Infinite dimensional controllability

Olivier Glass *

Contents

0 Glossary

Infinite dimensional control system. A infinite dimensional control system is a dynamical system whose state lies in an infinite dimensional vector space —typically a Partial Differential Equation (PDE)—, and depending on some parameter to be chosen, called the control.

Exact controllability. The exact controllability property is the possibility to steer the state of the system from any initial data to any target by choosing the control as a function of time in an appropriate way.

Approximate controllability. The approximate controllability property is the possibility to steer the state of the system from any initial data to a state arbitrarily close to a target by choosing a suitable control.

Controllability to trajectories. The controllability to trajectories is the possibility to make the state of the system join some prescribed trajectory by choosing a suitable control.

1 Definition of the subject and its importance

Controllability is a mathematical problem, which consists in determining the targets to which one can drive the state of some dynamical system, by means of a control parameter present in the equation. Many physical systems such as quantum systems, fluid mechanical systems, wave propagation, diffusion phenomena, etc. are represented by an infinite number of degrees of freedom, and their evolution follows some partial differential equation. Finding active controls in order to properly influence the dynamics of these systems generate highly involved problems. The control theory for PDEs, and among this

*Laboratoire Jacques-Louis Lions, Université Pierre et Marie Curie, Paris, France

theory, controllability problems, is a mathematical description of such situations. Any dynamical system represented by a PDE, and on which an external influence can be described, can be the object of a study from this point of view.

2 Introduction

The problem of controllability is a mathematical description of the general following situation. We are given an evolution system (typically a physical one), on which we can exert a certain influence. Is it possible to use this influence to make the system reach a certain state? More precisely, the system takes generally the following form:

$$\dot{y} = F(t, y, u), \tag{1}$$

where y is a description of the state of the system, $\dot{y}$ denotes its derivative with respect to the time t, and u is the control, that is, a parameter which can be chosen in a suitable range. The standard problem of controllability is the following. Given a time $T > 0$, an initial state y_0 and a target y_1, is it possible to find a control function u (depending on the time), such that the solution of the system, starting from y_0 and provided with this function u reaches the state y_1 at time T?

If the state of the system can be described by a finite number of degrees of freedom (typically, if it belongs to an Euclidean space or to a manifold), we call the problem finite dimensional. The present article deals with the case where y belongs to an infinite-dimensional space, typically a Banach space or a Hilbert space. Hence the systems described here have an infinite number of degree of freedom. The potential range of the applications of the theory is extremely wide: the models from fluid dynamics (see for instance [42]) to quantum systems (see [44, 62]), networks of structures (see [19, 37]), wave propagation, etc., are countless.

In the infinite dimensional setting, the equation (1) is typically a partial differential equation, where F acts as a differential operator on the function y, and the influence of u can take multiple different forms: typically, u can be an additional (force) term in the right-hand side of the equation, localized in a part of the domain; it can also appear in the boundary conditions; but other situations can clearly be envisaged (we will describe some of them).

Of course the possibilities to introduce a control problem for partial differential equation are virtually infinite. The number of results since the beginning of the theory in the 60's has been constantly growing and has reached huge dimensions: it is, in our opinion, hopeless to give a fair general view of all the activity in the theory. This paper will only try to present some basic techniques connected to the problem of infinite-dimensional controllability.

As in the finite dimensional setting, one can distinguish between the linear systems, where the partial differential equation under view is linear (as well as the action of the control), and the nonlinear one.

Structure of the paper. In Section 3, we will introduce the problems that are under view and give some examples. The main parts of this paper are Sections 4 and 5, where we consider linear and nonlinear systems, respectively.

3 First definitions and examples

3.1 General framework

We define a infinite-dimensional control system as the following data:

1. an evolution system (typically a PDE)

$$\dot{y} = F(t, y, u),$$

2. the unknown y is the state of the system, which is a function depending on time: $t \in [0, T] \mapsto y(t) \in \mathcal{Y}$, where the set $\mathcal{Y}$ is a functional space (for instance a Banach or a Hilbert space), or a part of a functional space,

3. a parameter u called the control, which is a time-dependent function $t \in [0, T] \mapsto u(t) \in \mathcal{U}$, where the set $\mathcal{U}$ of admissible controls is again some part of a functional space.

As a general rule, one expects that for any initial data $y_{|t=0}$ and any appropriate control function u there exists a unique solution of the system (at least locally in time). In some particular cases, one can find problems "of controllability" type for stationary problems (such as elliptic equations), see for instance [46].

3.2 Examples

Let us give two classical examples of the situation. These are two types of acting control frequently considered in the literature: in one case, the control acts as a localized source term in the equation, while in the second one, the control acts on a part of the boundary conditions. The examples below concern the wave and the heat equations with Dirichlet boundary conditions: these classical equations are reversible and irreversible, respectively, which, as we will see, is of high importance when considering controllability problems.

Example 1. *Distributed control for the wave/heat equation with Dirichlet boundary conditions.*

We consider:

- Ω *a regular domain in* $\mathbb{R}^n$*, which is in general required to be bounded,*
- ω *a nonempty open subdomain in* Ω*,*
- *the wave/heat equation is posed in* $[0,T]\times\Omega$*, with a localized source term in* ω *:*

wave equation $\qquad$ *heat equation*

$$\begin{cases} \Box v := \partial^2_{tt} v - \Delta v = \mathbf{1}_\omega u, \\ v_{|\partial\Omega} = 0, \end{cases} \qquad \begin{cases} \partial_t v - \Delta v = \mathbf{1}_\omega u, \\ v_{|\partial\Omega} = 0. \end{cases}$$

  - *In the first case, the state* y *of the system is given by the couple* $(v(t,\cdot),\partial_t v(t,\cdot))$*, for instance considered in the space* $H^1_0(\Omega)\times L^2(\Omega)$ *or in* $L^2(\Omega)\times H^{-1}(\Omega)$*.*
  - *In the second case, the state* y *of the system is given by the function* $v(t,\cdot)$*, for instance in the space* $L^2(\Omega)$*,*
  - *In both cases, the control is the function* u*, for instance considered in* $L^2([0,T];L^2(\omega))$*.*

Example 2. *Boundary control of the wave/heat equation with Dirichlet boundary conditions.*

We consider:

- Ω *a regular domain in* $\mathbb{R}^n$*, typically a bounded one,*
- Σ *an open nonempty subset of the boundary* $\partial\Omega$*,*
- *the heat/wave equation in* $[0,T]\times\Omega$*, with non homogeneous boundary conditions inside* Σ*:*

wave equation $\qquad$ *heat equation*

$$\begin{cases} \Box v := \partial^2_{tt} v - \Delta v = 0, \\ v_{|\partial\Omega} = \mathbf{1}_\Sigma u, \end{cases} \qquad \begin{cases} \partial_t v - \Delta v = 0, \\ v_{|\partial\Omega} = \mathbf{1}_\Sigma u. \end{cases}$$

