0$ and $(q,v) \in \mathcal{Q}_\delta \times \mathbb{R}^{3N}$, we define the function

$$\tilde{g}(q,v,\cdot) := \sum_{i=1}^{(3N+1)^2} \sqrt{\mu_i(q,v)}\, g_i(q,\cdot), \tag{7.22}$$

in $\mathcal{C}_b(q) \cap \mathcal{E}$, where we recall that the positive functions μ_i were defined in (7.16). We then define $\mathcal{T} : \mathcal{Q}_\delta \times \mathbb{R}^{3N} \to \mathcal{Q}_\delta \times \mathbb{R}^{3N}$ by

$$\mathcal{T}: \quad (q,v) \mapsto (\mathcal{T}_1, \mathcal{T}_2)(q,v) := \left(q, \left(\int_{\partial \mathcal{S}_\kappa(q)} |\nabla \mathcal{A}[q, \tilde{g}(q,v,\cdot)]|^2 \partial_n \boldsymbol{\varphi}_\kappa(q,\cdot)\, ds \right)_{\kappa=1,\dots,N} \right).$$

Recalling (7.9) and Q_q from (7.10), we may further expand

$$\begin{aligned}\mathcal{T}_2(q,v) &= \sum_{1\leqslant i,j\leqslant (3N+1)^2} \sqrt{\mu_i(q,v)\mu_j(q,v)}\left(\int_{\partial\mathcal{S}_\kappa(q)} \nabla\mathcal{A}[q,g_i(q,\cdot)]\cdot\nabla\mathcal{A}[q,g_j(q,\cdot)]\partial_n\boldsymbol{\varphi}_\kappa(q,\cdot)\,ds\right)_{\kappa\in\{1,\dots,N\}}\\ &= 2Q_q\left(\left(\sqrt{\mu_i(q,v)}\right)_{i=1,\dots,(3N+1)^2}\right).\end{aligned}$$

Considering $\mathcal{T}_2$ as a quadratic map of the variable $(\sqrt{\mu_i(q,v)})_{i=1\dots(3N+1)^2}$ with coefficients close to $\delta_{i,j}e_i$, relying on (7.17) and (7.21), we see that for suitably small $\nu>0$ one has for any $q_0\in Q_\delta$

$$\|\mathcal{T}_2(q_0,\cdot)-\mathrm{Id}\|_{C^1(B(0,1))}<\frac{1}{2}.$$

Consequently $\mathcal{T}_2$ constitutes a diffeomorphism from $B(0,1)$ onto its image, which contains at least $B(\mathcal{T}_2(q_0,0),1/2)$. Furthermore, since $\mathcal{T}_2(q_0,0)=\mathcal{O}(\nu)$, reducing $\nu>0$ if necessary, we have that $B(\mathcal{T}_2(q_0,0),1/2)$ contains $B(0,1/4)$. Therefore, for such ν, $\mathcal{T}$ is invertible at any $(q,v)\in\mathcal{Q}_\delta\times B(0,1/4)$.

Now we set for $1\leqslant i\leqslant(3N+1)^2$

$$\widetilde{X}_{q,i}:=\sqrt{\mu_i(q,\tilde{v})}>0 \text{ where } (q,\tilde{v}):=\mathcal{T}^{-1}(q,0), \quad\text{and}\quad \overline{X}_q:=\frac{\widetilde{X}_q}{\|\widetilde{X}_q\|}. \tag{7.23}$$

Using (7.22), we find $Q_q(\overline{X}_q)=0$. Moreover it is easy to check that for $1\leqslant i\leqslant(3N+1)^2$,

$$DQ_q(\overline{X}_q)(0,\dots,0,1,0,\dots,0)=\frac{1}{2}\overline{X}_{q,i}\,e_i+\mathcal{O}(\nu). \tag{7.24}$$

Hence for $\nu>0$ small enough, thanks to (7.17) and to the positivity of the coordinates $\overline{X}_{q,i}$, we see that $\mathrm{Range}(DQ_q(\overline{X}_q))=\mathbb{R}^{3N}$.

Using Lemma 7.6, (7.16), (7.23) and the regularity of $\mathcal{T}^{-1}$, we deduce that $\overline{X}_q$ is Lipschitz with respect to q and consequently $q\mapsto DQ_q(\overline{X}_q)$ is also Lipschitz. In order to apply Proposition 7.3, it remains to make a selection of right inverses of $DQ_q(\overline{X}_q)$ which are Lipschitz with respect to q. A possibility for that, relying on (7.17) is to define

$$A_q:\mathbb{R}^{3N}\longrightarrow\mathbb{R}^{(3N+1)^2}\quad\text{by}\quad A_q(v)_i=2\frac{\mu_i(q,v)}{\overline{X}_i(q,v)},$$

which is Lipschitz with respect to q as a quotient of Lipschitz maps with positive denominator. Then due to (7.24), $DQ_q(\overline{X}_q)\circ A_q=\mathrm{Id}_{\mathbb{R}^{3N}}+O(\nu)$. It is consequently invertible in $\mathbb{R}^{3N}$ through a Neumann series which is consequently also Lipschitz in q. This allows to define unambiguously a right-inverse to $DQ_q(\overline{X}_q)$ in a Lipschitz way with respect to q.

