On the controllability of the fifth order Korteweg-de Vries equation

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1 Introduction

In this paper, we study the controllability of the fifth order Korteweg-de Vries equation:

$$u_t + \alpha u_{5x} + \mu u_{xxx} + \beta u u_{xxx} + \delta u_x u_{xx} + P'(u) u_x = 0, \tag{1}$$

where α, μ, β and δ are real constants and P is a cubic polynomial:

$$P(u) = pu + qu^2 + ru^3. \tag{2}$$

This class of equations was introduced by Kichenassamy and Olver [16]. It contains in particular the Kawahara equation [14] introduced to model magneto-acoustic waves, the various models derived by Olver [18] for the unidirectional propagation of waves in shallow water when the third order term appearing in the Korteweg-de Vries equation is small, and many other models. See [16] for a discussion of them.

In this paper, we are interested in studying this equation in a bounded domain. We will both consider the Cauchy problem with boundary conditions and the boundary controllability problem. Note that there is an important literature concerning the Cauchy problem in the real line, see for instance [6, 9, 16, 15, 17, 20] and references therein. But for what concerns the boundary value problem, the problem was still completely open as far as we know. Note that the initial boundary value problem for the (third-order) Korteweg-de Vries equation has drained much attention (see in particular [1, 4, 5, 10, 13]). The controllability problem was also, up to our knowledge, completely open. The equivalent for the Korteweg-de Vries equation has also known many developments lately [2, 3, 7, 12, 21, 22, 23, 24, 25].

To be more precise, we will consider in the sequel that $\alpha > 0$: this is not a restriction since it suffices to make the change of variable $x' = 1 - x$ and to invert the role of the left and right boundaries. The spatial domain will be $[0, 1]$, which is not a restriction either in the present paper, since it will suffice to rescale in space to obtain a result on an interval of arbitrary length. (Note that this is not necessarily the case for the Korteweg-de Vries equation with Neumann boundary control, see [21]).

The boundary conditions that we will consider are the following:

$$u_{|x=0} = v_1, \quad u_{|x=1} = v_2, \quad u_{x|x=0} = v_3, \quad u_{x|x=1} = v_4, \quad u_{xx|x=0} = v_5. \tag{3}$$

The first and main result of this paper concerns a boundary controllability result for equation (1). To be more precise, we will control the system from the right endpoint (by using only v_2) and v_4 while maintaining v_1, v_3, v_5 to zero), and the type of controllability that we consider is the *local controllability to trajectories*. That is to say, we consider $T > 0$ and a fixed trajectory $\overline{u}$ of (1), and prove that for any initial state u_0 sufficiently close to $\overline{u}_{|t=0}$, there exist controls (v_2, v_4) which steer the system from u_0 to $\overline{u}_{|t=T}$.

The precise result is the following.

Theorem 1. *Let $T > 0$. Let $\overline{u} \in L^\infty(0, T; W^{3,\infty}(0, 1))$ be a trajectory of (1) with boundary conditions $\overline{u}_{|x=0} = \overline{u}_{x|x=0} = \overline{u}_{xx|x=0} = 0$. There exists $\varepsilon > 0$ such that for any $u_0 \in L^2(0, 1)$ such that*

$$\|u_0 - \overline{u}(0, \cdot)\|_{L^2(0,1)} \leq \varepsilon, \tag{4}$$

there exist two controls v_2, v_4 in $L^2(0, T)$ such that the solution u of (1) with initial condition

$$u_{|t=0} = u_0, \tag{5}$$

and boundary condition (3) *with controls* $(0, v_2, 0, v_4, 0)$, *belongs to* $C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^2(0,1))$ *and satisfies*

$$u_{|t=T} = \overline{u}(T, \cdot). \tag{6}$$

As we will see, the solution to the controllability problem that we construct is in fact more regular than $C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^2(0,1))$. Indeed, in order to prove Theorem 1, we will first take the controls (v_2, v_4) as zero and we will prove, thanks to Theorem 2 below, that the state becomes H^2_0. Then we will work with more regular solutions (belonging to $L^2(\epsilon, T; H^4(0,1)) \cap C^0([\epsilon, T]; H^2(0,1))$).

Remark 1. *Using the reversible character of this system stated on the whole real line, it is not difficult to deduce that equation* (1) *is locally exactly controllable near* 0 *when using the five controls (for sufficiently regular states).*

The second result of this paper concerns the Cauchy problem, where we can prove that the problem is well posed locally in time, and regularizes the state of the system.

Theorem 2. *Given* $u_0 \in L^2(0,1)$, *there exists* $T > 0$ *such that the nonlinear problem* (1) *with homogenous boundary conditions* (3) *(*$v_1 = v_2 = v_3 = v_4 = v_5 = 0$*) admits a unique solution* $u \in L^2(0,T; H^2(0,1)) \cap C^0([0,T]; L^2(0,1))$ *satisfying*

$$\|u\|_{L^2(0,T;H^2(0,1)) \cap C^0([0,T];L^2(0,1))} \leq C \|u_0\|_{L^2(0,1)}.$$

Moreover, this solution regularizes in the sense that for any $\tau \in (0,T]$, $u \in C^\infty([\tau, T] \times [0,1])$, *with*

$$\|u\|_{H^k(\tau,T;H^k(0,1))} \leq C(\tau, k) \|u_0\|_{L^2(0,1)}. \tag{7}$$

We conclude this introduction by a remark concerning the choice of the controls among $v_1, \ldots, v_5$. The controllability to the trajectories described in Theorem 1 may not take place if one chooses another set of controls, for instance when acting through v_5 only and keeping $v_1 = v_2 = v_3 = v_4 = 0$. This is given in the next example.

Proposition 1. *Let* $L > 0$ *be a solution of* $\tan(L) = L$. *The system*

$$\begin{cases} -\varphi_t - \varphi_{5x} - \varphi_{xxx} = 0 & \textit{in } (0,T) \times (-L, L), \\ \varphi_{|x=-L} = \varphi_{|x=L} = \varphi_{x|x=-L} = \varphi_{x|x=L} = \varphi_{xx|x=L} = 0 & \textit{in } (0,T), \\ \varphi_{|t=T} = \varphi_T & \textit{on } (-L, L), \end{cases} \tag{8}$$

has solutions satisfying

$$\varphi(0, \cdot) \not\equiv 0 \textit{ and } \varphi_{xx}(t, -L) = 0 \textit{ in } (0,T), \tag{9}$$

for all $T > 0$. *As a consequence the system*

$$\begin{cases} u_t + u_{5x} + u_{xxx} = 0 & \textit{in } (0,T) \times (-L, L), \\ u_{|x=-L} = u_{|x=L} = u_{x|x=-L} = u_{x|x=L} = 0 & \textit{in } (0,T), \\ u_{xx|x=-L} = v_5(t) & \textit{in } (0,T), \end{cases} \tag{10}$$

is not null approximately controllable by the control v_5.

Proof of Proposition 1. We introduce the following (time-independent) function

$$g(x) = \cos(x) + \mu x^2 - (\cos(L) + \mu L^2),$$

where

$$\mu := \frac{\sin(L)}{2L} = \frac{\cos(L)}{2}.$$

It is elementary to check that

$$\begin{cases} g_{5x} + g_{xxx} = 0 & \text{in } (0,T) \times (-L, L), \\ g_{|x=-L} = g_{|x=L} = g_{x|x=-L} = g_{x|x=L} = g_{xx|x=-L} = g_{xx|x=L} = 0 & \text{in } (0,T), \end{cases}$$

hence g satisfies (8) and (9). Now the equivalence between the unique continuation of (8) and the approximate controllability of (10) is an application of the standard duality in PDE control theory, see for instance [8].

Let us note that this phenomenon of critical values of the length of the domain was raised by Rosier [21] for the linearized KdV equation (see [2, 3, 7] for further developments on this subject). Hence, according to the values of the length of the domain and of the coefficients, a similar behaviour can take place here. We believe that this leads to many open and challenging problems.

The structure of the paper is the following. In Section 2, we study the initial boundary value problem for a linearized equation. This requires proving a regularizing effect on the equation $\zeta_t + \zeta_{5x} = g$. In Section 3, we study the controllability of a linearized equation. In Section 4, we use a fixed point argument to establish Theorem 2 and an inverse mapping theorem to establish Theorem 1. Finally Sections 5 and 6 are devoted to the most technical parts of the paper, namely, the proof of a Carleman estimate and a proof of the regularizing effect to the left.

2 Cauchy problem for the linearized equation

In this section, we study the well posedness of the following linearized equation:

$$y_t + \alpha y_{5x} = \sum_{k=0}^{3} a_k(t,x)\partial_x^k y + h. \tag{11}$$

Recall that we consider $\alpha > 0$. We will state the corresponding result in Paragraph 2.3.

For this, we will first study the adjoint system of (11):

$$\begin{cases} \psi_t + \alpha\psi_{5x} = \sum_{k=0}^{3}(-1)^{k+1}\partial_x^k(a_k(t,x)\psi) + f, & \text{in } (0,T)\times(0,1),\\ \psi_{|x=0} = \psi_{x|x=0} = 0 & \text{in } (0,T),\\ \psi_{|x=1} = \psi_{x|x=1} = \psi_{xx|x=1} = 0 & \text{in } (0,T),\\ \psi_{|t=T} = 0 & \text{in } (0,1). \end{cases} \tag{12}$$

2.1 The equation $\zeta_t + \alpha\zeta_{5x} = g$

We begin with the following proposition.

Proposition 2. *Consider $\alpha > 0$. Given $\zeta_T \in (H^5 \cap H_0^2)(0,1)$ satisfying $\zeta_{T,xx}(1) = 0$ and $g \in C^1([0,T]; L^2(0,1))$ there exists a unique solution $\zeta \in C^0([0,T]; H^5(0,1)) \cap C^1([0,T]; L^2(0,1))$ of*

$$\begin{cases} \zeta_t + \alpha\zeta_{5x} = g & \text{in } (0,T)\times(0,1),\\ \zeta_{|x=0} = \zeta_{x|x=0} = 0 & \text{in } (0,T),\\ \zeta_{|x=1} = \zeta_{x|x=1} = \zeta_{xx|x=1} = 0 & \text{in } (0,T),\\ \zeta_{|t=T} = \zeta_T & \text{in } (0,1). \end{cases} \tag{13}$$

Proof of Proposition 2.

This follows from the standard Lumer-Phillips theory. We can introduce the operator $A : D(A) \to L^2(0,1)$:

$$D(A) = \{\vartheta \in (H^5 \cap H_0^2)(0,1)/\vartheta_{xx}(1) = 0\} \text{ and } A\vartheta = \alpha\vartheta_{5x}.$$

Then one can see that its adjoint is defined via

$$D(A^*) = \{h \in (H^5 \cap H_0^2)(0,1)/h_{xx}(0) = 0\} \text{ and } A^*h = -\alpha h_{5x}.$$

Then it is elementary to check that $\langle A\vartheta, \vartheta\rangle_{L^2} \leq 0$ and $\langle A^*h, h\rangle_{L^2} \leq 0$ so that Proposition 2 follows from standard operator theory [19].

Now we prove some estimates for the solutions of (13). We define the spaces, for $k \in \mathbb{N}$,

$$X_k := \{y \in L^2(0,T; H^{k+2}(0,1)) \cap C^0([0,T]; H^k(0,1)), \;\; y_{xx|x=0} \in H^{k/5}(0,T)\},$$

endowed with their natural norm.

Proposition 3. *One has the following estimates on the solutions of* (13)*:*

$$\|\zeta\|_{X_s} \leq C\|g\|_{L^2(0,T;H_0^{s-2}(0,1))}, \;\; \textit{for } s \in [0,10], \tag{14}$$

$$\|\zeta\|_{X_s} \leq C\|g\|_{L^1(0,T;H_0^{s}(0,1))}, \;\; \textit{for } s \in [0,10], \tag{15}$$

$$\|\zeta_{xxx|x=0}\|_{H^{1/5}(0,T)} + \|\zeta_{xxx|x=1}\|_{H^{1/5}(0,T)} \leq C\|g\|_{L^2((0,T)\times(0,1))}, \tag{16}$$

and

$$\|\zeta_{4x|x=0}\|_{L^2(0,T)} + \|\zeta_{4x|x=1}\|_{L^2(0,T)} \leq C\|g\|_{L^2((0,T)\times(0,1))}. \tag{17}$$

Remark 2. *If we interpolate* (14) *and* (15)*, we also deduce*

$$\|\zeta\|_{X_s} \leq C\|g\|_{L^{4/3}(0,T;\tilde{H}^{s-1}(0,1))}, \;\; \textit{for } s \in [0,10]. \tag{18}$$

Proof of Proposition 3.

We consider a smooth solution of (13) and establish several estimates on it.

1. *Proof of* (14)-(15).

• *Estimate in* X_0. We multiply (13) by $(1+x)\zeta$:

$$-\frac{1}{2}\frac{d}{dt}\int_0^1 (1+x)\zeta^2\,dx + \alpha\zeta_{xx|x=0}^2 + \frac{5}{2}\alpha\int_0^1 \zeta_{xx}^2\,dx = -\int_0^1 (1+x)g\zeta\,dx. \tag{19}$$

It follows that

$$\|\zeta\|_{X_0} \leq C\|g\|_{L^2(0,T;H_0^{-2}(0,1))}. \tag{20}$$

It is also clear that from (19) it follows

$$\|\zeta\|_{X_0} \leq C\|g\|_{L^1(0,T;L^2(0,1))}. \tag{21}$$

• *Estimate in* X_5. Now we consider $g \in L^2(0,T;H_0^3(0,1))$. Observe that due to (13), for such a g, the traces of ζ_{5x} and ζ_{6x} on both sides, and the trace of ζ_{7x} on the right, vanish.