The states are the same as in the previous example, but here the control u *is imposed on a part of the boundary. One can for instance consider the set of controls as* $L^2([0,T];L^2(\Sigma))$ *in the first case, as* $C^\infty_0((0,T)\times\Sigma)$ *in the second one.*

Needless to say, one can consider other boundary conditions than Dirichlet's. Let us emphasize that, while these two types of control are very frequent, these are not by far the only ones: for instance, one can consider the following "affine control": the heat equation with a right hand side $u(t)g(x)$ where the control u depends only on the time, and g is a fixed function:

$$\begin{cases} \partial_t v - \Delta v = u(t)g(x), \\ v_{|\partial\Omega} = 0, \end{cases}$$

see an example of this below. Also, one could for instance consider the "bilinear control", which takes the form:

$$\begin{cases} \partial_t v - \Delta v = g(x)u(t)v, \\ v_{|\partial\Omega} = 0. \end{cases}$$

3.3 Main problems

Now let us give some definitions of the typical controllability problems, associated to a control system.

Definition 1. *A control system is said to be exactly controllable in time $T > 0$ if and only if, for all y_0 and y_1 in $\mathcal{Y}$, there is some control function $u : [0, T] \to \mathcal{U}$ such that the unique solution of the system*

$$\begin{cases} \dot{y} = F(t, y, u) \\ y_{|t=0} = y_0, \end{cases} \tag{2}$$

satisfies

$$y_{|t=T} = y_1.$$

Definition 2. *We suppose that the space $\mathcal{Y}$ is endowed with a metric d. The control system is said to be approximately controllable in time $T > 0$ if and only if, for all y_0 and y_1 in $\mathcal{Y}$, for any $\varepsilon > 0$, there exists a control function $u : [0, T] \to \mathcal{U}$ such that the unique solution of the system* (2) *satisfies*

$$d(y_{|t=T}, y_1) < \varepsilon.$$

Definition 3. *We consider a particular element* 0 *of $\mathcal{Y}$. A control system is said to be zero-controllable in time $T > 0$ if and only if, for all y_0 in $\mathcal{Y}$, there exists a control function $u : [0, T] \to \mathcal{U}$ such that the unique solution of the system* (2) *satisfies*

$$y_{|t=T} = 0.$$

Definition 4. *A control system is said to be controllable to trajectories in time $T > 0$ if and only if, for all y_0 in $\mathcal{Y}$ and any trajectory $\overline{y}$ of the system (typically but not necessarily satisfying* (2) *with $u = 0$), there exists a control function $u : [0, T] \to \mathcal{U}$ such that the unique solution of the system* (2) *satisfies*

$$y_{|t=T} = \overline{y}(T).$$

Definition 5. *All the above properties are said to be fulfilled locally, if they are proved for y_0 sufficiently close to the target y_1 or to the starting point of the trajectory $\overline{y}(0)$; they are said to be fulfilled globally if the property is established without such limitations.*

3.4 Remarks

We can already make some remarks concerning the problems that we described above.

1. The different problems of controllability should be distinguished from the problems of optimal control, which give another viewpoint on control theory. In general, problems of optimal control look for a control u minimizing some functional

$$J(u, y(u)),$$

where $y(u)$ is the trajectory associated to the control u.

2. It is important to notice that the above properties of controllability depend in a crucial way on the choice of the functional spaces $\mathcal{Y}$, $\mathcal{U}$. The approximate controllability in some space may be the exact controllability in another space. In the same way, we did not specify the regularity in time of the control functions in the above definitions: it should be specified for each problem.

3. A very important fact for controllability problems is that when a problem of controllability has a solution, it is almost never unique. For instance, if a time-invariant system is controllable regardless of the time T, it is clear that one can choose u arbitrarily in some interval $[0, T/2]$, and then choose an appropriate control (for instance driving the system to 0) during the interval $[T/2, T]$. In such a way, one has constructed a new control which fulfills the required task. The number of controls that one can construct in this way is clearly infinite. This is of course already true for finite dimensional systems.

4. Formally, the problem of interior control when the control zone ω is the whole domain Ω is not very difficult, since it suffices to consider the trajectory

$$v(t) := v_0 + \frac{t}{T}(v_1 - v_0),$$

to compute the left-hand side of the equation with this trajectory, and to choose it as the control. However, by doing so, one obtains in general a control with very low regularity. Note also that, on the contrary, as long as the boundary control problem is concerned, the case when Σ is equal to the whole boundary $\partial\Omega$ is not that simple.

5. Let us also point a "principle" which shows that interior and boundary control problems are not very different. We discuss this in a formal manner.

   Suppose for instance that controllability holds for any domain and subdomain Ω and ω. When considering the controllability problem on Ω via boundary control localized on Σ, one may introduce an extension $\tilde{\Omega}$ of Ω, obtained by gluing along Σ an "additional" open set Ω_2, so that

$$\tilde{\Omega} = \Omega \cup \Omega_2, \quad \overline{\Omega} \cap \overline{\Omega_2} \subset \Sigma \text{ and } \tilde{\Omega} \text{ is regular.}$$

   Consider now $\omega \subset \Omega_2$, and obtain a controllability result on $\tilde{\Omega}$ via interior control located in ω (one has, of course, to extend initial and final states from Ω to $\tilde{\Omega}$). Consider y a solution of this problem, driving the system from y_0 to y_1, in the case of exact controllability, for instance. Then one gets a solution of the boundary controllability problem on Ω, by taking the restriction of y on Ω, and by fixing the trace of y on Σ as the corresponding control (in the case of Dirichlet boundary conditions), the normal derivative in the case of Neumann boundary conditions, etc.

   Conversely, when one has some boundary controllability result, one can obtain an interior control result in the following way. Consider the problem in Ω with interior control distributed in ω. Solve the boundary control problem in $\Omega \setminus \omega$ via boundary controls in $\partial\omega$. Consider y the solution of this problem. Extend properly the solution y to Ω, and as previously, compute the left hand side for the extension, and consider it as a control (it is automatically distributed in ω).

   Of course, in both situations, the regularity of the control that we obtain has to be checked, and this might need a further treatment.

6. Let us also remark that for linear systems, there is no difference between controllability to zero and controllability to trajectories. In that case it is indeed equivalent to bring y_0 to $\overline{y}(T)$ or to bring $y_0 - \overline{y}(0)$ to zero. Note that even for linear systems, on the contrary, approximate controllability and exact controllability differ: an affine subspace of an infinite dimensional space can be dense without filling all the space.