This concludes the proof of Proposition 7.5. □

7.3. **Proof of Proposition 7.1.** Under the assumptions of Proposition 7.1, we first introduce $\mathcal{E}$ and the functions g_i given by Proposition 7.5. Next we set

$$d=3N,\quad E=\mathbb{R}^{(3N+1)^2},\quad F=\cup_{q\in\mathcal{Q}_\delta}\{q\}\times\tilde{\mathscr{K}}\times\mathscr{B}(q,r_\omega),\quad \mathfrak{p}=(q,q',\gamma,\omega),$$

where we recall that $\tilde{\mathscr{K}}$ is defined in (7.1), and for $X:=(X_i)_{1\leqslant i\leqslant 3N+1)^2}$, we set

$$Q_{\mathfrak{p}}(X)=\mathfrak{Q}(q)\left[\sum_{i=1}^{(3N+1)^2}X_ig_i(q,\cdot)\right]\text{ and } L_{\mathfrak{p}}(X)=\mathfrak{L}(q,q',\gamma,\omega)\left[\sum_{i=1}^{(3N+1)^2}X_ig_i(q,\cdot)\right],$$

recalling (6.10) and using (7.10). It is classical that the Kirchhoff potentials φ and the stream function ψ are C^∞ as functions of (q,x) on $\cup_{q\in\mathcal{Q}_\delta}\{q\}\times\mathcal{F}(q)$, see e.g. [5, 30, 39]. The Lipschitz continuity of

$\mathfrak{p} \mapsto L_{\mathfrak{p}}$ then follows from (6.10), the definitions given in Section 6 and the fact that $\mathfrak{L}$ is linear with respect to (q', γ, ω).

We further define τ by changing the topology in the ω component of F to the weak-L^3 topology on $\mathscr{B}(q, r_\omega)$. Using once more the linearity of $\mathfrak{L}$ with respect to ω and the definitions given in Section 6, one can deduce the continuity of the map $\mathfrak{p} \mapsto L_{\mathfrak{p}}$ with respect to the topology τ.

Therefore, the conditions of Proposition 7.3 are satisfied, and we apply it to obtain a Lipschitz map

$$R = (R_1, \dots, R_{(3N+1)^2}) : F \times \mathbb{R}^{3N} \longrightarrow \mathcal{E},$$

such that

$$\mathfrak{Q}(q) \left[\sum_{i=1}^{(3N+1)^2} R_i(q, q', \gamma, \omega, \cdot) g_i(q, \cdot) \right] + \mathfrak{L}(q, q', \gamma, \omega) \left[\sum_{i=1}^{(3N+1)^2} R_i(q, q', \gamma, \omega, \cdot) g_i(q, \cdot) \right] = \mathrm{Id}_{\mathbb{R}^{3N}}.$$

Finally one then sets

$$\mathfrak{R}(q, q', \gamma, \omega) := \sum_{i=1}^{(3N+1)^2} R_i(q, q', \gamma, \omega, \cdot) g_i(q, \cdot),$$

to conclude the proof of Proposition 7.1. □

8. Proof of the existence part of Theorem 3.1

In this section, we prove Theorem 3.1. Let $\delta > 0$ and $\mathcal{E}$ be a finite dimensional subspace of $\mathcal{C}$ as given by Proposition 7.1. Let $T > 0$, $r_\omega > 0$ and $\mathscr{K}$ be a compact subset of $\mathbb{R}^{3N} \times \mathbb{R}^{3N} \times \mathbb{R}^N$. Let $\mathscr{C}$ be given by (7.2). Let q in $C^2([0, T]; \mathcal{Q}_\delta)$ and γ in $\mathbb{R}^N$ such that for any t in $[0, T]$, the triple $(q'(t), q''(t), \gamma)$ is in $\mathscr{K}$. Let ω_0 in $L^\infty(\mathcal{F}(q(0))$ such that

$$\|\omega_0\|_{L^\infty(\mathcal{F}_0)} \leqslant r_\omega. \tag{8.1}$$

To prove the existence part of Theorem 3.1 (i.e. the first item) we look for a velocity field u in $LL(T)$ with $\operatorname{curl} u(0, \cdot) = \omega_0$ and for any t in $[0, T]$,

$$\omega(t, \cdot) := \operatorname{curl} u(t, \cdot) \in \mathscr{B}(q(t), r_\omega),$$

(recall the definition in (3.1)), satisfying, for t in $[0, T]$, the equations (1.1)-(1.2), (1.3)-(1.4) for κ in $\{1, 2, \dots, N\}$, (1.5), (2.2) with g given by (3.2), (2.4) and (2.5).

8.1. **Reduction to a fixed point problem for the vorticity.** Following the analysis of Section 6, we are going to look for a vorticity ω solution to the second equation of System (6.13) with g given by (3.2) and $\mathscr{C}$ by (7.2), i.e.

$$g(t) = \mathscr{C}(q(t), q'(t), q''(t), \gamma, \omega(t, \cdot)). \tag{8.2}$$

Once ω is determined, with this choice of g, the first equation of System (6.13) is satisfied thanks to Proposition 7.1, and according to Section 6, this entails that the fluid velocity u given by (5.13) satisfies (1.1)-(1.2), (1.3)-(1.4) for $\kappa \in \{1, 2, \dots, N\}$, (1.5), (2.2) with g given by (8.2), and (2.5).

Hence we look for a solution ω of the second equation of System (6.13) such that $\omega(t, \cdot)$ is in $\mathscr{B}(q(t), r_\omega)$ for any t in $[0, T]$, and satisfying the condition (2.4) on the entering vorticity and the initial condition $\omega(0, \cdot) = \omega_0$. We will use a fixed point argument. One option would be to look for a fixed point of a mapping which maps a vorticity ω to the solution $\tilde{\omega}$ of the transport equation:

$$\begin{aligned} &(\partial_t + U \cdot \nabla)\tilde{\omega} = 0 \text{ in } \mathcal{F}(q), \text{ for } t \in [0, T], \\ &\tilde{\omega} = 0 \text{ on } \Sigma_-, \text{ for } t \in [0, T], \\ &\tilde{\omega}(0) = \omega_0, \end{aligned} \tag{8.3}$$

where U in $LL(T)$ is the following vector field associated with ω:

$$U := \sum_{\kappa=1}^{N} \nabla(\boldsymbol{\varphi}_\kappa(q,\cdot)\cdot \boldsymbol{q}'_\kappa) + \nabla^\perp \psi_{\omega,\gamma}(q,\cdot) + \nabla\mathcal{A}[q, \mathscr{C}(q,q',q'',\gamma,\omega)], \tag{8.4}$$

cf. Section 5.