We apply the operator ∂_{5x} to the equation and we apply (20):

$$\|\zeta_{5x}\|_{X_0} \leq C\|g_{5x}\|_{L^2(0,T;H_0^{-2}(0,1))}.$$

Using the equation, this gives

$$\|\zeta_{xx|x=0}\|_{H^1(0,T)} \leq C\|g\|_{L^2(0,T;H_0^3(0,1))}.$$

This yields also

$$\|\zeta\|_{X_5} \leq C\|g\|_{L^2(0,T;H_0^3(0,1))}. \tag{22}$$

In the same way, we have

$$\|\zeta\|_{X_5} \leq C\|g\|_{L^1(0,T;H_0^5(0,1))}. \tag{23}$$

• *Estimate in* X_{10}. Here we consider $g \in L^2(0,T;H_0^8(0,1))$. We apply the operator ∂_{5x} to the equation and we apply (22) (since $g_{5x} \in L^2(0,T;H_0^3(0,1))$):

$$\|\zeta_{5x}\|_{X_5} \leq C\|g_{5x}\|_{L^2(0,T;H_0^3(0,1))}.$$

This yields as previously

$$\|\zeta\|_{X_{10}} \leq C\|g\|_{L^2(0,T;H_0^8(0,1))}. \tag{24}$$

Also we have

$$\|\zeta\|_{X_{10}} \leq C\|g\|_{L^1(0,T;H_0^{10}(0,1))}. \tag{25}$$

• *Interpolation argument.* By an interpolation argument, we deduce (14) and (15) for every $s \in [0, 10]$.

2. *Proof of* (16).

Let $\rho \in C^4([0,1];\mathbb{R})$ satisfying $\rho(x) = 0$ for $x \in [0, 1/2]$ and $\rho(x) = 1$ for $x \in [3/4, 1]$.

• We use estimate (14) for $s = 1$:

$$\|\zeta\|_{X_1} \leq C\|g\|_{L^2(0,T;H_0^{-1}(0,1))}. \tag{26}$$

Multiplying (13) with $\rho\zeta_{xx}$, integrating in space and integrating by parts, we get, for almost any $t \in [0, T]$,

$$\frac{\alpha}{2}|\zeta_{xxx|x=1}|^2 = \frac{3\alpha}{2}\int_0^1 \rho_x|\zeta_{xxx}|^2 dx - \frac{\alpha}{2}\int_0^1 \rho_{xxx}|\zeta_{xx}|^2 dx - \frac{1}{2}\frac{d}{dt}\int_0^1 \rho|\zeta_x|^2 dx \\ - \langle \rho_x\zeta_x, \zeta_t\rangle_{H_0^2(0,1)\times H^{-2}(0,1)} - \langle g, \rho\zeta_{xx}\rangle_{H^{-1}(0,1)\times H_0^1(0,1)}. \tag{27}$$

Integrating in time and thanks to (13)-(26), we get

$$\|\zeta\|_{X_1} + \|\zeta_{xxx|x=1}\|_{L^2(0,T)} + \|\zeta_{xxx|x=0}\|_{L^2(0,T)} \leq C\|g\|_{L^2(0,T;H_0^{-1}(0,1))}. \tag{28}$$

An estimate for $\zeta_{xxx|x=0}$ can be done in the same way, by employing the weight $1 - \rho$.

• Now, we use estimate (14) for $s = 6$:

$$\|\zeta\|_{X_6} \leq C\|g\|_{L^2(0,T;H_0^4(0,1))}. \tag{29}$$

In order to prove that $\zeta_{txxx|x=1} \in L^2(0, T)$, we multiply (13) by $\rho\partial_t\partial_x^7\zeta$, we integrate in space and we integrate by parts (using again what we know on the traces of ζ_{5x}, ζ_{6x} and ζ_{7x}):

$$\frac{1}{2}|\zeta_{txxx|x=1}|^2 = \frac{7}{2}\int_0^1 \rho_x|\zeta_{txxx}|^2\, dx + \int_0^1 \zeta_{txxx}(6\rho_{xx}\zeta_{txx} + 4\rho_{xxx}\zeta_{tx} + \rho_{4x}\zeta_t)\, dx \\ - \frac{\alpha}{2}\frac{d}{dt}\int_0^1 (\rho|\zeta_{6x}|^2 - \rho_{xx}|\zeta_{5x}|^2)\, dx + \alpha\langle \rho_x\zeta_{6x}, \zeta_{t5x}\rangle_{H_0^2\times H^{-2}} + \zeta_{txxx|x=1}g_{xxx|x=1} \\ - \int_0^1 \zeta_{txxx}\sum_{j=0}^4 \Big(\binom{4}{j}\,\partial_x^j g\,\partial_x^{4-j}\rho\Big)\, dx. \tag{30}$$

As previously, the same can be done for $\zeta_{txxx|x=0}$.

Integrating in time, using Cauchy-Schwarz inequality to estimate the last term in the second line and using (13)-(29), we get

$$\|\zeta\|_{X_6} + \|\zeta_{txxx|x=1}\|_{L^2(0,T)} + \|\zeta_{txxx|x=0}\|_{L^2(0,T)} \leq C\|g\|_{L^2(0,T;H_0^4(0,1))}. \tag{31}$$

• An interpolation argument applied to (28) and (31) provides (16).

3. *Proof of inequality* (17).

We multiply the equation of ζ by $\rho\zeta_{4x}$ and we integrate in space. After some integration by parts, we obtain:

$$\frac{\alpha}{2}|\zeta_{4x|x=1}|^2 = \frac{\alpha}{2}\int_0^1 \rho_x|\zeta_{4x}|^2\,dx - \frac{1}{2}\frac{d}{dt}\int_0^1 \rho|\zeta_{xx}|^2\,dx - \langle \rho_{xx}\zeta_{xx}, \zeta_t\rangle_{H_0^1\times H^{-1}} - 2\langle \rho_x\zeta_{xx}, \zeta_{tx}\rangle_{H_0^2\times H^{-2}} + \int_0^1 \rho\zeta_{4x} g\,dx. \quad (32)$$

Integrating in time this identity, we have

$$\|\zeta_{4x|x=1}\|_{L^2(0,T)} \leq C(\|\zeta\|_{L^\infty(0,T;H^2(0,1))} + \|\zeta_t\|_{L^2(0,T;H^{-1}(0,1))} + \|\zeta\|_{L^2(0,T;H^4(0,1))} + \|g\|_{L^2((0,T)\times(0,1))}).$$

Using (14), we have estimated $\zeta_{4x|x=1}$ as in (17). The estimate $\zeta_{4x|x=0}$ is similar by multiplying by $(1-\rho)\zeta_{4x}$.

2.2 Well posedness for the adjoint equation

Now we can state the following existence and regularity result for (12).

Proposition 4. *Given $a_k \in L^\infty(0,T;W^{k,\infty}(0,1))$ (for $k = 0\dots 3$), $f \in L^2(0,T;L^2(0,1))$, there exists a unique solution $\psi \in L^2(0,T;H^4(0,1)) \cap C^0([0,T];H^2(0,1))$ of (12).*

Proof of Proposition 4.
We use a fixed point scheme. Given $\hat{\psi} \in L^2(0,T;H^4(0,1)) \cap C^0([0,T];H^2(0,1))$, we consider the solution $\psi := \mathcal{T}\hat{\psi}$ of

$$\begin{cases} \psi_t + \alpha\psi_{5x} = \sum_{k=0}^3 (-1)^{k+1}\partial_x^k(a_k(t,x)\hat{\psi}) + f & \text{in } (\hat{T},T)\times(0,1), \\ \psi_{|x=0} = \psi_{x|x=0} = 0 & \text{in } (\hat{T},T), \\ \psi_{|x=1} = \psi_{x|x=1} = \psi_{xx|x=1} = 0 & \text{in } (\hat{T},T), \\ \psi_{|t=T} = 0 & \text{in } (0,1), \end{cases} \quad (33)$$

where $\hat{T} \in (0,T)$ is to be fixed later.

Using Proposition 3 (precisely (14) for $s = 2$), we infer that

$$\begin{aligned} \|\mathcal{T}\hat{\psi}_1 - \mathcal{T}\hat{\psi}_2\|_{X_2} &\leq C(\|a_k\|_{L^\infty(0,T;W^{k,\infty})})\|\hat{\psi}_1 - \hat{\psi}_2\|_{L^2(0,T;H^3)} \\ &\leq C(\|a_k\|_{L^\infty(0,T;W^{k,\infty})})\hat{T}^{1/4}\|\hat{\psi}_1 - \hat{\psi}_2\|_{L^4(0,T;H^3)}. \end{aligned} \quad (34)$$

Note that the constant in (14) is independent of $\hat{T} \in (0,T)$.

Then by interpolation we deduce that

$$\begin{aligned} \|\mathcal{T}\hat{\psi}_1 - \mathcal{T}\hat{\psi}_2\|_{X_2} &\leq C(\|a_k\|_{L^\infty(0,T;W^{k,\infty})})\hat{T}^{1/4}\|\hat{\psi}_1 - \hat{\psi}_2\|^{1/2}_{L^\infty(0,T;H^2)}\|\hat{\psi}_1 - \hat{\psi}_2\|^{1/2}_{L^2(0,T;H^4)} \\ &\leq C(\|a_k\|_{L^\infty(0,T;W^{k,\infty})})\hat{T}^{1/4}\|\hat{\psi}_1 - \hat{\psi}_2\|_{X_2}. \end{aligned} \quad (35)$$

It follows that $\mathcal{T}$ is contracting for sufficiently small time $\hat{T}$. Then extending the solution obtained in $(\hat{T},T)$ to a solution in $(0,T)$ is standard using the linear character of the equation.

Furthermore, the solutions described in Proposition 4 possess the following regularity property.

Proposition 5. *Under the assumptions of Proposition 4, the solution ψ has the following hidden regularity:*

$$\|\psi\|_{X_2} + \|\psi_{xx|x=0,1}\|_{H^{2/5}(0,T)} + \|\psi_{xxx|x=0,1}\|_{H^{1/5}(0,T)} + \|\psi_{4x|x=0,1}\|_{L^2(0,T)} \leq C\|f\|_{L^2((0,T)\times(0,1))}. \quad (36)$$

Proof of Proposition 5.

This is a consequence of Propositions 3 and 4. Note that due to the contracting character of $\mathcal{T}$ and using Proposition 3, we have

$$\|\psi\|_{X_2} \lesssim \|\mathcal{T}(0)\|_{X_2} \lesssim \|f\|_{L^2((0,T)\times(0,1))}.$$

Now we can use

$$g := \sum_{k=0}^{3}(-1)^{k+1}\partial_x^k(a_k(t,x)\psi) + f,$$

as a right hand side in (13) to deduce (36) from Proposition 3.

2.3 Well posedness for the initial boundary value problem

In this paragraph we give the notion of solution of

$$\begin{cases} y_t + \alpha y_{5x} = \sum_{k=0}^{3} a_k(t,x)\partial_x^k y + h & \text{in } (0,T)\times(0,1), \\ y_{|x=0} = v_1,\ y_{|x=1} = v_2,\ y_{x|x=0} = v_3 & \text{in } (0,T), \\ y_{x|x=1} = v_4,\ \ y_{xx|x=0} = v_5 & \text{in } (0,T), \\ y_{|t=0} = y_0 & \text{in } (0,1), \end{cases} \tag{37}$$

where y_0 , h, v_1, ..., v_5 are given function. The solution of (37) for homogeneous boundary conditions and $h \in L^2(0,T;L^2(0,1))$ is granted by Proposition 4 (replace t by $T-t$ and x by $1-x$). Hence we can suppose without loss of generality that $h = 0$.