4 Linear systems

In this section, we will briefly describe the theory of controllability for linear systems. The main tool here is the duality between controllability and observability of the adjoint system, see in particular the works by Lions, Russell, and Dolecki and Russell [H, I, 20, 41, 51]. This duality is also of primary importance for finite-dimensional systems.

Let us first describe informally the method of duality for partial differential equations in two cases given in example above.

4.1 Two examples

The two examples that we wish to discuss are the boundary controllability of the wave equation and the interior controllability of the heat equation. The complete answers to these problems have been given by Bardos, Lebeau and Rauch [7] for the wave equation (see also Burq and Gérard [11] and Burq [10] for another proof and a generalization), and simultaneously by Lebeau and Robbiano [39] and Fursikov and Imanuvilov, see [D] for the heat equation. The complete proofs of these deep results are clearly out of the reach of this short presentation, but one can explain rather easily on these examples how the corresponding controllability problems can be transformed into some observability problems. These observability problems consist into proving a certain inequality. We refer for instance to Lions [H] or Zuazua [J] for a more complete introduction to these problems.

First, we notice an important difference between the two equations, which will clearly have consequences concerning the controllability problems. It is indeed well-known that, while the wave equation is a reversible equation, the heat equation is on the contrary irreversible and has a strong regularizing effect. From the latter property, one sees that it is not possible to expect an exact controllability result for the heat equation: outside the control zone ω, the state $u(T)$ will be smooth, and in particular one cannot attain an arbitrary state. As a consequence, while it is natural to seek the exact controllability for the wave equation, it will be natural to look either for approximate controllability or controllability to zero as long as the heat equation is concerned.

For both systems we will introduce the adjoint system (typically obtained via integration by parts): it is central in the resolution of the control problems of linear equations. In both cases, the adjoint system is written in a backward in time form. We consider our two examples separately.

4.1.1 Wave equation

We first consider the case of the wave equation with boundary control on Σ and Dirichlet boundary conditions on the rest of the boundary:

$$\begin{cases} \partial^2_{tt} v - \Delta v = 0, \\ v_{|\partial\Omega} = \mathbf{1}_\Sigma u, \\ (v, v_t)_{|t=0} = (v_0, v_0'). \end{cases}$$

The problem considered is the exact controllability in $L^2(\Omega) \times H^{-1}(\Omega)$ (recall that the state of the system is (u, u_t)), by means of boundary controls in $L^2((0,T) \times \Sigma)$.

For this system, the adjoint system reads:

$$\begin{cases} -\partial^2_{tt} \psi - \Delta \psi = 0, \\ \psi_{|\partial\Omega} = 0, \\ (\psi, \psi_t)_{|t=T} = (\psi_T, \psi_T'). \end{cases} \tag{3}$$

Notice that this adjoint equation is well-posed: here it is trivial since the equation is reversible.

The key argument which connects the controllability problem of the equation with the study of the properties of the adjoint system is is the following duality formula. It is formally easily obtained by multiplying the equation with the adjoint state and in integrating by parts. One obtains

$$\Big[\int_\Omega \psi(\cdot, x) v_t(\cdot, x)\, dx - \int_\Omega \psi_t(\cdot, x) v(\cdot, x)\, dx \Big]_0^T = - \iint_{(0,T)\times\Sigma} \frac{\partial \psi}{\partial n} \mathbf{1}_\Sigma u \, dt\, d\sigma. \tag{4}$$

In other words, these central formula describes in a simple manner the jump in the evolution of the state of the system in terms of the control, when measured against the dual state.

To make the above computation more rigorous, one can consider for dual the state ψ "classical" solutions in $C^0([0,T];H^1_0(\Omega))\cap C^1([0,T];L^2(\Omega))$ (these solutions are typically obtained by using a diagonalizing basis for the Dirichlet laplacian, or by using evolution semi-group theory), while the solutions of the direct problem for $(v_0,v_1,u)\in L^2(\Omega)\times H^{-1}(\Omega)\times L^2(\Sigma)$ are defined in $C^0([0,T];L^2(\Omega))\cap C^1([0,T];H^{-1}(\Omega))$ via the transposition method; for more details we refer to the book of Lions [H].

Now, due to the linearity of the system and because we consider the problem of exact controllability (hence things will be different for what concerns the heat equation), it is not difficult to see that it is not restrictive to consider the problem of controllability starting from 0 (that is the problem of reaching any y_1 when starting from $y_0 := (v_0, v_0') = 0$). Denote indeed $R(T,y_0)$ the affine subspace made of states that can be reached from y_0 at time T for some control. Then calling $y(T)$ the final state of the system for $y_{|t=0}$ and $u=0$, then one has $R(T,y_0)=y(T)+R(T,0)$. Hence $R(T,y_0)=\mathcal{Y}\Leftrightarrow R(T,0)=\mathcal{Y}$.

From (4), we see that reaching (v_T, v_T') from $(0,0)$ will be achieved if and only if the relation

$$\int_\Omega \psi(T,x)v_T'\,dx - \int_\Omega \psi'(T,x)v_T\,dx = -\iint_{(0,T)\times\Sigma} \frac{\partial\psi}{\partial n}\mathbf{1}_\Sigma u\,dt\,d\sigma \tag{5}$$

is satisfied for all choice of (ψ_T,ψ_T').

On the left-hand side, we have a linear form on (ψ_T,ψ_T') in $H^1_0(\Omega)\times L^2(\Omega)$, while on the right hand side, we have a bilinear form on $((\psi_T,\psi_T'),u)$. Suppose that we make the *choice* of looking for a control in the form

$$u=\frac{\partial\overline{\psi}}{\partial n}\mathbf{1}_\Sigma, \tag{6}$$

for some $\overline{\psi}$ solution of (3).

Then one sees, using Riesz' theorem, that to solve this problem for $(v_T,v_T')\in L^2(\Omega)\times H^{-1}(\Omega)$, it is sufficient to prove that the map $(\psi_T,\psi_T')\mapsto \|\frac{\partial\psi}{\partial n}\mathbf{1}_\Sigma\|_{L^2((0,T)\times\Sigma)}$ is a norm equivalent to the $H^1_0(\Omega)\times L^2(\Omega)$ one: for some $C>0$,

$$\|(\psi_T,\psi_T')\|_{H^1_0(\Omega)\times L^2(\Omega)} \le C\|\frac{\partial\psi}{\partial n}\mathbf{1}_\Sigma\|_{L^2((0,T)\times\Sigma)}. \tag{7}$$

This is the observability inequality to be proved to prove the controllability of the wave equation. Let us mention that the inequality in the other sense, that is, the fact that the linear map $(\psi_T,\psi_T')\mapsto\frac{\partial\psi}{\partial n}\mathbf{1}_\Sigma$ is well-defined and continuous from $H^1_0(\Omega)\times L^2(\Omega)$ to $L^2(\partial\Omega)$ is true but not trivial: it is a "hidden" regularity result, see [H].