However, a main difficulty with this approach is that, since the vector field U is not tangent on Σ due to the presence of the control $\mathcal{A}$, this solution $\tilde{\omega}$ is a priori not properly defined. To define it appropriately, we will use the framework of Chapter VI.1 from [2], which concerns transport equations on fixed domains with open boundary. Therefore, since in our problem the domain is not fixed, but changes with the solid movement, we will first use a diffeomorphism which transforms equation (8.3) into a transport equation on a cylindrical domain (in space-time).

We will then consider our fixed point problem on the level of the transformed system on the cylindrical domain, see Section 8.4.

We will rely on the Schauder fixed point theorem which asserts as we recall that if $\mathcal{B}$ is a nonempty convex closed subset of a normed space $\mathcal{X}$ and $\mathscr{F} : \mathcal{B} \mapsto \mathcal{B}$ is a continuous mapping such that $\mathscr{F}(\mathcal{B})$ is contained in a compact subset of $\mathcal{B}$, then $\mathscr{F}$ has a fixed point.

8.2. **A priori bounds on the fluid velocity.**

We define the spaces

$$\mathcal{X} := L^\infty\left((0,t)\times\mathcal{F}_0\right), \quad \tilde{\mathcal{X}} := L^\infty\left(\cup_{t\in(0,T)}\{t\}\times\mathcal{F}(q(t))\right).$$

In order to define an appropriate set $\mathcal{B}$, we will make use of the following bound on U given by (8.4).

Lemma 8.1. *Suppose that the solid movement q and the circulation γ are fixed and that $\omega \in \tilde{\mathcal{X}}$ satisfies $\|\omega\|_{\tilde{\mathcal{X}}} \leqslant \|\omega_0\|_{L^\infty(\mathcal{F}_0)}$. Then there exists a constant $M > 0$ (which may depend on the solid movement, the circulation and $\|\omega_0\|_{L^\infty(\mathcal{F}_0)}$) such that the function U defined in* (8.4) *satisfies*

$$\|U(t,\cdot)\|_{L^2(\mathcal{F}(q(t)))} \leqslant M, \text{ for almost every } t\in[0,T].$$

The proof of the lemma above will rely on following result, see e.g. [27, Lemma 1].

Lemma 8.2. *There exists $C = C(\mathcal{Q}_\delta) > 0$ such that for any q in $\mathcal{Q}_\delta$, for any $u : \mathcal{F}(q) \to \mathbb{R}^2$, for any $p \geqslant 2$, there holds, with the convention $\|f\|_{W^{1-1/p,p}(\partial\mathcal{F})} := \inf\{\|\overline{f}\|_{W^{1,p}(\mathcal{F})},\ \overline{f} \in W^{1,p}(\mathcal{F}) \text{ and } \overline{f}_{|\partial\mathcal{F}} = f\}$:*

$$\begin{aligned}\|u\|_{W^{1,p}(\mathcal{F}(q))} \leqslant & Cp\left(\|\operatorname{div} u\|_{L^p(\mathcal{F}(q))} + \|\operatorname{curl} u\|_{L^p(\mathcal{F}(q))}\right)\\ &+ C\left(\|u\cdot n\|_{W^{1-1/p,p}(\partial\mathcal{F}(q))} + \sum_{\kappa=1}^N \left|\int_{\partial\mathcal{S}_\kappa(q)} u\cdot\tau \,\mathrm{d}s\right|\right).\end{aligned}$$

Proof of Lemma 8.1. Relying on the div-curl system satisfied by U, it follows from Lemma 8.2 that for almost any $t \in [0,T]$ one has

$$\|U(t,\cdot)\|_{L^2(\mathcal{F}(q(t)))} \leqslant C(1 + \|\omega\|_{\tilde{\mathcal{X}}} + \|\mathscr{C}(q,q',q'',\gamma,\omega)(t,\cdot)\|_{H^{1/2}(\Sigma)}),$$

where $C > 0$ is a constant which may depend on the solid movement q and the circulation γ. Therefore it suffices to find a bound on $\|\mathscr{C}(q,q',q'',\gamma,\omega)(t,\cdot)\|_{H^{1/2}(\Sigma)}$ for almost every $t \in [0,T]$. However, this follows from Propositions 6.1 and 7.1. $\square$

8.3. **Passing to a cylindrical domain.** We first introduce a time-dependent diffeomorphism which tranforms the fluid domain into a cylindrical domain.

Lemma 8.3. *Let $T > 0$ and $\delta > 0$ suitably small. Let $q \in C^2([0,T];\mathcal{Q}_\delta)$. There exists a C^2 time-dependent family of smooth area-preserving diffeomorphisms $(\varphi_t)_{t\in[0,T]}$ of $\overline{\Omega}$, such that: $\varphi_0 = Id$, $\varphi_t(x) = x$ on $\mathcal{V}_{\delta/4}(\partial\Omega)\cap\overline{\Omega}$, and for all $t \in [0,T]$ and $\kappa = 1,\dots,N$,*

$$\varphi_t(x) = h_\kappa(t) + R(\theta_\kappa(t))(x - h_\kappa(0)) \text{ on } \mathcal{V}_{\delta/4}(\mathcal{S}_{\kappa,0}).$$

where $\mathcal{V}_\delta(A)$ is the δ-neighborhood of a set $A \subset \mathbb{R}^2$ and where we used the notations of (1.6)-(1.7).

In particular, for any $t \in [0,T]$ and $\kappa = 1,\dots,N$, φ_t maps $\mathcal{S}_{\kappa,0}$ onto $\mathcal{S}_\kappa(q(t))$ and hence maps $\mathcal{F}_0$ onto $\mathcal{F}(q(t))$.

Proof of Lemma 8.3. The proof is essentially the same as the one of [27, Proposition 1]. Reducing $\delta > 0$ if necessary, we may suppose that $\mathcal{V}_\delta(\partial\Omega)$ and $\mathcal{V}_\delta(\partial\mathcal{S}_\kappa)$, $\kappa = 1,\dots,N$, are tubular neighborhoods of $\partial\Omega$ and $\partial\mathcal{S}_\kappa$, respectively. (This does not depend on $q(t)$ since $\mathcal{S}_\kappa(q(t))$ is obtained by a rigid movement from $\mathcal{S}_{\kappa,0}$.)