Definition 1. *Let $y_0 \in H^{-2}(0,1)$, v_1, $v_2 \in L^2(0,T)$, v_3, $v_4 \in H^{-1/5}(0,T)$ and $v_5 \in H^{-2/5}(0,T)$. We call y a solution by transposition of (37) with $h=0$, a function $y \in L^2((0,T)\times(0,1))$ such that*

$$\begin{aligned} &\int_0^T\int_0^1 y\, f\, dx\, dt = \langle u_0, \psi|_{t=0}\rangle_{H^{-2}(0,1)\times H_0^2(0,1)} + \alpha\int_0^T v_1\, \psi_{4x|x=0}\, dt - \alpha\int_0^T v_2\, \psi_{4x|x=1}\, dt \\ &-\alpha\langle v_3, \psi_{xxx|x=0}\rangle_{H^{-1/5}(0,T)\times H^{1/5}(0,T)} + \alpha\langle v_4, \psi_{xxx|x=1}\rangle_{H^{-1/5}(0,T)\times H^{1/5}(0,T)} \\ &+\alpha\langle v_5, \psi_{xx|x=0}\rangle_{H^{-2/5}(0,T)\times H^{2/5}(0,T)} + \int_0^T a_{3|x=0}\, v_1\, \psi_{xx|x=0}\, dt, \quad \forall f \in L^2((0,T)\times(0,1)), \end{aligned} \tag{38}$$

where ψ is the solution of (12) *associated to f.*

Proposition 6. *There exists a unique solution by transposition of system (37) with $h = 0$. Moreover, there exists $C > 0$ such that*

$$\|y\|_{L^2((0,T)\times(0,1))} \le C(\|y_0\|_{H^{-2}(\Omega)} + \|v_1\|_{L^2(0,T)} + \|v_2\|_{L^2(0,T)} + \|v_3\|_{H^{-1/5}(0,T)} + \|v_4\|_{H^{-1/5}(0,T)} + \|v_5\|_{H^{-2/5}(0,T)}).$$

Proof: All comes to prove

$$\psi \in C^0([0,T];H_0^2(0,1)),\ \psi_{4x|x=0,1} \in L^2(0,T),\ \psi_{xxx|x=0,1} \in H^{1/5}(0,T),\ \psi_{xx|x=0} \in H^{2/5}(0,T)$$

and the following inequality:

$$\begin{aligned} &\|\psi\|_{L^\infty(0,T;H^2(0,1))} + \|\psi_{4x|x=0,1}\|_{L^2(0,T)} + \|\psi_{xxx|x=0,1}\|_{H^{1/5}(0,T)} \\ &\qquad + \|\psi_{xx|x=0}\|_{H^{2/5}(0,T)} \le C\|h\|_{L^2((0,T)\times(0,T))}. \end{aligned} \tag{39}$$

This was established in Proposition (5).

3 Controllability of the linearized equation

3.1 Carleman estimate

We consider the following dual system

$$\begin{cases} \varphi_t + \alpha\varphi_{5x} = \sum_{k=0}^{3}(-1)^{k+1}\partial_x^k(a_k(t,x)\varphi) + f & \text{in } (0,T)\times(0,1), \\ \varphi_{|x=0} = \varphi_{|x=1} = \varphi_{x|x=0} = \varphi_{x|x=1} = \varphi_{xx|x=1} = 0 & \text{in } (0,T), \\ \varphi_{|t=T} = \varphi_T & \text{on } (0,1), \end{cases} \tag{40}$$

where

$$a_k \in L^\infty(0,T;W^{k,\infty}(0,1)) \text{ for } k = 0\dots 3. \tag{41}$$

A central argument in this paper consists in establishing a Carleman inequality for (40). For this let us set

$$\alpha(t,x) = \frac{\beta(x)}{t^{1/4}(T-t)^{1/4}}, \tag{42}$$

for $(t,x) \in Q$. Weight functions of this kind were first introduced by A. V. Fursikov and O. Yu. Imanuvilov; see [11]. In the above equation β is a positive, strictly decreasing and concave polynomial of degree 2 in $[0,1]$. Observe that the function α satisfies

$$C \le T^{1/2}\alpha, \quad C_0\alpha \le -\alpha_x \le C_1\alpha, \quad C_0\alpha \le -\alpha_{xx} \le C_1\alpha \quad \text{in } (0,T)\times[0,1], \tag{43}$$

$$|\alpha_t| + |\alpha_{xt}| + |\alpha_{xxt}| \le CT\alpha^5, \quad |\alpha_{tt}| \le C(T^2\alpha^9 + \alpha^5) \le CT^2\alpha^9 \quad \text{in } (0,T)\times[0,1], \tag{44}$$

where C, C_0 and C_1 are positive constants independent of T.

We have:

Proposition 7. *Suppose that* (41) *applies. There exists a positive constant C independent of T such that, for any $\varphi_T \in L^2(0,1)$ and $f \in L^2(0,T;L^2(0,1))$, we have*

$$\iint_Q e^{-2s\alpha}\alpha(|\varphi_{4x}|^2 + s^2\alpha^2|\varphi_{xxx}|^2 + s^4\alpha^4|\varphi_{xx}|^2 + s^6\alpha^6|\varphi_x|^2 + s^8\alpha^8|\varphi|^2)\,dt\,dx$$
$$\le C\Big(\int_0^T \alpha_{|x=1}e^{-2s\alpha_{|x=1}}(|\varphi_{4x|x=1}|^2 + s^2\alpha_{|x=1}^2|\varphi_{xxx|x=1}|^2)\,dt + s^{-1}\iint_Q e^{-2s\alpha}|f|^2 dt\,dx\Big), \tag{45}$$

for any $s \ge C(T^{1/4} + T^{1/2})$, where φ is the solution of (40).

The proof of this inequality is postponed to Section 5.

Remark 3. *We will also require β to satisfy*

$$\max_{x\in[0,1]} \beta(x) < \sqrt{2} \min_{x\in[0,1]} \beta(x). \tag{46}$$

This is not needed for Proposition 7 (nor to Proposition 8 below), but will be useful later.

3.2 Weighted observability estimate

Now let us deduce from Proposition 7 a slightly modified inequality, with a weight function not vanishing at $t = 0$.

We begin by introducing a new weight. Set ℓ on $[0,T]$ by

$$\ell(t) := \begin{cases} \dfrac{T^2}{4} & \text{if } t \le \dfrac{T}{2}, \\ t(T-t) & \text{otherwise.} \end{cases} \tag{47}$$

Now introduce

$$\gamma(t,x) = \frac{\beta(x)}{\ell(t)^{1/4}}. \tag{48}$$

Proposition 8. *Suppose that* (41) *applies. There exist two positive constants s_0 and $C > 0$ depending on T such that, for any $\varphi_T \in L^2(0,1)$ and any $f \in L^2(0,T;L^2(0,1))$, we have*

$$\iint_Q e^{-2s_0\gamma}\gamma^9|\varphi|^2\,dx\,dt + \int_0^1 |\varphi(0,x)|^2\,dx \le C\Big(\iint_Q e^{-2s_0\gamma}|f|^2 dt\,dx + \int_0^T \gamma_{|x=1}e^{-2s_0\gamma_{|x=1}}\big(|\varphi_{4x|x=1}|^2 + \gamma_{|x=1}^2|\varphi_{xxx|x=1}|^2\big)\,dt\Big), \quad (49)$$

where φ is the solution of (40).

Proof of Proposition 8.

We use the following energy estimate:

$$\|\varphi\|_{L^\infty(0,T/2;L^2(0,1))} \le C\exp\{C\|a_k\|_{L^\infty(0,T;W^{k,\infty}(0,1))}\}(\|f\|_{L^2(0,3T/4;L^2(0,1))} + \|\varphi\|_{L^2(T/2,3T/4;L^2(0,1))}). \quad (50)$$

To get (50), introduce $\eta \in C^\infty([0,T];\mathbb{R})$ such that $\eta = 1$ in $[0,T/2]$ and $\eta = 0$ in $[3T/4,T]$, multiply equation (40) by $\eta(t)(1+x)\varphi$, and perform several integration by parts as in (20).

Let us notice that the weight functions γ and $e^{-2s\gamma}$ are positive for $t \in [0,T/2]$. Hence there is a constant C such that

$$\|e^{-s\gamma}\gamma^{9/2}\varphi\|_{L^\infty(0,T/2;L^2(0,1))} \le C\exp\{C\|a_k\|_{L^\infty(0,T;W^{k,\infty}(0,1))}\} \times (\|e^{-s\gamma}f\|_{L^2(0,3T/4;L^2(0,1))} + \|e^{-s\gamma}\gamma^{9/2}\varphi\|_{L^2(T/2,3T/4;L^2(0,1))}). \quad (51)$$

Next, we use (45) and the choice of γ to deduce

$$\int_{\frac{T}{2}}^T\int_{(0,1)} e^{-2s\gamma}\gamma^9|\varphi|^2\,dx\,dt \le C\Big(\iint_Q e^{-2s\gamma}|f|^2 dt\,dx + \int_0^T \gamma_{|x=1}e^{-2s\gamma_{|x=1}}\big(|\varphi_{4x|x=1}|^2 + \gamma_{|x=1}^2|\varphi_{xxx|x=1}|^2\big)\,dt\Big), \quad (52)$$

for s large enough. Combining (51) and (52) we obtain (49).

Let us consider s_0 as in Proposition 8. We introduce

$$\kappa_0 := \frac{s_0\sqrt{2}}{T^{1/4}}\max_{x\in[0,1]}\beta(x) \quad \text{and} \quad \kappa_1 := \frac{s_0}{T^{1/4}}\min_{x\in[0,1]}\beta(x). \quad (53)$$

Corollary 1. *Under the assumptions of Proposition 8, one has*

$$\iint_Q e^{-\frac{2\kappa_0}{(T-t)^{1/4}}}(T-t)^{-9/4}|\varphi|^2\,dx\,dt + \int_0^1 |\varphi(0,x)|^2\,dx \le C\Big(\iint_Q e^{-\frac{2\kappa_1}{(T-t)^{1/4}}}|f|^2 dt\,dx + \int_0^T (T-t)^{-1/4}e^{-\frac{2\kappa_1}{(T-t)^{1/4}}}\big(|\varphi_{4x|x=1}|^2 + (T-t)^{-1/2}|\varphi_{xxx|x=1}|^2\big)\,dt\Big). \quad (54)$$

3.3 Controllability

We introduce the following space:

$$E_0 = \{y \in L^2(0,T;L^2(0,1)) \;/\; e^{\frac{\kappa_1}{(T-t)^{1/4}}}y \in L^2(0,T;L^2(0,1))\}. \quad (55)$$

We have the following controllability result.

Proposition 9. *Given h such that $(T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} h \in L^2((0,T)\times(0,1))$ and $y_0 \in L^2(0,1)$, there exist controls $v_2, v_4 \in L^2(0,T)$ satisfying*

$$(T-t)^{1/8} e^{\frac{\kappa_1}{(T-t)^{1/4}}} v_2 \in L^2(0,T) \quad \text{and} \quad (T-t)^{3/8} e^{\frac{\kappa_1}{(T-t)^{1/4}}} v_4 \in L^2(0,T), \tag{56}$$

such that if we call y the solution of (37) starting from y_0 with $v_1 = v_3 = v_5 = 0$, then y belongs to E_0. In particular y, which belongs to $C^0([0,T]; H^{-5}(0,1))$, satisfies

$$y_{|t=T} = 0 \text{ on } (0,1). \tag{57}$$

Besides, there exists a constant $C > 0$ such that

$$\begin{aligned} \| e^{\frac{\kappa_1}{(T-t)^{1/4}}} y \|_{L^2(0,T;L^2(0,1))} &+ \| (T-t)^{1/8} e^{\frac{\kappa_1}{(T-t)^{1/4}}} (v_2, (T-t)^{1/4} v_4) \|_{L^2(0,T)} \\ &\leq C \Big(\| y_0 \|_{L^2(0,1)} + \| (T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} h \|_{L^2((0,T)\times(0,1))} \Big). \end{aligned} \tag{58}$$

Proof of Proposition 9.

The proof is inspired by Fursikov and Imanuvilov's approach [11]. Define L

$$Ly := y_t + \alpha y_{5x} - \sum_{k=0}^{3} a_k(t,x) \partial_x^k y, \tag{59}$$

and L^* its dual operator:

$$L^*\phi := -\phi_t - \alpha \phi_{5x} - \sum_{k=0}^{3} (-1)^k \partial_x^k (a_k(t,x)\phi). \tag{60}$$

Let us set

$$F_0 = \{ \phi \in C^\infty([0,T]\times[0,1];\mathbb{R}) \ / \ \phi_{|x=0} = \phi_{|x=1} = \phi_{x|x=0} = \phi_{x|x=1} = \phi_{xx|x=1} = 0 \}.$$

Consider the bilinear form

$$\begin{aligned} a(\hat{\phi}, \phi) = & \iint_Q e^{-\frac{2\kappa_1}{(T-t)^{1/4}}} L^*\hat{\phi}\, L^*\phi \, dx\, dt \\ & + \int_0^T e^{-\frac{2\kappa_1}{(T-t)^{1/4}}} (T-t)^{-1/4} \big[\hat{\phi}_{4x|x=1} \phi_{4x|x=1} + (T-t)^{-1/2} \hat{\phi}_{xxx|x=1} \phi_{xxx|x=1} \big] \, dt \qquad \forall \hat{\phi}, \phi \in F_0. \end{aligned}$$

We also introduce the linear form

$$\langle \ell, \phi \rangle = \iint_Q h\phi \, dt\, dx + \int_0^1 u_0 \, \phi_{|t=0} \, dx. \tag{61}$$

Introduce $\overline{F}_0$ the completion of F_0 for the norm $\phi \mapsto a(\phi,\phi)^{1/2}$ (it is a norm from Corollary 1).

The next step in this proof is to demonstrate that there exists exactly one $\hat{\phi}$ in the class $\overline{F}_0$ satisfying

$$a(\hat{\phi}, \phi) = l(\phi), \ \forall \phi \in \overline{F}_0. \tag{62}$$

Now $\overline{F}_0$ is a Hilbert space for the scalar product $a(\cdot\,,\cdot)$, hence in order to get (62) it is sufficient to prove that ℓ is a continuous linear form on $\overline{F}_0$. From Cauchy-Schwarz inequality, we see that

$$\Big| \iint_Q h\phi \, dt\, dx \Big| \leq \| (T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} h \|_{L^2(0,T;L^2(0,1))} \| (T-t)^{-9/8} e^{-\frac{\kappa_0}{(T-t)^{1/4}}} \phi \|_{L^2(0,T;L^2(0,1))}. \tag{63}$$

Using the assumption on h and Corollary 1, one sees that ℓ is indeed a continuous linear form on $\overline{F}_0$. Hence there exists a unique $\hat{\phi} \in \overline{F}_0$ satisfying (62).