When this inequality is proved, a constructive way to select the control is to determine a minimum $(\overline{\psi}_T,\overline{\psi}_T')$ of the functional

$$(\psi_T,\psi_T')\mapsto J(\psi_T,\psi_T') := \frac{1}{2}\iint_\Sigma \left|\frac{\partial\psi}{\partial n}\right|^2 dt\,d\sigma + <\phi_T, v_1>_{H^1_0\times H^{-1}} - <\psi_T', v_0>_{L^2\times L^2}, \tag{8}$$

then to associate to $(\overline{\psi}_T,\overline{\psi}_T')$ the solution $\overline{\phi}$ of (5), and finally to set u as in (6).

The way described above to determine a particular control — it is clear that not all controls are in the form (6) — is Lions's HUM method (see [H]). This particular control can be proved to be optimal in the L^2 norm, that is, any other control answering to the controllability problem has a larger norm in $L^2((0,T)\times\Sigma)$. As a matter of fact, looking for the L^2 optimal control among those which answer to the problem is a way to justify the choice (6), see [H].

4.1.2 Heat equation

Now let us consider the heat equation with Dirichlet boundary conditions and localized distributed control:

$$\begin{cases} \partial_t v - \Delta v = \mathbf{1}_\omega u, \\ v_{|\partial\Omega} = 0, \\ v_{|t=0} = v_0. \end{cases}$$

In this case, we consider in the same way the dual problem:

$$\begin{cases} -\partial_t \phi - \Delta \phi = 0, \\ \phi_{|\partial\Omega} = 0, \\ \phi(T) = \phi_T. \end{cases} \tag{9}$$

Notice that this adjoint equation is well-posed. Here it is very important that the problem is formulated in a backward way: the backward in time setting compensates the opposite sign before the time derivative. It is clear that the above equation is ill-posed when considering initial data at $t = 0$.

In the same way as for the wave equation, multiplying the equation by the adjoint state and integrating by parts yields, at least formally:

$$\int_\Omega \phi_T \, v_{|t=T} \, dx - \int_\Omega \phi(0) \, v_0 \, dx = \iint_{(0,T)\times\omega} \phi \, u \, dt \, dx. \tag{10}$$

Note that standard methods yield regular solutions for both direct and adjoint equations when v_0 and ϕ_T belong to $L^2(\Omega)$, and u belongs to $L^2((0,T)\times\omega)$.

Now let us discuss the approximate controllability and the controllability to zero problems separately.

Approximate controllability. Due to linearity, the approximate controllability is equivalent to the approximate controllability starting from 0. Now the set $R(0,T)$ of all final states $v(T)$ which can be reached from 0, via controls $u \in L^2((0,T)\times\omega)$, is a vector subspace of $L^2(\Omega)$. The density of $R(0,T)$ in $L^2(\Omega)$ amounts to the existence of a non trivial element in $(R(0,T))^\perp$. Considering ϕ_T such an element, and introducing it in (10) together with $v_0 = 0$, we see that this involves the existence of a non-trivial solution of (9), satisfying

$$\phi_{|(0,T)\times\omega} = 0.$$

Hence to prove the approximate controllability, we have to prove that there is no such non-trivial solutions, that is, we have to establish a unique continuation result. In this case, this can be proved by using Holmgren's unique continuation principle (see for instance Hörmander [35]), which establishes the result. Note in passing that Holmgren's theorem is a very general and important tool to prove unique continuation results; however it requires the analyticity of the coefficients of the operator, and in many situations one cannot use it directly.

Let us also mention that as for the exact controllability of the wave equation above, and the zero-controllability for the heat equation below, one can single out a control for the approximate controllability by using a variational approach consisting in minimizing some functional as in (8): see Lions [43] and Fabre, Puel and Zuazua [22].

Controllability to zero. Now considering the problem of controllability to zero, we see that, in order that the control u brings the system to 0, it is necessary and sufficient that for all choice of $\phi_T \in L^2(\Omega)$, we have

$$-\int_\Omega \phi(0) \, v_0 \, dx = \iint_{(0,T)\times\omega} \phi \, u \, dt \, dx.$$

Here one would like to reason as for the wave equation, that is, make the *choice* that to look for u in the form

$$u := \phi \mathbf{1}_\omega,$$

for some solution ϕ of the adjoint system. But here the application

$$N : \phi_T \in L^2(\Omega) \mapsto \Big(\iint_{(0,T)\times\omega} \phi^2 \, dt \, dx \Big)^{\frac{1}{2}}.$$

determines a norm (as seen from the above unique continuation result), but this norm is no longer equivalent to the usual $L^2(\Omega)$ norm (if it was, one could establish an exact controllability result!). A way to overcome the problem is to introduce the Hilbert space X obtained by completing $L^2(\Omega)$ for the norm N. In this way, we see that to solve the zero-controllability problem, it is sufficient to prove that the linear mapping $\phi_T \mapsto \phi(0)$ is continuous with respect to the norm N: for some $C > 0$,

$$\|\phi(0)\|_{L^2(\Omega)} \leq C \|\phi \mathbf{1}_\omega\|_{L^2((0,T)\times\omega)}. \tag{11}$$

This is precisely the observability inequality which one has to prove to establish the zero controllability of the heat equation. It is weaker than the observability inequality when the left-hand side is $\|\phi(T)\|_{L^2(\Omega)}$ (which as we noticed is false).

When this inequality is proven, a constructive way to determine a suitable control is to determine a minimum $\overline{\phi}_T$ in X of the functional

$$\phi_T \mapsto \frac{1}{2} \iint_{(0,T)\times\omega} |\phi|^2 \, dt \, dx + \int_\Omega \phi(0) \, v_0 \, dx,$$

then to associate to $\overline{\phi}_T$ the solution $\overline{\phi}$ of (9), and finally to set

$$u := \overline{\phi} \mathbf{1}_\omega.$$

(That the mapping $\overline{\phi}_T \mapsto \overline{\phi} \mathbf{1}_\omega$ can be extended to a mapping from X to $L^2(\omega)$ comes from the definition of X.)

So in both situations of exact controllability and controllability to zero, one has to establish a certain inequality in order to get the result; and to prove approximate controllability, one has to establish a unique continuation result. This turns out to be very general, as described in Paragraph 4.2.