Now φ_t is obtained as the flow of a smooth time-dependent vector field. Let $\Lambda \in C^\infty(\mathbb{R};[0,1])$ such that $\Lambda \equiv 1$ on $(-\infty,\frac{1}{4}]$ and $\Lambda \equiv 0$ on $[\frac{1}{2},+\infty)$. For $t \in [0,T]$, we let

$$V(t,x) := \sum_{\kappa=1}^{N} \nabla^\perp \left[\Lambda\left(\frac{d_s(x,\mathcal{S}_\kappa(q(t)))}{\delta}\right)\left(x^\perp \cdot h'_\kappa(t) + \frac{|x-h_\kappa(t)|^2}{2}\theta'_\kappa(t)\right)\right],$$

where $d_s(\cdot,\partial\mathcal{S}_\kappa)$ denotes the signed distance to $\partial\mathcal{S}_\kappa$ (negative inside $\mathcal{S}_\kappa$), which is regular in a tubular neighborhood of $\partial\mathcal{S}_\kappa$. Note that the terms in the above sum have disjoint supports, due to the properties of Λ and the fact that q has values in Q_δ.

One can easily check that for each $t \in [0,T]$, $V(t,\cdot)$ is compactly supported in Ω, divergence-free, and that it coincides with the solid velocity $h'_\kappa(t) + \theta'_\kappa(t)(x - h_\kappa(t))^\perp$ on each $\mathcal{V}_{\delta/4}(\mathcal{S}_\kappa(q(t)))$. Hence considering $\varphi_t(\cdot)$ as the flow associated with V, this proves the claim. □

We then introduce $\psi_t := \varphi_t^{-1}$, so that ψ_t maps $\mathcal{F}(q(t))$ to $\mathcal{F}_0$ and is also area-preserving. Let $U \in \mathrm{LL}(T)$ be divergence-free and consider a function $\tilde\omega \in \tilde{\mathcal{X}}$. We may define

$$U_*(t,x) := [d\varphi_t(x)]^{-1}\cdot U(t,\varphi_t(x)), \quad \tilde\omega_*(t,x) := \tilde\omega(t,\varphi_t(x)), \text{ for } x \in \mathcal{F}_0. \tag{8.5}$$

Since φ is volume-preserving, one has that U_* is divergence-free. Furthermore, it is easy to check that the following hold in the weak sense on $\cup_{t\in(0,T)}\{t\}\times\mathcal{F}(q(t))$:

$$\begin{aligned}
&\partial_t\tilde\omega(t,x) = \partial_t\tilde\omega_*(t,\psi_t(x)) + \nabla\tilde\omega_*(t,\psi_t(x))\cdot\frac{d}{dt}\psi_t(x),\\
&\nabla\tilde\omega(t,x) = d\psi_t(x)\cdot\nabla\tilde\omega_*(t,\psi_t(x)).
\end{aligned}$$

It then follows, using the fact that $d\psi_t(\cdot) = [d\varphi_t(\cdot)]^{-1}$, that $\tilde\omega$ is a solution of (8.3) if and only if $\tilde\omega_*$ is a solution of the system

$$\begin{aligned}
&(\partial_t + V\cdot\nabla)\,\tilde\omega_*(t,x) = 0, \text{ for } t\in[0,T],\ x\in\mathcal{F}_0,\\
&\tilde\omega_* = 0 \text{ on } \Sigma_-, \text{ for } t\in[0,T],\\
&\tilde\omega(0)_* = \omega_0,
\end{aligned} \tag{8.6}$$

where $V(t,x) = U_*(t,x) + \frac{d}{dt}\psi_t(\varphi_t(x))$.

It is straightforward to see that since $U \in LL(T)$ and φ_t is a C^∞ diffeomorphism which is C^2 with respect to time, one has that $V \in L^1((0,T);W^{1,1}(\mathcal{F}_0;\mathbb{R}^2))$ and $\operatorname{div} V = 0$ due to φ_t being volume-preserving. Therefore, conditions (VI.4) to (VI.6) in Theorem VI.1.6 from [2] are satisfied, and it follows that there exists a unique weak solution $\tilde\omega_*$ in $\mathcal{X}$ of (8.6).

Note that inverting the transformations (8.5), one obtains a weak solution $\tilde{\omega} \in \tilde{\mathcal{X}}$ of (8.3).

8.4. **Definition of an appropriate operator.** We may now precise the functional setting for a fixed point problem involving system (8.6).

We set

$$\mathcal{B} := \left\{ \omega_* \in \mathcal{X} : \ \|\omega_*\|_{\mathcal{X}} \leqslant \|\omega_0\|_{L^\infty(\mathcal{F}_0)}, \ \|\partial_t \omega_*\|_{L^2((0,T);H^{-1}(\mathcal{F}_0))} \leqslant C_\varphi \,(M+1)\, \sqrt{T} \|\omega_0\|_{L^\infty(\mathcal{F}_0)} \right\},$$

where M is given by Lemma 8.1 and

$$C_\varphi := \sup_{t\in[0,T]} \max \left\{ \left\| \frac{d}{dt} \psi_t(\varphi_t(\cdot)) \right\|_{L^2(\mathcal{F}_0)}, \|d\psi_t(\varphi_t(\cdot))\|_{L^\infty(\mathcal{F}_0)} \right\}.$$

We endow $\mathcal{X} = L^\infty((0,T)\times\mathcal{F}_0)$ with the $C^0([0,T];\text{weak-}L^3(\mathcal{F}_0))$ topology. Since $\mathcal{B}$ is convex and closed in the $C^0([0,T];L^3(\mathcal{F}_0))$ topology, it follows that it is also closed in the $C^0([0,T];\text{weak-}L^3(\mathcal{F}_0))$ topology. We define the operator $\mathscr{F} : \mathcal{B} \longrightarrow \mathcal{B}$ as follows.