Let us set

$$y = e^{-\frac{2\kappa_1}{(T-t)^{1/4}}} L^* \hat{\phi} \quad v_2 = (T-t)^{-1/4} e^{-\frac{2\kappa_1}{(T-t)^{1/4}}} \hat{\phi}_{4x|x=1} \quad \text{and} \quad v_4 = (T-t)^{-3/4} e^{-\frac{2\kappa_1}{(T-t)^{1/4}}} \hat{\phi}_{xxx|x=1}. \tag{64}$$

Finally it is not difficult to see that $y \in E_0$, that (v_2, v_4) satisfies (56) and that y is a solution of (37) with $v_1 = v_3 = v_5 = 0$. This concludes the proof of Proposition 9.

Now we define the space

$$E_1 = \{y \in E_0 \ / \ (T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}} y \in L^2(0,T; H^4(0,1)) \cap C^0([0,T]; H_0^2(0,1)),$$
$$y_{|x=0} = y_{x|x=0} = y_{xx|x=0} = 0, \ (T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} Ly \in L^2(0,T; L^2(0,1))\}. \tag{65}$$

Proposition 10. *Given h such that $(T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} h \in L^2((0,T)\times(0,1))$ and $y_0 \in H_0^2(0,1)$, there exist controls $(v_2, v_4) \in L^2(0,T)^2$ such that the associated solution y of (37) with $v_1 = v_3 = v_5 = 0$ belongs to E_1 and moreover satisfies*

$$\|(T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}} y\|_{L^2(0,T;H^4(0,1)) \cap C^0([0,T];H^2(0,1))} \leq C\Big(\|y_0\|_{H_0^2(0,1)} + \|(T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} h\|_{L^2((0,T)\times(0,1))}\Big), \tag{66}$$

for some $C > 0$.

Proof of Proposition 10.

We extend the problem to the interval $[0,2]$. We extend y_0 (resp. h) by 0 in $[1,2]$ (resp. $[0,T]\times[1,2]$), we call $\tilde{y}_0$ (resp. $\tilde{h}$) the resulting function. We also extend a_k in $[0,T]\times[1,2]$ in a way that keeps the $L^\infty(0,T;W^{k,\infty})$ regularity (in a continuous way), and in such a way that

$$a_k(t,x) = 0 \text{ in } [0,T] \times [\frac{3}{2}, 2].$$

We now consider the following control problem

$$\begin{cases} \tilde{y}_t + \alpha \tilde{y}_{5x} = \sum_{k=0}^{3} \tilde{a}_k(t,x) \partial_x^k \tilde{y} + \tilde{h} & \text{in } (0,T)\times(0,2), \\ \tilde{y}_{|x=0} = \tilde{y}_{x|x=0} = \tilde{y}_{xx|x=0} = 0 & \text{in } (0,T), \\ \tilde{y}_{|x=2} = \tilde{v}_2, \ \tilde{y}_{x|x=2} = \tilde{v}_4, & \text{in } (0,T), \\ \tilde{y}_{|t=0} = \tilde{y}_0 & \text{in } (0,2). \end{cases} \tag{67}$$

According to Proposition 9, there exist $\tilde{v}_2, \tilde{v}_4$ fulfilling (56) such that the corresponding solution $\tilde{y}$ belongs to E_0 (adapted to the interval $[0,2]$ of course). Now we claim that the restriction of $\tilde{y}$ to $[0,T]\times[0,1]$ satisfies the required properties. We have to establish that

$$(T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}} y \in L^2(0,T; H^4(0,1)) \cap C^0([0,T]; H^2(0,1)).$$

For that, we introduce

$$y^*(t,x) := (T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}} \tilde{y}(t,x). \tag{68}$$

This function satisfies

$$\begin{cases} y^*_t + \alpha y^*_{5x} = \sum_{k=0}^{3} \tilde{a}_k(t,x) \partial_x^k y^* + h^* & \text{in } (0,T)\times(0,2), \\ y^*_{|x=0} = y^*_{x|x=0} = y^*_{xx|x=0} = 0 & \text{in } (0,T), \\ y^*_{|x=2} = v_2^*, \ y^*_{x|x=2} = v_4^* & \text{in } (0,T), \\ y^*_{|t=0} = y_0^* & \text{in } (0,2), \end{cases} \tag{69}$$

where

$$y_0^* = T^{5/4} e^{\frac{\kappa_1}{T^{1/4}}} \tilde{y}_0, \ (v_2^*, v_4^*) = (T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}} (\tilde{v}_2, \tilde{v}_4)$$
$$\text{and } h^* = (T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}} \tilde{h} + \frac{d}{dt}[(T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}}]\tilde{y}.$$

These data are in $H_0^2(0,2)$, in $L^2(0,T)^2$ and in $L^2(0,T;L^2(0,2))$ respectively, thanks to Proposition 9. We will use the following lemma, whose proof is postponed to the Appendix.

Lemma 1. *For k large enough, one has $(2-x)^k y^* \in L^2(0,T;H^4(0,2)) \cap C^0([0,T];H^2(0,2))$ with the estimate*

$$\|(2-x)^{k+\frac{1}{2}} y^*\|_{L^\infty(0,T;H^2(0,2))} + \sum_{j=0}^{4} \|(2-x)^{k+j-4} \partial_x^j y^*\|_{L^2(0,T;L^2(0,2))}$$
$$\leq C\big(\|h^*\|_{L^2(0,T;L^2(0,2))} + \|y_0^*\|_{H^2(0,2)} + \|v_2^*\|_{L^2(0,T)} + \|v_4^*\|_{L^2(0,T)}\big), \quad (70)$$

for some positive constant C.

Now we use (70) and the continuity of the previous extensions from $(0,1)$ to $(0,2)$ to deduce

$$\|(T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}} y\|_{L^2(0,T;H^4(0,1)) \cap C^0([0,T];H^2(0,1))}$$
$$\leq C\Big(\|y_0\|_{H_0^2(0,1)} + \|(T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}} h\|_{L^2((0,T)\times(0,1))} + \|\frac{d}{dt}[(T-t)^{5/4} e^{\frac{\kappa_1}{(T-t)^{1/4}}}] y\|_{L^2((0,T)\times(0,1))}$$
$$+ \|(T-t)^{1/8} e^{\frac{\kappa_1}{(T-t)^{1/4}}} (v_2, (T-t)^{1/4} v_4)\|_{L^2(0,T)}\Big),$$

for some $C > 0$. Finally, we use $\kappa_1 < \kappa_0$ to estimate the second term in the right hand side, and (58) to estimate the last two terms. We deduce (66).

4 Nonlinear problem

4.1 Proof of Theorem 2

We use a fixed point scheme to prove local existence and uniqueness in $X := L^2(0,T;H_0^2(0,1)) \cap C^0([0,T];L^2(0,1))$. Given $z \in X$, we introduce the solution of

$$\begin{cases} u_t + \alpha u_{5x} + \mu u_{xxx} + \beta z u_{xxx} + \delta z_x u_{xx} + P'(z) u_x = 0 & \text{in } (0,T)\times(0,1), \\ u_{|x=0} = u_{|x=1} = u_{x|x=0} = u_{x|x=1} = u_{xx|x=0} = 0 & \text{in } (0,T), \\ u_{|t=0} = u_0 & \text{in } (0,1). \end{cases} \quad (71)$$

Call $\mathcal{T}$ the corresponding operator. The existence and uniqueness of u is obtained as in Proposition 4: one associates to $\hat{\psi} \in X$ the solution of

$$\begin{cases} u_t + \alpha u_{5x} = -\mu \hat{\psi}_{xxx} - \beta z \hat{\psi}_{xxx} - \delta z_x \hat{\psi}_{xx} - P'(z) \hat{\psi}_x & \text{in } (0,T)\times(0,1), \\ u_{|x=0} = u_{|x=1} = u_{x|x=0} = u_{x|x=1} = u_{xx|x=0} = 0 & \text{in } (0,T), \\ u_{|t=0} = u_0 & \text{in } (0,1). \end{cases} \quad (72)$$

Let us notice that $\hat{\psi}_{xxx} \in L^2(0,T;H^{-1}(0,1))$ while (by interpolation) $z \in L^4(0,T;H^1(0,1))$, and hence $z\hat{\psi}_{xxx} \in L^{4/3}(0,T;H^{-1}(0,1))$; on the other hand, one also sees that $z_x \hat{\psi}_{xx} \in L^{4/3}(0,T;H^{-1}(0,1))$. It follows then from Remark (2) that (72) defines a solution in $L^2(0,T;H^2(0,1)) \cap C^0([0,T];L^2(0,1))$.

Now consider $\hat{\psi}_1$ and $\hat{\psi}_2$ in X, and their images u_1 and u_2 by the above mapping. Making the difference of the two equations, multiplying by $(2-x)(u_1 - u_2)$ and performing the same operations as in Proposition 3, we infer

$$\|u_1-u_2\|_X^2 \le C\Big[\iint_Q \Big|(u_1-u_2)_x(\hat\psi_1-\hat\psi_2)_{xx}\Big|\,dt\,dx + \iint_Q \Big|(u_1-u_2)z_x(\hat\psi_1-\hat\psi_2)_{xx}\Big|\,dt\,dx \\ + \iint_Q \Big|z(u_1-u_2)_x(\hat\psi_1-\hat\psi_2)_{xx}\Big|\,dt\,dx + \iint_Q \Big|(u_1-u_2)(1+z^2)(\hat\psi_1-\hat\psi_2)_x\Big|\,dt\,dx\Big]. \quad (73)$$

We deduce

$$\|u_1-u_2\|_X^2 \le C\|\hat\psi_1-\hat\psi_2\|_{L^2(0,T;H^2(0,1))}\Big[\|u_1-u_2\|_{L^2(0,T;H^1(0,1))} \\ + \|z_x\|_{L^{8/3}(0,T;L^\infty(0,1))}\|u_1-u_2\|_{L^8(0,T;L^2(0,1))} + \|z\|_{L^8(0,T;L^2(0,1))}\|u_1-u_2\|_{L^{8/3}(0,T;W^{1,\infty}(0,1))}\Big] \\ + C\|\hat\psi_1-\hat\psi_2\|_{L^4(0,T;H^1(0,1))}\Big[\|u_1-u_2\|_{L^{4/3}(0,T;L^2(0,1))} + \|u_1-u_2\|_{L^4(0,T;L^6(0,1))}\|z\|_{L^4(0,T;L^6(0,1))}^2\Big]. \quad (74)$$

Note that by interpolation and Sobolev imbedding we have

$$L^2(0,T;H_0^2(0,1))\cap L^\infty(0,T;L^2(0,1)) \hookrightarrow L^{12}(0,T;L^6(0,1))\cap L^{8/3}(0,T;W^{1,\infty}(0,1)).$$

We infer that (at least if $T\le 1$)

$$\|u_1-u_2\|_X^2 \le CT^{1/8}(1+\|z\|_X^2)\|\hat\psi_1-\hat\psi_2\|_X\|u_1-u_2\|_X.$$

Hence the operator is contractive for sufficiently small time T, which proves the local well posedness of (71).

Now let us prove that $\mathcal{T}$ has a fixed point. First, let us prove that for some constant $C>0$, the solution u of (71) satisfies

$$\|u\|_X \le \exp(CT^{1/8}(1+\|z\|_X^2))\|u_0\|_{L^2(0,1)}. \quad (75)$$

For that, we multiply again (71) by $(2-x)u$; after some integration by parts, we can deduce

$$\frac{d}{dt}\|u\|_{L^2(0,1)}^2 + \frac{5\alpha}{2}\int_0^1 |u_{xx}|^2\,dx \le C(1+\|z_x(t,\cdot)\|_\infty^2 + \|z(t,\cdot)\|_\infty^2)\|u\|_{L^2}^2 + \epsilon\int_0^1 |u_{xx}|^2\,dx,$$

for arbitrarily small ϵ. Then choosing ϵ small enough and applying Gronwall's lemma yields (75).

Now let us show that $\mathcal{T}$ is contractive on

$$B := \{u\in X \ / \ \|u\|_X \le 2\|u_0\|_{L^2(0,1)}\},$$

for sufficiently small T.

From (75), we see that, provided that T is suitably small, B is stable by $\mathcal{T}$. We now consider that this is the case. Now consider $z_1,z_2\in B$ and denote $u_1 := \mathcal{T}z_1$, $u_2 := \mathcal{T}z_2$, $z := z_1-z_2$, $u := u_1-u_2$. We have

$$u_t + \alpha u_{5x} + \mu u_{xxx} + \beta z_1 u_{xxx} + \beta z u_{2,xxx} + \delta z_{1,x}u_{xx} + \delta z_x u_{2,xx} \\ + P'(z_1)u_x + [P'(z_1)-P'(z_2)]u_{2,x} = 0 \text{ in } (0,T)\times(0,1), \quad (76)$$

We multiply (76) by $(2-x)u$, integrate in both time and space and perform the same reasoning as in (73)-(74). After lengthy but straightforward computations, we deduce (if $T\le 1$)

$$\|u\|_X^2 \le CT^{1/8}[\|u_2\|_X\|u\|_X\|z\|_X + (1+\|z_1\|_X^2)\|u\|_X^2]. \quad (77)$$

Using that both u_2 and z_1 belong to B, this establishes that $\mathcal{T}$ is contractive on B for sufficiently small time T.