4.1.3 Remarks

Let us mention that concerning the heat equation, the zero-controllability holds for any time $T > 0$ and for any non trivial control zone ω, as shown by means of a spectral method — see Lebeau and Robbiano [39], or by using a global Carleman estimate in order to prove the observability property — see Fursikov and Imanuvilov [D]. That the zero-controllability property does not require the time T or the control zone ω to be large enough is natural since parabolic equations have an infinite speed of propagation; hence one should not require much time for the information to propagate from the control zone to the whole domain. The zero controllability of the heat equation can be extended to a very wide class of parabolic equations; for such results global Carleman estimates play a central role, see for instance the reference book by Fursikov and Imanuvilov [D] and the review article by Fernández-Cara and Guerrero [24]. Note also that Carleman estimates are not used only in the context of parabolic equations: see for instance [D] and Zhang [56].

Let us also mention two other approaches for the controllability of the one-dimensional heat equation: the method of moments of Fattorini and Russell (see [23] and a brief description below), and the method of Laroche, Martin and Rouchon [38] to get an explicit approximate control, based on the idea of "flatness" (as introduced by Fliess, Lévine, Rouchon and Martin [29]).

On the other hand, the controllability for the wave equation does not hold for any T or any Σ. Roughly speaking, the result of Bardos, Lebeau and Rauch [7] states that the controllability property holds if and only if every ray of the geometric optics in the domain (reflecting on the boundary) meets the control zone during the time interval $[0, T]$. In this case, it is also natural that the time should be large enough, because of the finite speed of propagation of the equation: one has to wait for the information coming from the control zone to influence the whole domain. Let us emphasize that in some cases, the geometry of the domain and the control zone is such that the controllability property does not hold, no matter how long the time T is. An example of this is for instance a circle in which some antipodal regions both belong to the uncontrolled part of the boundary. This is due to the existence of Gaussian beams, that is, solutions which are concentrated along some ray of the geometrical optics (and decay exponentially away from it); when this ray does not meet the control zone, this contradicts in particular (7). The result of [7] relies on microlocal analysis. Note that another important tool used for proving observability inequalities is a multiplier method (see in particular Lions [H], Komornik [E], Osses [49]): however in the case of the wave equation, this can be done only in particular geometric situations.

4.2 Abstract approach

The duality between the controllability of a system and the observability of its adjoint system turns out to be very general, and can be described in an abstract form due to Dolecki and Russell [20]. Here we will only describe a particular simplified form of the general setting of [20]. Consider the following system

$$\dot{y} = Ay + Bu, \tag{12}$$

where the state y belongs to a certain Hilbert space H, on which A, which is densely defined and closed, generates a strongly continuous semi-group of bounded operators. The operator B belongs to $\mathcal{L}(U; H)$, where U is also a Hilbert space. The solutions of (12) are given by the method of variation of constants so that

$$y(T) = e^{tA}y(0) + \int_0^T e^{(T-\tau)A}Bu(\tau)d\tau.$$

We naturally associate with the control equation (12), the following observation system:

$$\begin{cases} \dot{z} = -A^* z, \\ z(T) = z_T, \end{cases} \\ c := B^* z. \tag{13}$$

The operator $-A^*$ generates a strongly continuous semi-group for negative times, and c in $L^2(0, T; U)$ is obtained by

$$c(t) = B^* e^{(T-t)A^*} z_T.$$

In the above system, the dynamics of the (adjoint) state z is free, and c is called the observed quantity of the system. The core of the method is to connect the controllability properties of (12) with observability properties of (13) such as described below.

Definition 6. *The system* (13) *is said to satisfy the unique continuation property if and only if the following implication holds true:*

$$c = 0 \textit{ in } [0, T] \Longrightarrow z = 0 \textit{ in } [0, T].$$

Definition 7. *The system* (13) *is said to be observable if and only if there exists* $C > 0$ *such that the following inequality is valid for all solutions* z *of* (13)*:*

$$\|z(T)\|_H \leq C\|c\|_{L^2(0,T;U)}. \tag{14}$$

Definition 8. *The system* (13) *is said to be observable at time* 0 *if and only if there exists* $C > 0$ *such that the following inequality is valid for all solutions* z *of* (13)*:*

$$\|z(0)\|_H \leq C\|c\|_{L^2(0,T;U)}. \tag{15}$$

The main property is the following.

Duality property.
1. The exact controllability of (12) is equivalent to the observability of (13).
2. The zero controllability of (12) is equivalent to the observability at time 0 of (13).
3. The approximate controllability of (12) in H is equivalent to unique continuation property for (13).

Brief explanation of the duality property.

The main fact is the following duality formula

$$< y(T), z(T) >_H - < y(0), z(0) >_H = \int_0^T < u(\tau), B^* e^{(T-\tau)A^*} z(T) >_U d\tau, \tag{16}$$

which in fact can be used to define the solutions of (12) by transposition.

1 & 3. Now it is rather clear that by linearity, we can reduce the problems of exact and approximate

controllability to the ones when $y(0) = 0$. Now the property of exact controllability for system (12) is equivalent to the surjectivity of the operator

$$S : u \in L^2(0,T;U) \mapsto \int_0^T e^{(T-\tau)A} Bu(\tau) d\tau \in H,$$

and the approximate controllability is equivalent to the density of its range. By (16) its adjoint operator is

$$S^* : z_T \in H \mapsto \int_0^T B^* e^{(T-\tau)A^*} z_T d\tau \in L^2(0,T;U).$$

Hence the equivalences 1 and 3 are come from a classical result from functional analysis (see for instance the book of Brezis [9]): the range of S is dense if and only if S^* is one-to-one; S is surjective if and only if S^* satisfies for some $C > 0$:

$$\|z_T\| \leq C \|S^* z_T\|,$$

that is, when (14) is valid.

2. In that case, still due to linearity, the zero-controllability for system (12) is equivalent to the following inclusion:

$$\text{Range}(e^{TA}) \subset \text{Range}(S).$$

In that case there is also a functional analysis result (generalizing the one cited above) which asserts the equivalence of this property with the existence of $C > 0$ such that

$$\|(e^{TA})^* h\| \leq C \|S^* h\|,$$

see Dolecki and Russell [20] and Douglas [21] for more general results.

It follows that in many situations, the exact controllability of a system is proved by establishing an observability inequality on the adjoint system. But this general method is not the final point of the theory: not only, this is only valid for linear systems, but importantly as well, it turns out that these type of inequalities are in general very difficult to establish. In general when a result of exact controllability is established by using this duality, the largest part of the proof is devoted to establishing the observability inequality.