Let ω_* in $\mathcal{B}$ and set $\omega(t,x) := \omega_*(t,\psi_t(x))$ for $t\in[0,T]$, $x \in \mathcal{F}(q(t))$. Hence, $\omega \in \tilde{\mathcal{X}}$. We associate with ω the function U in (8.4), and implicitly U_* via (8.5). Thus we may define

$$\mathscr{F}(\omega_*) := \tilde{\omega}_*,$$

where $\tilde{\omega}_*$ is the solution of (8.6).

Let us check that $\mathscr{F}(\mathcal{B}) \subset \mathcal{B}$. The bound $\|\tilde{\omega}_*\|_{\mathcal{X}} \leqslant \|\omega_0\|_{L^\infty(\mathcal{F}_0)}$ is immediate via Theorem VI.1.6 from [2]. For the bound on the time derivative, we have for almost any $t \in [0,T]$, for any $\Phi \in H^1_0(\mathcal{F}_0)$ with $\|\Phi\|_{H^1_0(\mathcal{F}_0)} = 1$, that

$$\int_{\mathcal{F}_0} \partial_t \tilde{\omega}_* \ \Phi \, dx = -\int_{\mathcal{F}_0} \operatorname{div}(\tilde{\omega}_* V) \ \Phi \, dx = \int_{\mathcal{F}_0} \tilde{\omega}_* V \cdot \nabla \Phi \, dx \leqslant \|\omega_0\|_{L^\infty(\mathcal{F}_0)} \|V(t,\cdot)\|_{L^2(\mathcal{F}_0)}.$$

It follows that

$$\begin{aligned}
\|\partial_t \tilde{\omega}(t,\cdot)\|_{H^{-1}(\mathcal{F}(q(t)))} &\leqslant \|\omega_0\|_{L^\infty(\mathcal{F}_0)} \left(\|U_*(t,\cdot)\|_{L^2(\mathcal{F}_0)} + \left\| \frac{d}{dt}\psi_t(\varphi_t(\cdot)) \right\|_{L^2(\mathcal{F}_0)} \right) \\
&\leqslant \|\omega_0\|_{L^\infty(\mathcal{F}_0)} \left(\sup_{t\in[0,T]} \|d\psi_t(\varphi_t(\cdot))\|_{L^\infty(\mathcal{F}_0)} \|U(t,\cdot)\|_{L^2(\mathcal{F}(q(t)))} + \left\| \frac{d}{dt}\psi_t(\varphi_t(\cdot)) \right\|_{L^2(\mathcal{F}_0)} \right) \\
&\leqslant C_\varphi \|\omega_0\|_{L^\infty(\mathcal{F}_0)} \left(\|U(t,\cdot)\|_{L^2(\mathcal{F}(q(t)))} + 1 \right),
\end{aligned}$$

from where we may conclude that $\mathscr{F}(\mathcal{B}) \subset \mathcal{B}$ by using Lemma 8.1.

To prove that $\mathscr{F}$ has a fixed point by Schauder's theorem, it remains to show that $\mathscr{F}$ is continuous and that $\mathscr{F}(\mathcal{B})$ is relatively compact (with respect to the $C^0_t(\text{weak-}L^3_x)$ topology).

8.5. **Continuity.** The following result follows from Lemma 8.2 by using the classical Yudovich argument (see [53, Lemma 2.2]):

Lemma 8.4. *There exists $C = C(\mathcal{Q}_\delta) > 0$ such that for any q in $\mathcal{Q}_\delta$, for any $u : \mathcal{F}(q) \to \mathbb{R}^2$ with* $\operatorname{div} u = 0$*, there holds*

$$\|u\|_{\textit{log-Lip}(\mathcal{F}(q))} \leqslant C \left(\|\operatorname{curl} u\|_{L^\infty(\mathcal{F}(q))} + \|u\cdot n\|_{C^{1,1/2}(\partial\mathcal{F}(q))} + \sum_{\kappa=1}^N \left| \int_{\partial\mathcal{S}_\kappa(q)} u\cdot\tau \, \mathrm{d}s \right| \right).$$

Now let $(\omega_{*,m})_{m\geqslant 1} \in \mathcal{B}^{\mathbb{N}}$, converging to some ω_* in $\mathcal{B}$ with respect to the $C^0([0,T];\text{weak-}L^3(\mathcal{F}_0))$ topology. We define

$$\omega_m(t,x) := \omega_{*,m}(t,\psi_t(x)) \text{ for } t\in[0,T], \ x\in\mathcal{F}(q(t)), \tag{8.7}$$

and

$$U_m := \sum_{\kappa=1}^{N} \nabla(\boldsymbol{\varphi}_\kappa(q,\cdot)\cdot \boldsymbol{q}'_\kappa) + \nabla^\perp \psi_{\omega_m,\gamma}(q,\cdot) + \nabla \mathcal{A}[q, \mathscr{C}(q,q',q'',\gamma,\omega_m)], \text{ for } m \geqslant 1, \tag{8.8}$$

U via (8.4) and correspondingly $\tilde{\omega}_{*,m} = \mathscr{F}(\omega_{*,m})$, respectively $\tilde{\omega}_* = \mathscr{F}(\omega_*)$ as above. Using Lemma 8.2 it follows that U and U_m are uniformly bounded in $\mathrm{LL}(T)$. Furthermore, it is easy to check that $\omega_m \to \omega$ in $C^0([0,T];\text{weak-}L^3(\mathcal{F}(q(t))))$. Using the continuity property at the end of Proposition 7.1, (7.2) and Lemma 7.2, one may deduce that

$$\mathscr{C}(q(t),q'(t),q''(t),\gamma,\omega_m(t,\cdot)) \to \mathscr{C}(q(t),q'(t),q''(t),\gamma,\omega(t,\cdot)) \text{ in } \mathcal{E}, \text{ for all } t\in[0,T].$$

Hence, using (8.4), (8.8) and Lemma 8.2, it is easy to see that $U_m \to U$ in $C^0([0,T];L^1(\mathcal{F}(q(t))))$.