Now that we have shown the existence of the solution of the nonlinear equation, we can prove its regularizing effect through a bootstrap argument.

First, since u is in $L^2(0,T;H^2(0,1))$ and using (75), one can find $T' \in (T/4, T/2)$ such that $u(T',\cdot) \in H^2(0,1)$ with

$$\|u(T',\cdot)\|_{H^2(0,1)} \le \frac{C}{T^{1/2}}\|u_0\|_{L^2}. \tag{78}$$

From now on, we consider T' as the initial time and look at the Cauchy problem starting from time T'. We introduce $\tilde{X} := L^2(T',T;H^3(0,1)) \cap C^0([T',T];H^1(0,1)) \cap H^1(T',T;H^{-2}(0,1))$ and

$$\tilde{B} := \{u \in \tilde{X} \ / \ \|u\|_{\tilde{X}} \le 2\|u(T',\cdot)\|_{H^1(0,1)}\}.$$

Observe that $\tilde{B}$ is a compact convex subset of $X_{T'} := L^2(T',T;H_0^2(0,1)) \cap C^0([T',T];L^2(0,1))$.

Now we see that $\tilde{X}$ is an invariant space for $\mathcal{T}$: the right hand side $\mu u_{xxx} + \beta z u_{xxx} + \delta z_x u_{xx} + P'(z)u_x$ belongs to $L^2(0,T;H^{-1}(0,1))$ when $u \in X$ and $z \in \tilde{X}$; then the statement follows from Proposition 3. Moreover, we have

$$\begin{aligned} \|u\|_{\tilde{X}} &\le C\big(\|\mu u_{xxx} + \beta z u_{xxx} + \delta z_x u_{xx} + P'(z)u_x\|_{L^2(0,T;H^{-1}(0,1))} + \|u(T',\cdot)\|_{H^1(0,1)}\big) \\ &\le C\big((1+\|z\|_{\tilde{X}}^2)\|u\|_X + \|u(T',\cdot)\|_{H^1(0,1)}\big). \end{aligned} \tag{79}$$

We use (75) and (78) to finally get

$$\|u\|_{\tilde{X}} \le C(1+\|z\|_{\tilde{X}}^2)\|u_0\|_{L^2(0,1)}. \tag{80}$$

This proves that for small u_0, $\mathcal{T}$ sends $\tilde{B}$ into itself. Then by Schauder's fixed point theorem, there exists a solution of the nonlinear problem in $\tilde{B}$. Due to the uniqueness of the solution constructed by contraction in $X_{T'}$, this solution coincides with the solution which we constructed in X.

We introduce $\eta \in C^\infty([0,T];\mathbb{R})$ such that $\eta = 0$ in $[0,T/4]$ and $\eta = 1$ in $[T/2,T]$. We consider the equation satisfied by ηu and use Proposition 3. It is not difficult to see that, since $u \in \tilde{X}$, the right hand side in

$$(\eta u)_t + \alpha(\eta u)_{5x} = -\mu\eta u_{xxx} - \beta\eta u u_{xxx} - \delta\eta u_x u_{xx} - \eta P'(u)u_x + \eta' u \text{ in } (0,T)\times(0,1),$$

can be estimated in $L^2(0,T;L^2(0,1))$. Hence we deduce that $\eta u \in L^2(0,T;H^4(0,1)) \cap C^0([0,T];H^2(0,1)) \cap H^1(0,T;H^{-1}(0,1))$. Then repeating the above steps we can show that the solution u becomes C^∞ in time and space in arbitrary small time.

4.2 Proof of Theorem 1

We consider a trajectory $\overline{u}$ as indicated in the statement; then $u = \overline{u} + y$ satisfies (1) if and only if y satisfies

$$\begin{aligned} y_t + y_{5x} + \mu y_{3x} + \beta((\overline{u}+y)y_{3x} + \overline{u}_{3x}y) + \delta((\overline{u}+y)_x y_{xx} + \overline{u}_{xx}y_x) \\ + p y_x + 2q((\overline{u}+y)y_x + \overline{u}_x y) + 3r(y(2\overline{u}+y)(\overline{u}+y)_x + \overline{u}^2 y_x) = 0. \end{aligned} \tag{81}$$

Conspicuously, the controllability of (1) to the trajectory $\overline{u}$ is equivalent to the null controllability of (81).

Now we have the following result for (81).

Proposition 11. *Given $y_0 \in L^2(0,1)$ and $\overline{u} \in L^\infty(0,T;W^{3,\infty}(0,1))$, there exists $T > 0$ such that the nonlinear problem (81) with homogenous boundary conditions (3) ($v_1 = v_2 = v_3 = v_4 = v_5 = 0$) admits a unique solution $y \in L^2(0,T;H^2(0,1)) \cap C^0([0,T];L^2(0,1))$, which regularizes in the sense that for any $\tau \in (0,T]$, $u \in L^2([\tau,T];H^4(0,1)) \cap C^0([\tau,T];H^2(0,1))$, with moreover*

$$\|y\|_{C^0([\tau,T];H^2(0,1))} \le C(\tau,\overline{u})\|u_0\|_{L^2(0,1)}. \tag{82}$$

The proof of Proposition 11 follows the steps of the proof of Theorem 2; all the computations are justified thanks to $\overline{u} \in L^\infty(0,T;W^{3,\infty}(0,1))$. We omit the details.

We now turn to the proof of Theorem 1. The solution of the controllability problem is obtained in two successive steps. In a first step, we set the controls (v_2, v_4) to $(0,0)$. According to Proposition 11, this regularizes the state of the system, so that we may consider that the initial state y_0 belongs to $H_0^2(0,1)$ and is small (see (82)). From now, we consider that this is the case, and proceed to the proof of the null-controllability of system (81) with such an initial state, by using the inverse mapping theorem.

We introduce the coefficients a_k as follows

$$\begin{aligned} a_0 &= \beta \overline{u}_{3x} + 2q\overline{u}_x + 6r\overline{u}\,\overline{u}_x, \\ a_1 &= \delta \overline{u}_{xx} + p + 2q\overline{u} + 3r\overline{u}^2, \\ a_2 &= \delta \overline{u}_x, \\ a_3 &= \mu + \beta\overline{u}. \end{aligned}$$

Recall that L is expressed by (59). Define

$$Y_1 := \{f \in L^2(0,T;L^2(0,1)) \ / \ (T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} f \in L^2(0,T;L^2(0,1))\}, \tag{83}$$

equipped with the clear corresponding norm. We consider the following map

$$\Lambda : \left\{ \begin{array}{l} E_1 \longrightarrow H_0^2(0,1) \times Y_1 \\ y \mapsto (y(0), Ly + \beta y y_{xxx} + \delta y_x y_{xx} + (2q + 6r\overline{u}) y y_x + 3r\overline{u} y^2 + 3ry^2 y_x). \end{array} \right. \tag{84}$$

Recall that the definition of E_1 was given in (65). Note that the mapping Λ is well defined and C^1. Indeed, from $y \in E_1$, we find out that

$$\beta (T-t)^{5/2} e^{\frac{2\kappa_1}{(T-t)^{1/4}}} y y_{xxx} \in L^2(0,T;L^2(0,1)).$$

Then, thanks to (46) and (53), we have that

$$\beta (T-t)^{9/8} e^{\frac{\kappa_0}{(T-t)^{1/4}}} y y_{xxx} \in L^2(0,T;L^2(0,1)).$$

The same can be done for all the other terms (since they are bilinear or trilinear). Now using Proposition 10, we see that $\Lambda'(0)$ is a surjective map. Hence there exists a neighborhood of $(0,0)$ in $H_0^2(0,1) \times Y_1$ on which Λ is onto. This gives the desired result.

5 Proof of Proposition 7

Let $\psi := e^{-s\alpha}\varphi$, where α is given by (42) and φ fulfills system (40). We deduce that

$$L_1\psi + L_2\psi = L_3\psi,$$

with

$$L_1\psi = \psi_t + \psi_{5x} + 10s^2\alpha_x^2\psi_{xxx} + 5s^4\alpha_x^4\psi_x, \tag{85}$$

$$L_2\psi = 5s\alpha_x\psi_{4x} + 10s^3\alpha_x^3\psi_{xx} + s^5\alpha_x^5\psi + s\alpha_t\psi + 10s\alpha_{xx}\psi_{xxx} + 30s^3\alpha_x^2\alpha_{xx}\psi_x, \tag{86}$$

and

$$\begin{aligned} L_3\psi = \ & -e^{-s\alpha}\Big\{\big[e^{s\alpha}(3s\alpha_{xx}\psi_x + 3s^2\alpha_x\alpha_{xx}\psi)\big]_{xx} + \big[e^{s\alpha}(3s\alpha_{xx}\psi_{xx} + 6s^2\alpha_x\alpha_{xx}\psi_x)\big]_x\Big\} \\ & -\Big\{ -6s\alpha_{xx}\psi_{xxx} + 12s^2\alpha_x\alpha_{xx}\psi_{xx} - 15s^3\alpha_x^2\alpha_{xx}\psi_x + 7s^4\alpha_x^3\alpha_{xx}\psi + 6s^3\alpha_x\alpha_{xx}^2\psi\Big\} \\ & + e^{-s\alpha}\Big\{f + \textstyle\sum_{k=0}^3 (-1)^{k+1}\partial_x^k(a_k(t,x)e^{s\alpha}\psi)\Big\}. \end{aligned} \tag{87}$$

(We recall that $\alpha_{xxx} = 0$.) Then, we have

$$\|L_1\psi\|^2_{L^2(Q)} + \|L_2\psi\|^2_{L^2(Q)} + 2\iint_Q L_1\psi\, L_2\psi\, dx\, dt = \|L_3\psi\|^2_{L^2(Q)}. \tag{88}$$

The main part of what follows consists in evaluating the double product term. We will denote by $(L_i\psi)_j$ $(1 \le i \le 4,\ 1 \le j \le 6)$ the j-th term in the expression of $L_i\psi$. We recall that $\alpha > 0$, $\alpha_x < 0$, $\alpha_{xx} < 0$ and $\alpha_{xxx} = 0$. In the sequel we will repeatedly use that $\psi_{|x=0,1} = \psi_{x|x=0,1} = 0$.

• First, integrating by parts with respect to x and t, we have

$$\begin{aligned}
((L_1\psi)_1,(L_2\psi)_1)_{L^2(Q)} &= -5s\iint_Q \alpha_x\psi_{tx}\psi_{xxx}\,dt\,dx - 5s\iint_Q \alpha_{xx}\psi_t\psi_{xxx}\,dt\,dx\\
&= \frac{5}{2}s\iint_Q \alpha_x(|\psi_{xx}|^2)_t\,dt\,dx + 10s\iint_Q \alpha_{xx}\psi_{tx}\psi_{xx}\,dt\,dx + 5s\iint_Q \alpha_{xxx}\psi_t\psi_{xx}\,dt\,dx\\
&\ge -CsT\iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx + 10s\iint_Q \alpha_{xx}\psi_{tx}\psi_{xx}\,dt\,dx.
\end{aligned} \tag{89}$$

For the second term, we get

$$\begin{aligned}
((L_1\psi)_1,(L_2\psi)_2)_{L^2(Q)} = &= -5s^3\iint_Q \alpha_x^3(|\psi_x|^2)_t\,dt\,dx - 30s^3\iint_Q \alpha_x^2\alpha_{xx}\psi_t\psi_x\,dt\,dx\\
&\ge -Cs^3T\iint_Q \alpha^7|\psi_x|^2\,dt\,dx - 30s^3\iint_Q \alpha_x^2\alpha_{xx}\psi_t\psi_x\,dt\,dx.
\end{aligned} \tag{90}$$

For the third term, we obtain

$$\begin{aligned}
((L_1\psi)_1,(L_2\psi)_3)_{L^2(Q)} &= \frac{1}{2}s^5\iint_Q \alpha_x^5(\psi^2)_t\,dt\,dx\\
&\ge -Cs^5T\iint_Q \alpha^9\psi^2\,dt\,dx.
\end{aligned} \tag{91}$$

We consider now the fourth term of $L_2\psi$ and using (44) we readily get

$$\begin{aligned}
((L_1\psi)_1,(L_2\psi)_4)_{L^2(Q)} &= \frac{s}{2}\iint_Q \alpha_t(\psi^2)_t\,dt\,dx\\
&\ge -CsT^2\iint_Q \alpha^9\psi^2\,dt\,dx.
\end{aligned} \tag{92}$$

The next term gives

$$((L_1\psi)_1,(L_2\psi)_5)_{L^2(Q)} = -10s\iint_Q \alpha_{xx}\psi_{tx}\psi_{xx}\,dt\,dx. \tag{93}$$

The last term gives

$$((L_1\psi)_1,(L_2\psi)_6)_{L^2(Q)} = 30s^3\iint_Q \alpha_x^2\alpha_{xx}\psi_x\psi_t\,dt\,dx. \tag{94}$$

All these computations ((89)-(94)) show that

$$\begin{aligned}
((L_1\psi)_1,(L_2\psi))_{L^2(Q)} &\ge -CsT\iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx - Cs^3T\iint_Q \alpha^7|\psi_x|^2\,dt\,dx\\
&\quad -C(s^5T+sT^2)\iint_Q \alpha^9\psi^2\,dt\,dx\\
&\ge -\epsilon s^5\iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx - \epsilon s^7\iint_Q \alpha^7|\psi_x|^2\,dt\,dx\\
&\quad -\epsilon s^9\iint_Q \alpha^9\psi^2\,dt\,dx,
\end{aligned} \tag{95}$$

for any $\epsilon > 0$, provided that $s \ge CT^{1/4}$, where C depends on ϵ.