Another information given by the observability is an estimate of the size of the control. Indeed, following the lines of the proof of the correspondence between observability and controllability, one can see that one can find a control u which satisfies:

- Case of exact controllability:

$$\|u\|_{L^2(0,T;\mathcal{U})} \leq C_{obs} \|y_1 - e^{TA} y_0\|_H,$$

- Case of zero controllability:

$$\|u\|_{L^2(0,T;\mathcal{U})} \leq C_{obs} \|y_0\|_H,$$

where in both cases C_{obs} determines a constant for which the corresponding observability inequality is true. (Obviously, not *all* the controls answering to the question satisfy this estimate.) This gives an upper bound on the size of a possible control. This can give some precise estimates on the cost of the control in terms of the parameters of the problem, see for instance the works of Fernández-Cara and Zuazua [26] and Miller [48] concerning the heat equation.

4.3 Some different methods

In this paragraph, we will discuss two methods that do not rely on the controllability/observability duality. This does not pretend to give a general vision of all the techniques that can be used in problems of controllability of linear partial differential equations. We just mention them in order to show that in some situations, the duality may not be the only tool available.

4.3.1 Characteristics

First, let us mention that in certain situations, one may use a characteristics method (see e.g. [18]). A very simple example is the first dimensional wave equation,

$$\begin{cases} v_{tt} - v_{xx} = 0, \\ v_{x|x=0} = 0, \\ v_{x|x=1} = u(t), \end{cases}$$

where the control is $u(t) \in L^2(0,T)$. This is taken from Russell [53]. The problem is transformed into

$$\frac{\partial w}{\partial t} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial w}{\partial x} \text{ with } w := \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} := \begin{pmatrix} v_t \\ v_x \end{pmatrix},$$
$$w_2(t,0) = 0 \text{ and } w_2(t,0) = u(t).$$

Of course, this means that $w_1 - w_2$ and $w_1 + w_2$ are constant along characteristics, which are straight lines of slope 1 and -1 respectively (this is d'Alembert's decomposition).

Now one can deduce an explicit appropriate control for the controllability problem in $[0,1]$ for $T > 2$, by constructing the solution w of the problem directly (and one takes the values of w_2 at $x = 1$ as the "resulting" control u).

The function w is completely determined from the initial and final values in the domains of determinacy D_1 and D_2:

$$D_1 := \{(t,x) \in [0,T] \times [0,1] \ / \ x + t \leq 1\} \text{ and } D_2 := \{(t,x) \in [0,T] \times [0,1] \ / \ t - x \geq T - 1\}.$$

That $T > 2$ involves that these two domains do not intersect. Now it suffices to complete the solution w in $D_3 := [0,T] \times [0,1] \setminus (D_1 \cup D_2)$. For that, one chooses $w_1(0,t)$ *arbitrarily* in the part ℓ of the axis $\{0\} \times [0,1]$ located between D_1 and D_2, that is for

$$x = 0 \text{ and } 1 \leq t \leq T - 1.$$

Once this choice is made, it is not difficult to solve the Goursat problem consisting in extending w in D_3: using the symmetric role of x and t, one considers x as the time. Then the initial condition is prescribed on ℓ, as well as the boundary conditions on the two characteristic lines $x + t = 1$ and $x - t = T - 1$. One can solve elementarily this problem by using the characteristics, and this finishes the construction of w, and hence of u.

Note that as a matter of fact, the observability inequality in this (one-dimensional) case is also elementary to establish, by relying on Fourier series or on d'Alembert's decomposition, see for instance [19].

The method of characteristics described above can be generalized in broader situations, including for instance the problem of boundary controllability of one-dimensional linear hyperbolic systems

$$v_t + A(x)v_x + B(x)v = 0,$$

where A is a real symmetric matrix with eigenvalues bounded away from zero, and A and B are smooth; see for instance Russell [I]. As a matter of fact, in some cases this method can be used to establish the observability inequality from the controllability result (while in most cases the other implication in the equivalence is used).

Of course, the method of characteristics may be very useful for transport equations

$$\frac{\partial f}{\partial t} + v.\nabla f = g \quad \text{or} \quad \frac{\partial f}{\partial t} + \operatorname{div}(vf) = g.$$

An example of this is the controllability of the Vlasov-Poisson equation, see [31].

Let us finally mention that this method be found also to tackle directly several nonlinear problems, as we will see later.

4.3.2 Moments

Another method which we would like to briefly discuss is the method of moments (see for instance Avdonin and Ivanov [A] for a general reference, see also Russell [I]), which can appear in many situations; in particular this method was used by Fattorini and Russell to prove the controllability of the heat equation in one space dimension, see [23]. Consider for instance the problem of controllability of the one-dimensional heat equation

$$\left\{ \begin{array}{l} v_t - v_{xx} = g(x)u(t), \\ v_{|[0,T]\times\{0,1\}} = 0. \end{array} \right.$$

Actually in [23], much more general situations are considered: in particular it concerns more general parabolic equations and boundary conditions, and boundary controls can also be included in the discussion.

It is elementary to develop the solution in the $L^2(0,1)$ orthonormal basis $(\sin(k\pi x))_{k\in\mathbb{N}^*}$. One obtains that the state zero is reached at time $T>0$ if and only if

$$\sum_{k>0} e^{-k^2 T} v_k \sin(k\pi x) = -\sum_{k>0} \int_0^T e^{-k^2(T-t)} g_k u(t) \sin(k\pi x) dt,$$

where v_k and g_k are the coordinates of v_0 and g in the basis. Clearly, this means that we have to find u such that that for all $k \in \mathbb{N}$,

$$\int_0^T e^{-k^2(T-t)} g_k u(t) dt = -e^{-k^2 T} v_k.$$

The classical Muntz-Szász theorem states that for a increasing family $(\lambda_n)_{n\in\mathbb{N}^*}$ of positive numbers, the family $\{e^{-\lambda_n t},\ n\in\mathbb{N}^*\}$ is dense in $L^2(0,T)$ if and only if

$$\sum_{n\geq 0} \frac{1}{\lambda_n} = +\infty,$$

and in the opposite case, the family is independent and spans a proper closed subspace of $L^2(0,T)$. Here the exponential family which we consider is $\lambda_n = n^2$ and we are in the second situation. The same method applies for other problems in which λ_n cannot be completely computed but is a perturbation of n^2; this allows to treat a wider class of problems. Now in this situation (see e.g. [I]), one can construct in $L^2(0,T)$ a biorthogonal family to $\{e^{-\lambda_n t},\ n\in\mathbb{N}^*\}$, that is a family $(p_n(t))_{n\in\mathbb{N}^*} \in (L^2(0,T))^{\mathbb{N}}$ satisfying

$$\int_0^T p_n(t) e^{-\lambda_k^2 t} dt = \delta_{kn}.$$

Once such a family is obtained, one has formally the following solution, under the natural assumption that $|g_k| \geq ck^{-\alpha}$:

$$u(t) = \sum_{k\in\mathbb{N}^*} -\frac{e^{-\lambda_k T} v_k}{g_k} p_k(t).$$

To actually get a control in $L^2(0,T)$, one has to estimate $\|p_k\|_{L^2}$, in order that the above sum is well-defined in L^2. In [23] it is proven that one can construct p_k in such a way that

$$\|p_k\|_{L^2(0,T)} \leq K_0 \exp(K_1 \omega_k), \quad \omega_k := \sqrt{\lambda_k},$$

which allows to conclude.