To obtain the convergence of $\mathscr{F}(\omega_{*,m})$ to $\mathscr{F}(\omega_*)$ in the $C^0([0,T];\text{weak-}L^3(\mathcal{F}_0))$ topology, we will use the stability result Theorem VI.1.9 from [2]. To do so, we further consider the transformations

$$U_{*,m}(t,x) := [d\varphi_t(x)]^{-1}\cdot U_m(t,\varphi_t(x)), \text{ for } t\in[0,t],\ x\in\mathcal{F}_0,\ m\geqslant 1. \tag{8.9}$$

Thus, one obtains the systems

$$\begin{aligned} &(\partial_t + V_m\cdot\nabla)\,\tilde{\omega}_{*,m}(t,x) = 0, \text{ for } t\in[0,T],\ x\in\mathcal{F}_0,\\ &\tilde{\omega}_{*,m} = 0 \text{ on } \Sigma_-, \text{ for } t\in[0,T],\\ &\tilde{\omega}_{*,m}(0) = \omega_0, \end{aligned} \tag{8.10}$$

where $V_m(t,x) = U_{*,m}(t,x) + \frac{d}{dt}\psi_t(\varphi_t(x))$.

Out of the conditions (VI.28a) to (VI.28g) from Theorem VI.1.9 in [2], the only ones that need checking are (VI.28a) and (VI.28c), the others are trivially satisfied. More precisely these two conditions are

$$V_m \to V \text{ in } L^1((0,T)\times\mathcal{F}_0) \text{ and } V_m\cdot n \to V\cdot n \text{ in } L^1((0,T)\times\Sigma).$$

However, these follow from $U_m \to U$ in $C^0([0,T];L^1(\mathcal{F}(q(t))))$ and the continuity of $\mathscr{C}$ already mentioned above.

Therefore we may apply Theorem VI.1.9 from [2] to obtain that $\tilde{\omega}_{*,m} \to \tilde{\omega}_*$ in $C^0([0,T];L^3(\mathcal{F}_0))$, and hence also in $C^0([0,T];\text{weak-}L^3(\mathcal{F}_0))$.

8.6. **Relative compactness.** Let $(\omega_{*,m})_{m\geqslant1} \in \mathcal{B}^{\mathbb{N}}$ and associate once more $(\omega_m)_{m\geqslant1}$, $(U_m)_{m\geqslant1}$ and $(U_{*,m})_{m\geqslant1}$ via (8.7), (8.8), respectively (8.9). Since $(\omega_{*,m})_{m\geqslant1}$ is uniformly bounded in L^∞ and $(\partial_t\omega_m)_{m\geqslant1}$ is uniformly bounded in $L^2_t(H^{-1}_x)$, it follows from the Aubin-Lions lemma (c.f. Appendix C in [38]) that $(\omega_{*,m})_{m\geqslant1}$ converges to some ω_* with respect to the $C^0([0,T];\text{weak-}L^3(\mathcal{F}_0))$ topology. Clearly, this limit is also in $\mathcal{B}$.

One may once again set $\omega(t,x) := \omega_*(t,\psi_t(x))$ and associate U via (8.4). Now one proceeds as in the previous section, to obtain that $U_m \to U$ in $C^0([0,T];L^1(\mathcal{F}(q(t))))$. Then, one may associate the systems (8.6) respectively (8.10). It is easy to see that $V_m \to V$ in $L^1((0,T)\times\mathcal{F}_0)$ and $V_m\cdot n \to V\cdot n$ in $L^1((0,T)\times\Sigma)$ holds once more, hence due to Theorem VI.1.9 from [2] we have that $\mathscr{F}(\omega_{*,m}) \to \mathscr{F}(\omega_*)$ in the $C^0([0,T];\text{weak-}L^3(\mathcal{F}_0))$ topology.

8.7. **Conclusion.** Schauder's fixed point theorem then implies that $\mathscr{F}$ has a fixed point in $\omega_*\in\mathcal{B}$. Setting $\omega(t,x) := \omega_*(t,\psi_t(x))$ for $t\in[0,T]$, $x\in\mathcal{F}(q(t))$ then gives a solution of the second equation of System (6.13), and due to (8.1), $\omega(t,\cdot)$ is in $\mathscr{B}(q(t),r_\omega)$ for any t in $[0,T]$. Using Lemma 8.2, one may in fact also deduce that the associated fluid velocity field is in $C^0([0,T];W^{1,p}(\mathcal{F}(q(t))))$, for all $p\in[1,+\infty)$. The Lipschitz dependence of the control (in particular in the strong L^3_x topology with respect to the vorticity) follows directly from Proposition 7.1. This concludes the proof of the first part of Theorem 3.1.

9. Proof of the uniqueness part of Theorem 3.1

This section is devoted to the uniqueness part of Theorem 3.1. Let $\delta > 0$. We consider the finite dimensional subspace $\mathcal{E}$ of $\mathcal{C}$ given by Proposition 7.1, as well as $T > 0$, $r_\omega > 0$, a compact subset $\mathscr{K}$ of $\mathbb{R}^{3N} \times \mathbb{R}^{3N} \times \mathbb{R}^N$, and the control law

$$\mathscr{C} \in \operatorname{Lip}(\cup_{q \in \mathcal{Q}_\delta}\{q\} \times \mathscr{K} \times \mathscr{B}(q, r_\omega); \mathcal{E}),$$

given by (7.2). We also consider a trajectory q in $C^2([0,T]; \mathcal{Q}_\delta)$ and γ in $\mathbb{R}^N$ such that for any t in $[0,T]$, the triple $(q'(t), q''(t), \gamma)$ is in $\mathscr{K}$,

$$(\tilde{q}, \tilde{u}) \in C^2([0,T]; \mathcal{Q}_\delta) \times [LL(T) \cap C^0([0,T]; W^{1,p}(\mathcal{F}(t)))], \text{ for all } p \in [1, +\infty),$$