• Now we consider the second term of L_1. The product with the first term of L_2 gives

$$\begin{aligned}((L_1\psi)_2,(L_2\psi)_1)_{L^2(Q)} = & -\frac{5}{2}s\iint_Q \alpha_{xx}|\psi_{4x}|^2\,dt\,dx+\frac{5}{2}s\int_0^T \alpha_{x|x=1}|\psi_{4x|x=1}|^2\,dt \\ & -\frac{5}{2}s\int_0^T \alpha_{x|x=0}|\psi_{4x|x=0}|^2\,dt.\end{aligned} \tag{96}$$

Similar computations give the following for the second term:

$$\begin{aligned}((L_1\psi)_2,(L_2\psi)_2)_{L^2(Q)} &= -5s^3\iint_Q \alpha_x^3(|\psi_{xxx}|^2)_x\,dt\,dx-10s^3\int_0^T \alpha^3_{x|x=0}\psi_{4x|x=0}\psi_{xx|x=0}\,dt \\ &\quad -30s^3\iint_Q \alpha_x^2\alpha_{xx}\psi_{4x}\psi_{xx}\,dt\,dx \\ &\geq 45s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx-5s^3\int_0^T \alpha^3_{x|x=1}|\psi_{xxx|x=1}|^2\,dt \\ &\quad +5s^3\int_0^T \alpha^3_{x|x=0}|\psi_{xxx|x=0}|^2\,dt-10s^3\int_0^T \alpha^3_{x|x=0}\psi_{4x|x=0}\psi_{xx|x=0}\,dt \\ &\quad +30s^3\int_0^T \alpha^2_{x|x=0}\alpha_{xx|x=0}\psi_{xxx|x=0}\psi_{xx|x=0}\,dt-Cs^3\iint_Q \alpha^3|\psi_{xxx}||\psi_{xx}|\,dt\,dx.\end{aligned} \tag{97}$$

For the third one we have

$$\begin{aligned}((L_1\psi)_2,(L_2\psi)_3)_{L^2(Q)} &= -s^5\iint_Q \alpha_x^5\psi_{4x}\psi_x\,dt\,dx-5s^5\iint_Q \alpha_x^4\alpha_{xx}\psi_{4x}\psi\,dt\,dx \\ &= \frac{s^5}{2}\iint_Q \alpha_x^5(|\psi_{xx}|^2)_x\,dt\,dx+10s^5\iint_Q \alpha_x^4\alpha_{xx}\psi_{xxx}\psi_x\,dt\,dx \\ &\quad +5s^5\iint_Q (\alpha_x^4\alpha_{xx})_x\psi_{xxx}\psi\,dt\,dx \\ &\geq -\frac{s^5}{2}\int_0^T \alpha^5_{x|x=0}|\psi_{xx|x=0}|^2\,dt-\frac{25}{2}s^5\iint_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx \\ &\quad -Cs^5\iint_Q \alpha^5(|\psi_{xxx}||\psi|+|\psi_{xx}||\psi_x|)\,dt\,dx.\end{aligned} \tag{98}$$

Then, we see that

$$\begin{aligned}((L_1\psi)_2,(L_2\psi)_4)_{L^2(Q)} &= -s\iint_Q \alpha_t\psi_{4x}\psi_x\,dt\,dx-s\iint_Q \alpha_{tx}\psi_{4x}\psi\,dt\,dx \\ &= \frac{s}{2}\iint_Q \alpha_t(|\psi_{xx}|^2)_x\,dt\,dx+2s\iint_Q \alpha_{tx}\psi_{3x}\psi_x\,dt\,dx \\ &\quad +s\iint_Q \alpha_{txx}\psi_{xxx}\psi\,dt\,dx \\ &\geq -CsT\iint_Q \alpha^5(|\psi_{xx}|^2+|\psi_{xxx}|(|\psi|+|\psi_x|))\,dt\,dx \\ &\quad -CsT\int_0^T \alpha^5_{|x=0}|\psi_{xx|x=0}|^2\,dt.\end{aligned} \tag{99}$$

Next,

$$\begin{aligned}((L_1\psi)_2,(L_2\psi)_5)_{L^2(Q)} = & -10s\iint_Q \alpha_{xx}|\psi_{4x}|^2\,dt\,dx+10s\int_0^T \alpha_{xx|x=1}\psi_{4x|x=1}\psi_{xxx|x=1}\,dt \\ & -10s\int_0^T \alpha_{xx|x=0}\psi_{4x|x=0}\psi_{xxx|x=0}\,dt.\end{aligned} \tag{100}$$

We used that $\alpha_{xxx}=0$.

Finally,

$$\begin{aligned}((L_1\psi)_2,(L_2\psi)_6)_{L^2(Q)} &= -30s^3\iint_Q \alpha_x^2\alpha_{xx}\psi_{4x}\psi_{xx}\,dt\,dx - 30s^3\iint_Q(\alpha_x^2\alpha_{xx})_x\psi_{4x}\psi_x\,dt\,dx\\ &= 30s^3\iint_Q \alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx + 30s^3\int_0^T(\alpha_x^2\alpha_{xx})_{|x=0}\psi_{xxx|x=0}\psi_{xx|x=0}\,dt\\ &\quad -Cs^3\iint_Q\alpha^3(|\psi_{4x}||\psi_x|+|\psi_{xxx}||\psi_{xx}|)\,dt\,dx.\end{aligned} \tag{101}$$

Putting together all the computations concerning the second term of $L_1\psi$ ((96)-(101)), we obtain

$$\begin{aligned}((L_1\psi)_2,L_2\psi)_{L^2(Q)} \geq\ & -\frac{25}{2}s\iint_Q\alpha_{xx}|\psi_{4x}|^2\,dt\,dx + 75s^3\iint_Q\alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx\\ & -\frac{25}{2}s^5\iint_Q\alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx - \frac{5}{2}s\int_0^T\alpha_{x|x=0}|\psi_{4x|x=0}|^2\,dt\\ & -\frac{s^5}{2}\int_0^T\alpha_{x|x=0}^5|\psi_{xx|x=0}|^2\,dt - 10s^3\int_0^T\alpha_{x|x=0}^3\psi_{4x|x=0}\psi_{xx|x=0}\,dt\\ & +5s^3\int_0^T\alpha_{x|x=0}^3|\psi_{xxx|x=0}|^2\,dt - \epsilon s^9\iint_Q\alpha^9\psi^2\,dt\,dx\\ & -\epsilon s^7\iint_Q\alpha^7|\psi_x|^2\,dt\,dx - \epsilon s^5\iint_Q\alpha^5|\psi_{xx}|^2\,dt\,dx\\ & -\epsilon s^3\iint_Q\alpha^3|\psi_{xxx}|^2\,dt\,dx - \epsilon s\iint_Q\alpha|\psi_{4x}|^2\,dt\,dx\\ & -\epsilon s^5\int_0^T\alpha_{|x=0}^5|\psi_{xx|x=0}|^2\,dt - \epsilon s^3\int_0^T\alpha_{|x=0}^3|\psi_{xxx|x=0}|^2\,dt\\ & -\epsilon s\int_0^T\alpha_{|x=0}|\psi_{4x|x=0}|^2\,dt - Cs\int_0^T\alpha_{|x=1}|\psi_{4x|x=1}|^2\,dt\\ & -Cs^3\int_0^T\alpha_{|x=1}^3|\psi_{xxx|x=1}|^2\,dt,\end{aligned} \tag{102}$$

for any $\epsilon>0$, provided that $s\geq C(T^{1/4}+T^{1/2})$, where C depends on ϵ. (We used that $s\geq C(\epsilon)T^{1/2}$ for appropriate $C(\epsilon)$ and $\alpha\leq CT\alpha^3$.)

• We consider now the products concerning the third term of $L_1\psi$. First, we have

$$\begin{aligned}((L_1\psi)_3,(L_2\psi)_1)_{L^2(Q)} &= -75s^3\iint_Q\alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx + 25s^3\int_0^T\alpha_{x|x=1}^3|\psi_{xxx|x=1}|^2\,dt\\ &\quad -25s^3\int_0^T\alpha_{x|x=0}^3|\psi_{xxx|x=0}|^2\,dt.\end{aligned} \tag{103}$$

Secondly

$$((L_1\psi)_3,(L_2\psi)_2)_{L^2(Q)} = -250s^5\iint_Q\alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx - 50s^5\int_0^T\alpha_{x|x=0}^5|\psi_{xx|x=0}|^2\,dt. \tag{104}$$

Third,

$$\begin{aligned}((L_1\psi)_3,(L_2\psi)_3)_{L^2(Q)} &= -5s^7\iint_Q\alpha_x^7(|\psi_x|^2)_x\,dt\,dx - 70s^7\iint_Q\alpha_x^6\alpha_{xx}\psi_{xx}\psi\,dt\,dx\\ &\geq 105s^7\iint_Q\alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx - Cs^7\iint_Q\alpha^7|\psi_x||\psi|\,dt\,dx.\end{aligned} \tag{105}$$

For the fourth term, we have

$$\begin{aligned}((L_1\psi)_3,(L_2\psi)_4)_{L^2(Q)} &= -10s^3\iint_Q\alpha_x^2\alpha_t\psi_{xx}\psi_x\,dt\,dx - 10s^3\iint_Q(\alpha_x^2\alpha_t)_x\psi_{xx}\psi\,dt\,dx\\ &\geq -Cs^3T\iint_Q\alpha^7(|\psi_x|^2+|\psi||\psi_{xx}|)\,dt\,dx.\end{aligned} \tag{106}$$

We obtain the following for the fifth term:

$$((L_1\psi)_3,(L_2\psi)_5)_{L^2(Q)} = 100s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx. \tag{107}$$

Finally,

$$((L_1\psi)_3,(L_2\psi)_6)_{L^2(Q)} = -300s^5 \iint_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dx\,dt - Cs^5 \iint_Q \alpha^5|\psi_{xx}||\psi_x|\,dt\,dx. \tag{108}$$

Consequently, we get the following for the third term of $L_1\psi$ ((103)-(108)):

$$\begin{aligned}((L_1\psi)_3, L_2\psi)_{L^2(Q)} \geq\ & 25s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx - 550s^5 \iint_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx \\ & +105s^7 \iint_Q \alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx - 25s^3\int_0^T \alpha^3_{x|x=0}|\psi_{xxx|x=0}|^2\,dt \\ & -50s^5\int_0^T \alpha^5_{x|x=0}|\psi_{xx|x=0}|^2\,dt - \epsilon s^9 \iint_Q \alpha^9\psi^2\,dt\,dx \\ & -\epsilon s^7 \iint_Q \alpha^7|\psi_x|^2\,dt\,dx - \epsilon s^5 \iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx \\ & -Cs^3\int_0^T \alpha^3_{|x=1}|\psi_{xxx|x=1}|^2\,dt,\end{aligned} \tag{109}$$

for any $\epsilon > 0$, where again $s \geq C(T^{1/4}+T^{1/2})$ and C depends on ϵ.

• Now, we compute the fourth term. First, we have:

$$\begin{aligned}((L_1\psi)_4,(L_2\psi)_1)_{L^2(Q)} =\ & -\frac{25}{2}s^5 \iint_Q \alpha_x^5(|\psi_{xx}|^2)_x\,dx\,dt - 125s^5 \iint_Q \alpha_x^4\alpha_{xx}\psi_{xxx}\psi_x\,dx\,dt \\ \geq\ & \frac{375}{2}s^5 \iint_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx + \frac{25}{2}s^5\int_0^T \alpha^5_{x|x=0}|\psi_{xx|x=0}|^2\,dt \\ & -Cs^5 \iint_Q \alpha^5|\psi_x||\psi_{xx}|\,dx\,dt.\end{aligned} \tag{110}$$

Next, we obtain

$$((L_1\psi)_4,(L_2\psi)_2)_{L^2(Q)} = -175s^7 \iint_Q \alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx. \tag{111}$$

For the third term, we get

$$((L_1\psi)_4,(L_2\psi)_3)_{L^2(Q)} = -\frac{45}{2}s^9 \iint_Q \alpha_x^8\alpha_{xx}|\psi|^2\,dt\,dx. \tag{112}$$

Then,

$$((L_1\psi)_4,(L_2\psi)_4)_{L^2(Q)} \geq -Cs^5T \iint_Q \alpha^9|\psi|^2\,dt\,dx. \tag{113}$$

The fifth term gives

$$((L_1\psi)_4,(L_2\psi)_5)_{L^2(Q)} = -50s^5 \iint_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx - Cs^5 \iint_Q \alpha^5|\psi_{xx}||\psi_x|\,dt\,dx. \tag{114}$$

Direct computations for the last term provides

$$((L_1\psi)_4,(L_2\psi)_6)_{L^2(Q)} = 150s^7 \iint_Q \alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx. \tag{115}$$

All these computations ((110)-(115)) gives

$$\begin{aligned}((L_1\psi)_4, L_2\psi)_{L^2(Q)} \quad \geq \quad & \frac{275}{2}s^5 \iint_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx - 25s^7 \iint_Q \alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx \\ & -\frac{45}{2}s^9 \iint_Q \alpha_x^8\alpha_{xx}|\psi|^2\,dt\,dx + \frac{25}{2}s^5 \int_0^T \alpha^5_{x|x=0}|\psi_{xx|x=0}|^2\,dt \\ & -\epsilon s^9 \iint_Q \alpha^9\psi^2\,dt\,dx - \epsilon s^7 \iint_Q \alpha^7|\psi_x|^2\,dt\,dx \\ & -\epsilon s^5 \iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx,\end{aligned} \tag{116}$$

for any $\epsilon > 0$, where again $s \geq C(T^{1/4} + T^{1/2})$ and C depends on ϵ.