5 Nonlinear systems

The most frequent method in dealing with control problems of nonlinear systems is the natural one (as for the Cauchy problem): one has to linearize (at least in some sense) the equation, try to prove some controllability result on the linear equation (for instance by using the duality principle), and then try to pass to the nonlinear system via typical methods such as inverse mapping theorem, fixed point theory, iterative schemes...

As for the usual inverse mapping theorem, it is natural to hope a local result from the controllability of the linear problem. Here we find an important difference between linear and nonlinear systems: while for linear systems, no distinction has to be made between local and global results, concerning the nonlinear systems, the two problems are really of different nature.

One should probably not expect a very general result indicating that the controllability of the linearized question involves the local controllability of the nonlinear system. The linearization principle is a general approach which one has to adapt to the different situations that one can meet. We give below some examples that can give a reader ideas of some existing approaches.

5.1 Some linearization situations

Let us first discuss some typical situations where the linearized equation has good controllability properties, and one can hope to get a result (in general local) from this information. In some situations where the nonlinearity is not too strong, one can hope to get global results. As previously, the situation where the underlying linear equation is reversible and the situation when it is not have to be distinguished. We briefly describe this in two different examples.

5.1.1 Semilinear wave equation

Let us discuss first an example with the wave equation. This is borrowed from the work of Zuazua [57, 58], where the equation considered is

$$\begin{cases} v_{tt} - v_{xx} + f(v) = u\mathbf{1}_\omega \text{ in } [0,1], \\ v_{|x=0} = 0, \ v_{|x=1} = 0, \end{cases}$$

where $\omega := (l_1, l_2)$ is the control zone and the nonlinearity $f \in C^1(\mathbb{R};\mathbb{R})$ is at most linear at infinity (see however the remark below) in the sense that for some $C > 0$,

$$|f(x)| \leq C(1 + |x|) \text{ on } \mathbb{R}. \tag{17}$$

The global exact controllability in $H^1_0 \times L^2$ (recall that the state of the system is (v, v_t)) by means of a control in $L^2((0,T)\times\omega)$ is proved by using the following linearization technique. In [58], it is proven that the following linearized equation:

$$\begin{cases} v_{tt} - v_{xx} + a(x)v = u\mathbf{1}_\omega \text{ in } [0,1], \\ v_{|x=0} = 0, v_{|x=1} = 0. \end{cases}$$

is controllable in $H^1_0(0,1) \times L^2(0,1)$ through $u(t) \in L^2((0,T)\times\omega)$, for times $T > 2\max(l_1, 1-l_2)$, for any $a \in L^\infty((0,T)\times(0,1))$. The corresponding observability inequality is

$$\|(\psi_T, \psi'_T)\|_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C_{obs}\|\psi\mathbf{1}_\omega\|_{L^2((0,T)\times\omega)}.$$

Moreover, the observability constant that one can obtain can be bounded in the following way:

$$C_{obs} \leq \alpha(T, \|a\|_\infty), \tag{18}$$

where α in non-decreasing in the second variable. One would like to describe a fixed-point scheme as follows. Given v, one considers the linearized problem around v, solves this problem and deduces a solution $\hat{v}$ of the problem of controllability from (v_0, v'_0) to (v_1, v'_1) in time T. More precisely, the scheme is constructed in the following way. We write

$$f(x) = f(0) + xg(x),$$

where g is continuous and bounded. The idea is to associate to any $v \in L^\infty((0,T)\times(0,1))$ a solution of the linear control problem

$$\begin{cases} \hat{v}_{tt} - \hat{v}_{xx} + \hat{v}g(v) = -f(0) + u\mathbf{1}_\omega \text{ in } [0,1], \\ \hat{v}_{|x=0} = 0, \ \hat{v}_{|x=1} = 0. \end{cases}$$

(Note that the "drift" term $f(0)$ on the right hand side can be integrated in the final state —just consider solution of the above system with $u = 0$ and $\hat{v}_{|t=0} = 0$ and withdraw it—, or integrated in the formulation (5).)

Here an issue is that, as we recalled earlier, a controllability problem has almost never a unique solution. The method in [58] consists in selecting a particular control, which is the one of smallest L^2-norm. Taking the fact that g is bounded and (18) into account, this particular control satisfies

$$\|u\|_{L^2(0,T)} \leq C(\|v_0\|_{H^1_0} + \|v_0'\|_{L^2} + \|v_1\|_{H^1_0} + \|v_1'\|_{L^2} + |f(0)|),$$

for some $C > 0$ independent of v.

Using the above information, one can deduce estimates for $\hat{v}$ in $C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_0)$ independently of v, and then show by Schauder's fixed point theorem that the above process has a fixed point, which shows a global controllability result for this semilinear wave equation.

Remark 1. *As a matter of fact, [58] proves the global exact controllability for f satisfying the weaker assumption*

$$\lim_{|x|\to+\infty} \frac{|f(x)|}{(1+|x|)\log^2(|x|)} = 0.$$

This is optimal since it is also proven in [58] that if

$$\liminf_{|x|\to+\infty} \frac{|f(x)|}{(1+|x|)\log^p(|x|)} > 0$$

for some $p > 2$ (and $\omega \neq (0,1)$) then the system is not globally controllable due to blow-up phenomena. To get this results one has to use Leray-Schauder's degree theory instead of Schauder's fixed point theorem.

For analogous conditions on the semilinear heat equation, see Fernández-Cara and Zuazua [27].

5.1.2 Burgers equation

Now let us discuss a parabolic example, namely the local controllability to trajectories of the viscous Burgers equation:

$$\begin{cases} v_t + (v^2)_x - v_{xx} = 0 \text{ in } (0,T) \times (0,1), \\ v_{|x=0} = u_0(t) \text{ and } v_{|x=1} = u_1(t), \end{cases} \tag{19}$$

controlled on the boundary via u_0 and u_1. This is taken from Fursikov and Imanuvilov [30].