and $\tilde{\gamma}$ in $\mathbb{R}^N$ such that for any t in $[0,T]$, the triple $(\tilde{q}'(t), \tilde{q}''(t), \tilde{\gamma})$ is in $\mathscr{K}$ and $\operatorname{curl} \tilde{u}(t, \cdot)$ is in $\mathscr{B}(\tilde{q}(t), r_\omega)$. We assume that $(\tilde{q}, \tilde{u})$ satisfies the Euler equations (1.1)-(1.2), the Newton equations (1.3)-(1.4) for κ in $\{1, 2, \ldots, N\}$, the interface condition (1.5), the boundary condition (2.2) on the normal velocity with g given by (3.3), the boundary condition (2.4) on the entering vorticity, the circulation conditions (2.5) (with $\tilde{\gamma}$ instead of γ) and the initial conditions

$$\tilde{q}(0) = q(0) \text{ and } \tilde{q}'(0) = q'(0). \tag{9.1}$$

Then it follows from the analysis performed in Section 6, in particular from the first equation of (6.13), applied to the solution $(\tilde{q}, \tilde{u})$, that

$$\mathfrak{Q}(\tilde{q})[g] + \mathfrak{L}(\tilde{q}, \tilde{q}', \tilde{\gamma}, \operatorname{curl} \tilde{u})[g] = \mathfrak{F}(\tilde{q}, \tilde{q}', \tilde{q}'', \tilde{\gamma}, \operatorname{curl} \tilde{u}). \tag{9.2}$$

Then by (3.3), (7.2) and Proposition 7.1, we infer that

$$\mathfrak{Q}(\tilde{q})[g] + \mathfrak{L}(\tilde{q}, \tilde{q}', \tilde{\gamma}, \operatorname{curl} \tilde{u})[g] = \mathfrak{F}(\tilde{q}, \tilde{q}', q'', \tilde{\gamma}, \operatorname{curl} \tilde{u})$$

and therefore we arrive at

$$\mathfrak{F}(\tilde{q}, \tilde{q}', q'', \tilde{\gamma}, \operatorname{curl} \tilde{u}) = \mathfrak{F}(\tilde{q}, \tilde{q}', \tilde{q}'', \tilde{\gamma}, \operatorname{curl} \tilde{u}). \tag{9.3}$$

Now, it follows from Proposition (6.1) that, for any admissible $(q, q', q'', \gamma, \omega)$, the term $\mathfrak{F}(q, q', q'', \gamma, \omega)$ can be decomposed into

$$\mathfrak{F}(q, q', q'', \gamma, \omega) = \mathcal{M}^a(q) q'' + \widetilde{\mathfrak{F}}(q, q', \gamma, \omega), \tag{9.4}$$

where the second term needs not to be written explicitly here. Then, using the decomposition (9.4) for both sides of (9.3), simplifying by $\widetilde{\mathfrak{F}}(\tilde{q}, \tilde{q}', \tilde{\gamma}, \operatorname{curl} \tilde{u})$, using the invertibility of the matrices $\mathcal{M}^a(q)$, and integrating twice in time using the initial data (9.1), we deduce that $q = \tilde{q}$ on $[0,T]$ and the proof of Theorem 3.1 is over.

For sake of completeness, let us explain how the proof above can be adapted to deal with the case where the initial positions and velocities of the rigid bodies do not match, that is if $(\tilde{q}(0), \tilde{q}'(0)) \neq (q(0), q'(0))$, but $\tilde{q}(0)$ is sufficiently close to $q(0)$, as mentioned in the comment regarding this part below the statement of Theorem 3.1. If $\tilde{q}(0)$ and $q(0)$ are not close, one may first use Theorem 3.1 to drive $\tilde{q}$ close to $q(0)$.

In this case we replace the control law (3.3) by (3.4) and the desired result is that the error $q(t) - \tilde{q}(t)$ exponentially decays to 0 as the time t goes to $+\infty$. Indeed in this case, proceeding as above, instead of (9.3), we obtain the following identity:

$$\mathfrak{F}(\tilde{q}, \tilde{q}', q'' + K_P(q - \tilde{q}) + K_D(q' - \tilde{q}'), \tilde{\gamma}, \operatorname{curl} \tilde{u}) = \mathfrak{F}(\tilde{q}, \tilde{q}', \tilde{q}'', \tilde{\gamma}, \operatorname{curl} \tilde{u}). \tag{9.5}$$

Using the decomposition (9.4) and the invertibility of the matrices $\mathcal{M}^a(q)$ we deduce that the error $e := q - \tilde{q}$ satisfies the linear differential equation $e'' + K_P\, e + K_D\, e' = 0$. Since the matrices K_P and K_D are positive definite symmetric it follows that $e(t)$ exponentially decays to 0 as the time t goes to

$+\infty$, with a rate which can be made arbitrarily fast by appropriate choices of K_P and K_D, see [44, Proposition 4.8].

10. Some extra comments on the issue of energy saving

As mentioned in the paragraph on the energy saving in the commentary below Theorem 3.1 one may wonder whether it is possible to turn on the control only when this becomes necessary. Since an uncontrolled equation is characterized by the equation $\mathfrak{F}(q, q', q'', \gamma, \omega) = 0$, see Section 6, this issue can be formulated as the following open problem where the targeted trajectory satisfies the uncontrolled equation at the initial time but perhaps not for positive times.

Open problem 10.1. *For any $T > 0$, ω_0 in $L^\infty(\mathcal{F}(q(0))$, γ in $\mathbb{R}^N$ and q in $C^2([0,T];\mathcal{Q})$ such that $\mathfrak{F}(q(0), q'(0), q''(0), \gamma, \omega_0) = 0$, is there a boundary control g in $C^\infty([0,T];\mathcal{C})$ with $g(0,\cdot) = 0$ and a velocity field u in $LL(T) \cap C^0([0,T]; W^{1,p}(\mathcal{F}(t)))$, for all $p \in [1,+\infty)$, with $\operatorname{curl} u(0,\cdot) = \omega_0$ such that, for t in $[0,T]$, (1.1)-(1.2), (1.3)-(1.4) for κ in $\{1, 2, \ldots, N\}$, (1.5), (2.2), (2.4) and (2.5) hold true ?*

Several comments are in order.