Let us now gather all the product $(L_1\psi, L_2\psi)_{L^2(Q)}$ coming from (95), (102), (109) and (116):

$$\begin{aligned}(L_1\psi, L_2\psi)_{L^2(Q)} \quad \geq \quad & -\frac{25}{2}s \iint_Q \alpha_{xx}|\psi_{4x}|^2\,dt\,dx + 100s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx \\ & -425s^5 \iint_Q \alpha_x^4\alpha_{xx}|\psi_{xx}|^2\,dt\,dx + 80s^7 \iint_Q \alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx \\ & -\frac{45}{2}s^9 \iint_Q \alpha_x^8\alpha_{xx}|\psi|^2\,dt\,dx - \frac{5}{2}s \int_0^T \alpha_{x|x=0}|\psi_{4x|x=0}|^2\,dt \\ & -20s^3 \int_0^T \alpha^3_{x|x=0}|\psi_{xxx|x=0}|^2\,dt - 38s^5 \int_0^T \alpha^5_{x|x=0}|\psi_{xx|x=0}|^2\,dt \\ & -10s^3 \int_0^T \alpha^3_{x|x=0}\psi_{4x|x=0}\psi_{xx|x=0}\,dt \\ & -\epsilon s^9 \iint_Q \alpha^9\psi^2\,dt\,dx - \epsilon s^7 \iint_Q \alpha^7|\psi_x|^2\,dt\,dx \\ & -\epsilon s^5 \iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx - \epsilon s^3 \iint_Q \alpha^3|\psi_{xxx}|^2\,dt\,dx \\ & -\epsilon s \iint_Q \alpha|\psi_{4x}|^2\,dt\,dx - \epsilon s^5 \int_0^T \alpha^5_{|x=0}|\psi_{xx|x=0}|^2\,dt \\ & -\epsilon s^3 \int_0^T \alpha^3_{|x=0}|\psi_{xxx|x=0}|^2\,dt - \epsilon s \int_0^T \alpha_{|x=0}|\psi_{4x|x=0}|^2\,dt \\ & -Cs \int_0^T \alpha_{|x=1}|\psi_{4x|x=1}|^2\,dt - Cs^3 \int_0^T \alpha^3_{|x=1}|\psi_{xxx|x=1}|^2\,dt,\end{aligned} \tag{117}$$

for $s \geq C(T^{1/4} + T^{1/2})$.

Let us explain how we handle the wrongly signed terms in $|\psi_{xxx}|^2$ and $|\psi_x|^2$. After integration by parts, we get

$$\begin{aligned}100s^3 \iint_Q \alpha_x^2\alpha_{xx}|\psi_{xxx}|^2\,dt\,dx \quad \geq \quad & -100s^3 \iint_Q \alpha_x^2\alpha_{xx}\psi_{xx}\psi_{4x}\,dt\,dx - \epsilon s^5 \iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx \\ & -\epsilon s^3 \int_0^T \alpha^3_{|x=0}|\psi_{xxx|x=0}|^2 - \epsilon s^5 \int_0^T \alpha^5_{|x=0}|\psi_{xx|x=0}|^2\,dt,\end{aligned} \tag{118}$$

by taking $s \geq CT^{1/2}$. The last two terms in the right hand side are already in (117), while the first one is estimated as follows, by using Cauchy-Schwarz's inequality:

$$100s^3\Big|\iint_Q \alpha_x^2|\alpha_{xx}|\psi_{xx}\psi_{4x}\,dt\,dx\Big| \leq 12s \iint_Q \alpha_{xx}|\psi_{4x}|^2\,dt\,dx + \frac{625}{3}s^5 \iint_Q \alpha_x^4|\alpha_{xx}||\psi_{xx}|^2\,dt\,dx. \tag{119}$$

On the other hand, by integration by parts and Cauchy-Schwarz's inequality, we have for $s \geq CT^{1/2}$:

$$80s^7\Big|\iint_Q \alpha_x^6\alpha_{xx}|\psi_x|^2\,dt\,dx\Big| \leq (22+\epsilon)s^9 \iint_Q \alpha_x^8|\alpha_{xx}||\psi|^2\,dt\,dx + \frac{800}{11}s^5 \iint_Q \alpha_x^4|\alpha_{xx}||\psi_{xx}|^2\,dt\,dx. \tag{120}$$

Observe that

$$\frac{625}{3} + \frac{800}{11} < 425. \tag{121}$$

Finally, we have

$$10s^3\Big|\int_0^T \alpha^3_{x|x=0}\psi_{4x|x=0}\psi_{xx|x=0}\,dt\Big| \leq 2s\int_0^T |\alpha_{x|x=0}||\psi_{4x|x=0}|^2\,dt + \frac{25}{2}s^5\int_0^T |\alpha_{x|x=0}|^5|\psi_{xx|x=0}|^2\,dt. \tag{122}$$

Now we observe that thanks to (43) we can absorb all the "ϵ terms" in (117) provided that $s \geq CT^{1/2}$. Finally, using again (43) we deduce from (88) the following inequality for ψ:

$$\begin{aligned} s\iint_Q \alpha|\psi_{4x}|^2\,dt\,dx + s^3\iint_Q \alpha^3|\psi_{xxx}|^2\,dt\,dx + s^5\iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx \\ + s^7\iint_Q \alpha^7|\psi_x|^2\,dt\,dx + s^9\iint_Q \alpha^9|\psi|^2\,dt\,dx \\ \leq C\Big(\|L_3\psi\|^2_{L^2(Q)} + s\int_0^T \alpha_{|x=1}|\psi_{4x|x=1}|^2\,dt + s^3\int_0^T \alpha^3_{|x=1}|\psi_{xxx|x=1}|^2\,dt\Big). \end{aligned} \tag{123}$$

Now it is not difficult to see that all the terms in $L_3\psi$ yield a L^2-norm estimated by

$$\begin{aligned} \|L_3\psi\|^2_{L^2(Q)} \leq C\Big(s^2\iint_Q \alpha^2|\psi_{xxx}|^2\,dt\,dx + s^4\iint_Q \alpha^4|\psi_{xx}|^2\,dt\,dx + s^6\iint_Q \alpha^6|\psi_x|^2\,dt\,dx \\ + s^8\iint_Q \alpha^8|\psi|^2\,dt\,dx + \iint_Q e^{-2s\alpha}|f|^2\,dt\,dx\Big), \end{aligned} \tag{124}$$

for $s \geq CT^{1/2}$. Here we have used that $a_k \in L^\infty(0,T;W^{k,\infty}(0,1))$. Hence they can be absorbed by the left hand side of (123) provided that $s \geq CT^{1/2}$. We deduce the Carleman inequality for ψ

$$\begin{aligned} s\iint_Q \alpha|\psi_{4x}|^2\,dt\,dx + s^3\iint_Q \alpha^3|\psi_{xxx}|^2\,dt\,dx + s^5\iint_Q \alpha^5|\psi_{xx}|^2\,dt\,dx \\ + s^7\iint_Q \alpha^7|\psi_x|^2\,dt\,dx + s^9\iint_Q \alpha^9|\psi|^2\,dt\,dx \\ \leq C\Big(s\int_0^T \alpha_{|x=1}|\psi_{4x|x=1}|^2\,dt + s^3\int_0^T \alpha^3_{|x=1}|\psi_{xxx|x=1}|^2\,dt + \iint_Q e^{-2s\alpha}|f|^2\,dt\,dx\Big). \end{aligned} \tag{125}$$

It remains to replace ψ by φ, to use (43) and $s \geq CT^{1/2}$ in order to deduce (45).

6 Proof of Lemma 1

We first establish two lemmas before turning to the core of the proof.

Lemma 2. *Let p satisfy*

$$\begin{cases} p_t + \alpha p_{5x} = g & \text{in } (0,T)\times(0,2), \\ p_{|x=0} = p_{x|x=0} = p_{xx|x=0} = 0 & \text{in } (0,T), \\ p_{|x=1} = \hat{v}_2,\ p_{x|x=1} = \hat{v}_4 & \text{in } (0,T), \\ p_{|t=0} = p_0 & \text{in } (0,2). \end{cases} \tag{126}$$

Then for $k \geq 2$, one has

$$\begin{aligned} \|(2-x)^{k+\frac12}p\|_{L^\infty(0,T;L^2(0,2))} + \|(2-x)^k p_{xx}\|_{L^2(0,T;L^2(0,2))} + \|(2-x)^{k-1}p_x\|_{L^2(0,T;L^2(0,2))} \\ \lesssim \|(2-x)^{k+1}g\|_{L^2(0,T;H^{-2}(0,2))} + \|(2-x)^{k-2}p\|_{L^2(0,T;L^2(0,2))} + \|(2-x)^{k+\frac12}p_0\|_{L^2(0,2)}. \end{aligned} \tag{127}$$

Proof of Lemma 2. As previously, we multiply by $(2-x)^{2k+1}p$; we get

$$\begin{aligned} &\frac12\frac{d}{dt}\int_0^2 (2-x)^{2k+1}|p|^2\,dx + 5\frac{2k+1}{2}\alpha\int_0^2 (2-x)^{2k}|p_{xx}|^2\,dx \\ &= \int_0^2 (2-x)^{2k+1}pg\,dx + 6k(2k+1)\alpha\int_0^2 (2-x)^{2k-1}p_xp_{xx}\,dx - 2k(2k+1)(2k-1)\alpha\int_0^2 (2-x)^{2k-2}pp_{xx}\,dx. \end{aligned}$$

We utilize Young's inequality:

$$\Big|\int_0^2 (2-x)^{2k-2} p p_{xx}\Big| \le \epsilon \int_0^2 (2-x)^{2k}|p_{xx}|^2\,dx + \frac{1}{\epsilon}\int_0^2 (2-x)^{2k-4}|p|^2\,dx,$$
$$\Big|\int_0^2 (2-x)^{2k-1} p_x p_{xx}\Big| \le \epsilon \int_0^2 (2-x)^{2k}|p_{xx}|^2\,dx + \frac{1}{\epsilon}\int_0^2 (2-x)^{2k-2}|p_x|^2\,dx.$$

Integrate by parts in the last term, we deduce (127).

Lemma 3. *Let p satisfy* (126)*. Then for $k \ge 7$, one has*

$$\|(2-x)^{k+\frac{1}{2}} p\|_{L^\infty(0,T;H^2(0,2))} \sum_{j=0}^{4} \|(2-x)^{k+j-4}\partial_x^j y^*\|_{L^2(0,T;L^2(0,2))} \lesssim \|g\|_{L^2(0,T;L^2(0,2))} + \|p_0\|_{H^2(0,2)} + \|\hat{v}_2\|_{L^2(0,T)} + \|\hat{v}_4\|_{L^2(0,T)}. \quad (128)$$

Proof of Lemma 3.

First step. *Higher order estimates.* Let $g \in L^2(0,T;H_0^3(0,2))$. We apply Lemma 2 to p_{5x} (which satisfies the boundary conditions), and get

$$\|(2-x)^{k+\frac{1}{2}} p_{5x}\|_{L^\infty(0,T;L^2(0,2))} + \|(2-x)^k p_{5x}\|_{L^2(0,T;H^2(0,2))} \lesssim \|(2-x)^{k+1} g_{5x}\|_{L^2(0,T;H^{-2}(0,2))} + \|(2-x)^{k-2} p_{5x}\|_{L^2(0,T;L^2(0,2))} + \|(2-x)^{k+\frac{1}{2}} p_{0,5x}\|_{L^2(0,2)}. \quad (129)$$

By an integration by parts, this inequality yields

$$\|(2-x)^{k+\frac{1}{2}} p_{5x}\|_{L^\infty(0,T;L^2(0,2))} + \|(2-x)^k p_{7x}\|_{L^2(0,T;L^2(0,2))} + \|(2-x)^{k-1} p_{6x}\|_{L^2(0,T;L^2(0,2))} \lesssim \|(2-x)^{k+1} g_{5x}\|_{L^2(0,T;H^{-2}(0,2))} + \|(2-x)^{k-2} p_{5x}\|_{L^2(0,T;L^2(0,2))} + \|(2-x)^{k+\frac{1}{2}} p_{0,5x}\|_{L^2(0,2)}. \quad (130)$$

Now in order to estimate the term concerning p_{5x} in the right hand side, we observe that

$$\int_0^T\int_0^2 (2-x)^{2k-4} p_{5x}p_{5x}\,dx\,dt = -\int_0^T\int_0^2 (2-x)^{2k-4} p_{6x}p_{4x}\,dx\,dt + (k-2)(2k-5)\int_0^T\int_0^2 (2-x)^{2k-6}|p_{4x}|^2\,dx\,dt, \quad (131)$$
$$\int_0^T\int_0^2 (2-x)^{2k-2} p_{6x}p_{6x}\,dx\,dt = -\int_0^T\int_0^2 (2-x)^{2k-2} p_{5x}p_{7x}\,dx\,dt + (k-1)(2k-3)\int_0^T\int_0^2 (2-x)^{2k-4} p_{5x}^2\,dx\,dt. \quad (132)$$

The identity (132) may be used to estimate the first integral in the right hand side of (131) (with $ab \le \epsilon a^2 + b^2/\epsilon$). Now injecting in (130), we obtain

$$\|(2-x)^{k+\frac{1}{2}} p_{5x}\|_{L^\infty(0,T;L^2(0,2))} + \sum_{j=0}^{2} \|(2-x)^{k-j}\partial_x^{7-j} p\|_{L^2(0,T;L^2(0,2))} \lesssim \|(2-x)^{k+1} g_{5x}\|_{L^2(0,T;H^{-2}(0,2))} + \|(2-x)^{k-3} p_{4x}\|_{L^2(0,T;L^2(0,2))} + \|(2-x)^{k+\frac{1}{2}} p_{0,5x}\|_{L^2(0,2)}. \quad (133)$$

Now to absorb the term concerning p_{4x} in the right hand side, we operate in the same way, but here a boundary term appears:

$$\int_0^T\int_0^2 (2-x)^{2k-6} p_{4x}p_{4x}\,dx\,dt = -\int_0^T\int_0^2 (2-x)^{2k-6} p_{5x}p_{3x}\,dx\,dt + (k-3)(2k-7)\int_0^T\int_0^2 (2-x)^{2k-8}|p_{3x}|^2\,dx\,dt + 2^{2k-6}\int_0^T p_{3x|x=0}p_{4x|x=0}\,dt. \quad (134)$$

This latter term is treated as follows:

$$\Big|\int_0^T p_{3x|x=0}p_{4x|x=0}\,dt\Big| \le \epsilon\int_0^T |p_{4x|x=0}|^2\,dt + \frac{1}{\epsilon}\int_0^T |p_{3x|x=0}|^2\,dt.$$

Now

$$\int_0^T |p_{4x|x=0}|^2\,dt = -\frac{1}{2^{2k-6}}\int_0^T\int_0^2 (2-x)^{2k-5}p_{4x}p_{5x}\,dt\,dx - \frac{(2k-5)}{2^{2k-5}}\int_0^T\int_0^2 (2-x)^{2k-6}|p_{4x}|^2\,dt\,dx,$$

which can be treated as above, while

$$\int_0^T |p_{3x|x=0}|^2\,dt = -\frac{1}{2^{2k-8}}\int_0^T\int_0^2 (2-x)^{2k-7}p_{3x}p_{4x}\,dt\,dx - \frac{(2k-7)}{2^{2k-7}}\int_0^T\int_0^2 (2-x)^{2k-8}|p_{3x}|^2\,dt\,dx,$$

which leads us to

$$\begin{aligned}&\|(2-x)^{k+\frac12}p_{5x}\|_{L^\infty(0,T;L^2(0,2))} + \sum_{j=0}^{3}\|(2-x)^{k-j}\partial_x^{7-j}p\|_{L^2(0,T;L^2(0,2))}\\ &\quad\lesssim \|(2-x)^{k+1}g_{5x}\|_{L^2(0,T;H^{-2}(0,2))} + \|(2-x)^{k-4}p_{3x}\|_{L^2(0,T;L^2(0,2))} + \|(2-x)^{k+\frac12}p_{0,5x}\|_{L^2(0,2)}. \quad (135)\end{aligned}$$

Then one follows the same steps as previously (note that $p_{|x=0} = p_{x|x=0} = p_{xx|x=0} = 0$) and finally get

$$\begin{aligned}&\|(2-x)^{k+\frac12}p_{5x}\|_{L^\infty(0,T;L^2(0,2))} + \sum_{j=0}^{6}\|(2-x)^{k-7+j}\partial_x^{j}p\|_{L^2(0,T;L^2(0,2))}\\ &\quad\lesssim \|(2-x)^{k+1}g_{5x}\|_{L^2(0,T;H^{-2}(0,2))} + \|(2-x)^{k-7}p\|_{L^2(0,T;L^2(0,2))} + \|(2-x)^{k+\frac12}p_{0,5x}\|_{L^2(0,2)}, \quad (136)\end{aligned}$$

and consequently, using Proposition 6,

$$\begin{aligned}&\|(2-x)^{k+\frac12}p_{5x}\|_{L^\infty(0,T;L^2(0,2))} + \sum_{j=0}^{6}\|(2-x)^{k-7+j}\partial_x^{j}p\|_{L^2(0,T;L^2(0,2))}\\ &\quad\lesssim \|(2-x)^{k+1}g\|_{L^2(0,T;H^3(0,2))} + \|(2-x)^{k+\frac12}p_{0,5x}\|_{L^2(0,2)} + \|\hat v_2\|_{L^2(0,T)} + \|\hat v_4\|_{L^2(0,T)}. \quad (137)\end{aligned}$$

Second step. *Interpolation.* Now we consider the operator which maps $(p_0, g, \hat v_2, \hat v_4)$ to $(2-x)^{k+1}p$: it is continuous from

$$L^2(0,2)\times L^2(0,T;H^{-2}(0,2))\times L^2(0,T)^2 \text{ to } L^2(0,T;H^2(0,2))\cap C^0([0,T];L^2(0,2)),$$

respectively

$$H_0^5(0,2)\times L^2(0,T;H_0^3(0,2))\times L^2(0,T)^2 \text{ to } L^2(0,T;H^7(0,2))\cap C^0([0,T];H^5(0,2)).$$

By interpolation, it is hence continuous from

$$H_0^2(0,2)\times L^2(0,T;L^2(0,2))\times L^2(0,T)^2 \text{ to } L^2(0,T;H^4(0,2))\cap C^0([0,T];H^2(0,2)).$$

This concludes the proof of Lemma 3.

Proof of Lemma 1.

We apply Lemma 3 with $p = y^*$ and $g = \sum_{j=0}^{3}\tilde a_j(t,x)\partial_x^j y^* + h^*$. We infer

$$\begin{aligned}&\|(2-x)^{k+\frac12}y^*\|_{L^\infty(0,T;H^2(0,2))} + \sum_{j=0}^{4}\|(2-x)^{k+j-4}\partial_x^j y^*\|_{L^2(0,T;L^2(0,2))} \lesssim \|h^*\|_{L^2(0,T;L^2(0,2))}\\ &\qquad + \sum_{j=0}^{3}\|\tilde a_j\partial_x^j y^*\|_{L^2(0,T;L^2(0,2))} + \|y_0^*\|_{H^2(0,2)} + \|v_2^*\|_{L^2(0,T)} + \|v_4^*\|_{L^2(0,T)}. \quad (138)\end{aligned}$$

Now, using that the supports of $\tilde{a}_j$ are away from 2, we can estimate the terms $\sum_{j=0}^{3} \|\tilde{a}_j \partial_x^j y^*\|_{L^2(0,T;L^2(0,2))}$ as follows

$$\sum_{j=0}^{3} \|\tilde{a}_j \partial_x^j y^*\|_{L^2(0,T;L^2(0,2))} \leq \epsilon \sum_{j=0}^{4} \|(2-x)^{k+j-4} \partial_x^j y^*\|_{L^2(0,T;L^2(0,2))} + C\|y^*\|_{L^2(0,T;L^2(0,2))},$$

exactly as in Lemma 3. We get

$$\|(2-x)^{k+\frac{1}{2}} y^*\|_{L^\infty(0,T;H^2(0,2))} + \sum_{j=0}^{4} \|(2-x)^{k+j-4} \partial_x^j y^*\|_{L^2(0,T;L^2(0,2))}$$
$$\lesssim \|h^*\|_{L^2(0,T;L^2(0,2))} + \|y^*\|_{L^2(0,T;L^2(0,2))} + \|y_0^*\|_{H^2(0,2)} + \|v_2^*\|_{L^2(0,T)} + \|v_4^*\|_{L^2(0,T)}.$$

Using again Proposition 6, and thanks to (58), this gives (70), hence completing the argument.

References

[1] Bona J., Sun S. M., Zhang B.-Y., A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain, Comm. Partial Differential Equations 28 (2003), no. 7-8, pp. 1391–1436.

[2] Cerpa E., Exact controllability of a nonlinear Korteweg-de Vries equation on a critical spatial domain, SIAM J. Control Optim. 46 (2007), pp. 877–899.

[3] Cerpa E., Crépeau E., Boundary controllability for the non linear Korteweg-de Vries equation on any critical domain, preprint 2007, to appear in Ann. Inst. H. Poincaré Anal. Non Linéaire.

[4] Colin T., Ghidaglia J.-M., An initial-boundary value problem for the Korteweg-de Vries equation posed on a finite interval, Adv. Differential Equations 6 (2001), no. 12, pp. 1463–1492.

[5] Colliander J. E., Kenig C. E., The generalized Korteweg-de Vries equation on the half line, Comm. Partial Diff. Eq. 27 (2002), no. 11-12, pp. 2187–2266.

[6] Cui S. B., Deng D. G., Tao S. P., Global Existence of Solutions for the Cauchy Problem of the Kawahara Equation with L^2 Initial Data, Acta Math. Sin. (Engl. Ser.) 22 (2006), no. 5, pp. 1457–1466.

[7] Coron J.-M., Crépeau E., Exact boundary controllability of a nonlinear KdV equation with critical lengths, J. Eur. Math. Soc, 6, 2004, pp. 367–398.

[8] Coron J.-M., *Control and Nonlinearity*, Mathematical Surveys and Monographs 136, American Mathematical Society, Providence, RI, 2007.

[9] Dawson L., Uniqueness properties of higher order dispersive equations, J. Differential Equations 236 (2007), no. 1, pp. 199–236.

[10] Faminskii A. V., On Two Initial Boundary Value Problems for the Generalized KdV Equation, Nonlinear Boundary Problems 14 (2004), pp. 58–71.

[11] Fursikov A., Imanuvilov O. Yu., *Controllability of Evolution Equations*, Lecture Notes #34, Seoul National University, Korea, 1996.

[12] Glass O., Guerrero S., Some exact controllability results for the linear KdV equation and uniform controllability in the zero-dispersion limit, preprint 2007, to appear in Asymp. Anal.

[13] Holmer J., The initial-boundary value problem for the Korteweg-de Vries equation, Communications in Partial Differential Equations, 31 (2006), pp. 1151-1190.

[14] Kawahara R., Oscillatory solitary waves in dispersive media, J. Phys. Soc. Japan, 33 (1972), pp. 260–264.

[15] Kenig C., Ponce G., Vega L., Higher-order nonlinear dispersive equations, Proc. Amer. Math. Soc. 122 (1994), no. 1, pp. 157–166.

[16] Kichenassamy S., Olver P. J., Existence and nonexistence of solitary wave solutions to higher-order model evolution equations, SIAM J. Math. Anal. 23 (1992), no. 5, pp. 1141–1166.

[17] Kwon S., Well-posedness and ill-posedness of the fifth-order modified KdV equation, Electron. J. Differential Equations 2008, No. 01, pp. 1–15.

[18] Olver P. J., Hamiltonian and non-Hamiltonian models for water waves, in Trends and Applications of Pure Mathematics to Mechanics, P. G. Ciarlet and M. Roseau, eds., Lecture Notes in Physics No. 195, Springer-Verlag, New York, 1984, pp. 273–290.

[19] Pazy A. *Semigroups of linear operators and applications to partial differential equations.* Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

[20] Ponce G., Lax pairs and higher order models for water waves, J. Differential Equations 102 (1993), no. 2, pp. 360–381.

[21] Rosier L., Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. ESAIM Control Optim. Calc. Var. 2 (1997), pp. 33–55.

[22] Rosier L., Exact boundary controllability for the linear Korteweg-de Vries equation on the half-line. SIAM J. Control Optim. 39 (2000), no. 2, pp. 331–351.

[23] Rosier L., Control of the surface of a fluid by a wavemaker. ESAIM Control Optim. Calc. Var. 10 (2004), no. 3, pp. 346–380.

[24] Russell D. L., Zhang B. Y., Controllability and stabilizability of the third-order linear dispersion equation on a periodic domain. SIAM J. Control Optim. 31 (1993), no. 3, pp. 659–676.

[25] Russell D. L., Zhang B. Y., Exact controllability and stabilizability of the Korteweg-de Vries equation. Trans. Amer. Math. Soc. 348 (1996), no. 9, 3643–3672.