First observe that, taking into account the decomposition (5.13), the issue stated in Open problem 10.1 is related to the question of whether it is possible to prescribe the initial fluid velocity rather than the initial fluid vorticity (as we actually did in the first part of Theorem 3.1). Since the fluid velocity in the fluid domain depends on the trace of its normal component on the boundary, it is necessary to require a compatibility condition between the initial value of the control g and the initial value of the fluid velocity u. Indeed a positive answer to Open problem 10.1 would entail that for any $T > 0$, for any log-Lipschitz vector field u_0 such that $\operatorname{curl} u_0$ in $L^\infty(\mathcal{F}(q(0))$,

$$\operatorname{div} u_0 = 0 \text{ in } \mathcal{F}(q_0), \quad u_0 \cdot n = 0 \text{ on } \partial\Omega, \quad u_0 \cdot n = \big(\theta'_\kappa(0)(\cdot - h_\kappa(0))^\perp + h'_\kappa(0)\big) \cdot n \text{ on } \partial\mathcal{S}_\kappa(0),$$

and q in $C^2([0,T];\mathcal{Q})$ such that $\mathfrak{F}(q(0), q'(0), q''(0), \gamma, \operatorname{curl} u_0) = 0$, where

$$\gamma := (\gamma_\kappa)_{\kappa=1,\ldots,N}, \quad \text{ with } \quad \gamma_\kappa = \int_{\partial\mathcal{S}_\kappa(0)} u_0 \cdot \tau \, \mathrm{d}s, \quad \text{ for all } \ \kappa \in \{1, 2, \ldots, N\},$$

there is a boundary control g in $C^\infty([0,T];\mathcal{C})$ with $g(0,\cdot) = 0$ and a velocity field u in $LL(T) \cap C^0([0,T]; W^{1,p}(\mathcal{F}(t)))$, for all $p \in [1,+\infty)$, with $u(0,\cdot) = u_0$ such that, for t in $[0,T]$, (1.1)-(1.2), (1.3)-(1.4) for κ in $\{1, 2, \ldots, N\}$, (1.5), (2.2), (2.4) and (2.5) hold true.

Let us also observe that if one is able to answer by a positive result to Open problem 10.1, then by using the time-reversibility of the system, one can deduce the following result where the targeted trajectory is an uncontrolled solution, associated with a vanishing vorticity, at the initial and final times, that is the result that for any $T > 0$, γ in $\mathbb{R}^N$ and q in $C^2([0,T];\mathcal{Q})$ such that

$$\mathfrak{F}(q(0), q'(0), q''(0), \gamma, 0) = 0 \text{ and } \mathfrak{F}(q(T), q'(T), q''(T), \gamma, 0) = 0,$$

there is a boundary control g in $C^\infty([0,T];\mathcal{C})$ with $g(0,\cdot) = g(T,\cdot) = 0$ and a velocity field u in $LL(T) \cap C^0([0,T]; W^{1,p}(\mathcal{F}(t)))$, for all $p \in [1,+\infty)$, with $\operatorname{curl} u(t,\cdot) = 0$ for any t in $[0,T]$, such that, for t in $[0,T]$, (1.1)-(1.2), (1.3)-(1.4) for κ in $\{1, 2, \ldots, N\}$, (1.5), (2.2), (2.4) and (2.5) hold true. Here we have restricted the issue to the setting of irrotational flows since it is the only case where the vorticity dynamics is under control. Indeed it seems difficult to reach a targeted trajectory which is at time $T > 0$ an uncontrolled solution corresponding to a non vanishing given vorticity with a control vanishing at time T, because the dynamics of the vorticity which remains close to the rigid bodies seems difficult to control from the external boundary.

Inspecting the proof of Proposition 7.3 we observe that the mapping R which is constructed there satisfies $R(\mathfrak{p}, 0) = \overline{X}_{\mathfrak{p}}$ (for any $\mathfrak{p}$). In particular since $\|\overline{X}_{\mathfrak{p}}\| = 1$, we have $R(\mathfrak{p}, 0) \neq 0$. It would be interesting to investigate alternative constructions of similar mappings R with the additional condition $R(\mathfrak{p}, 0) = 0$ since this would entail that the corresponding mappings $\mathscr{C}(q, q', q'', \gamma, \omega)$ defined by (7.2) vanishes when $\mathfrak{F}(q, q', q'', \gamma, \omega) = 0$. Perhaps tools from algebraic geometry could be useful, see [18].

If one looks for a control g of a different form than $g = \mathscr{C}(q, q', q'', \gamma, \omega)$, potentially not in the set $\mathcal{C}_b(q)$, one may wonder whether it is possible to take advantage of the term with the time derivative in (6.5) to control the motion, with the idea to determine the control as the solution of a first order ODE in time. If the quadratic term does not vanish for the controls chosen in this strategy, then it is a nonlinear ODE which may lead to a blow-up in finite time. In our construction, because of the rigidity of harmonic functions, it seems difficult to find controls for which the term in the parenthesis in the first term of the right hand side of (6.5) reaches arbitrary value while corresponding to a vanishing quadratic term. Therefore this seems limited to the case where the targeted motion for the rigid bodies is close to an uncontrolled solution for which the right hand side of (6.12) vanishes. However, it could be that one may start with such a control before switching to the quadratic control constructed in this paper.

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CEREMADE, UMR CNRS 7534, Université Paris-Dauphine, PSL Research University, Place du Maréchal de Lattre de Tassigny, 75775 Paris Cedex 16, France

Institut für Mathematik, Universität Leipzig, D-04109, Leipzig, Germany

Institut de Mathématiques de Bordeaux, UMR CNRS 5251, Université de Bordeaux, 351 cours de la Libération, F33405 Talence Cedex, France & Institut Universitaire de France