Now the linear result concerns the zero-controllability via boundary controls of the system

$$\begin{cases} v_t + (zv)_x - v_{xx} = 0 \text{ in } (0,T) \times (0,1), \\ v_{|x=0} = u_0(t) \text{ and } v_{|x=1} = u_1(t). \end{cases} \tag{20}$$

Consider X the Hilbert space composed of functions z in $L^2(0,T; H^2(0,1))$ such that $z_t \in L^2(0,1; L^2(0,1))$. In [30] it is proved that given $v_0 \in H^1(0,1)$ and $T > 0$, one can construct a map which to any z in X, associates $v \in X$ such that v is a solution of (20) which drives v_0 to 0 during time interval $[0,T]$, and moreover this map is compact from X to X. As in the previous situation, the particular control (u_0, u_1) has to be singled out in order for the above mapping to be single-valued (and compact). But here, the criterion is not quite the optimality in L^2-norm of the control. The idea is the following: first, one transforms the controllability problem for (20) from y_0 to 0 into a problem of "driving" 0 to 0 for a problem with right-hand side:

$$\begin{cases} w_t + (zw)_x - w_{xx} = f_0 \text{ in } (0,T) \times (0,1), \\ w_{|x=0} = u_0(t) \text{ and } w_{|x=1} = u_1(t), \end{cases} \tag{21}$$

for some f_0 supported in $(T/3, 2T/3) \times (0,1)$. For this, one introduces $\chi \in C^\infty([0,T]; \mathbb{R})$ such that $\chi = 1$ during $[0,T/3]$ and $\chi = 0$ during $[2T/3,T]$, and $\hat{v}$ the solution of (20) starting from v_0 with $u_0 = u_1 = 0$, and considers $w := v - \chi\hat{v}$.

Now the operator mentionned above is the one which to z associates the solution of the controllability problem which minimizes the L^2-norm of w among all the solutions of this controllability problem. The optimality criterion yields a certain form for the solution. That the corresponding control exists (and is unique) relies on a Carleman estimate, see [30] (moreover this allows estimates on the size of w). As a matter of fact, to get the compactness of the operator, one extends the domain, solves the above problem in this extended domain, and then uses an interior parabolic regularity result to have bounds in smaller spaces, we refer to [30] for more details.

Once the operator is obtained, the local controllability to trajectories is obtained as follows. One considers $\overline{v}$ a trajectory of the system (19), belonging to X. Withdrawing (19) for $\overline{v}$ to (19) for the unknown, we see that the problem to solve through boundary controls is

$$\begin{cases} y_t - y_{xx} + [(2\overline{v} + y)y]_x = f_0 \text{ in } (0,T) \times (0,1), \\ y_{|x=0} = v_0 - \overline{v}(0) \text{ and } y_{|t=T} = 0. \end{cases}$$

Now consider $v_0 \in H^1(0,1)$ such that

$$\|v_0 - \overline{v}(0)\|_{H^1(0,1)} < r,$$

for $r > 0$ to be chosen. To any $y \in X$, one associates the solution of the controllability problem (21) constructed above, driving $v_0 - \hat{v}(0)$ to 0, for $z := (2\hat{v} + y)$. The estimates on the solution of the control problem allow to establish that the unit ball of X is stable by this process provided r is small enough. The compactness of the process is already proved, so Schauder's theorem allows to conclude.

Note that it is also proved in [30] that the global approximate controllability does not hold.

5.1.3 Some other examples

Let us also mention two other approaches which may be useful in this type of situations.

The first one is the use of Kakutani-Tikhonov fixed-point theorem for multivalued maps (cf. for instance Smart [55]), see in particular Henry [33, 34] and Fabre, Puel and Zuazua [22]. Such technique is particularly useful, because it avoids the selection process of a particular control. One associates to v the set $T(v)$ of all $\hat{v}$ solving the controllability problem for the equation linearized around v, with all possible controls (in a suitable class). Then under appropriate conditions, one can find a fixed point in the sense that $v \in T(v)$.

Another approach that is very promising is the use of a Nash-Moser process, see in particular the work [8] by Beauchard. In this paper the controllability of a Schrödinger equation via a bilinear control is considered. In that case, one can solve some particular linearized equation (as a matter of fact, the return method described in the next paragraph is used), but with a loss of derivative; as a consequence the approaches described above fail, but the use of Nash-Moser's theorem allows to get a result. Note finally that in certain other functional settings, the controllability of this system fails, as shown by using a general result on bilinear control by Ball, Marsden and Slemrod [4].

5.2 The return method

It occurs that in some situations, the linearized equation is not systematically controllable, and one cannot hope by applying directly the above process to get even local exact controllability. The return method has been introduced by Coron to deal with such situations (see in particular [C]). As a matter of fact, this method can be useful even when the linearized equation is controllable. The principle of the method is the following: find a particular trajectory $\overline{y}$ of the nonlinear system, starting at some base point (typically 0) and returning to it, such that the linearized equation around this is controllable. In that case, one can hope to find a solution of the nonlinear local controllability problem close to $\overline{y}$.

A typical situation of this is the two-dimensional Euler equation for incompressible inviscid fluids (see Coron [13]), which reads

$$\begin{cases} \partial_t y + (y.\nabla)y = -\nabla p \text{ in } \Omega, \\ \operatorname{div} y = 0 \text{ in } \Omega, \\ y.n = 0 \text{ on } \partial\Omega \setminus \Sigma, \end{cases} \tag{22}$$

where the unknown is the velocity field $y : \Omega \to \mathbb{R}^2$ (the pressure p can be eliminated from the equation), Ω is a regular bounded domain (simply connected to simplify), n is the unit outward normal on the boundary, and $\Sigma \subset \partial\Omega$ is the control zone. On Σ, the natural control which can be assigned is the normal velocity $y.n$ and the vorticity $\omega := \operatorname{curl} y := \partial_1 y^2 - \partial_2 y^1$ at "entering" points, that is points where $y.n < 0$.

Let us discuss this example in an informal way. As noticed by J.-L. Lions, the linearized equation around the null state

$$\begin{cases} \partial_t y = -\nabla p \text{ in } \Omega, \\ \operatorname{div} y = 0 \text{ in } \Omega, \\ y.n = 0 \text{ on } \partial\Omega \setminus \Sigma, \end{cases}$$

is trivially not controllable (even approximately). Now the main goal is to find the trajectory $\overline{y}$ such that the linearized equation near $\overline{y}$ is controllable. It will be easier to work with the vorticity formulation

$$\begin{cases} \partial_t \omega + (y.\nabla)\omega = 0 \text{ in } \Omega, \\ \operatorname{curl} y = \omega, \ \operatorname{div} y = 0. \end{cases} \tag{23}$$

In fact, one can show that assigning $y(T)$ is equivalent to assign both $\omega(T)$ in Ω and $y(T).n$ on Σ, and since this latter is a part the control, it is sufficient to know how to assign the vorticity of the final state.

We can linearize (23) in the following way: to y one associates $\hat{y}$ through

$$\begin{cases} \partial_t \omega + (y.\nabla)\omega = 0 \text{ in } \Omega, \\ \operatorname{curl} \hat{y} = \omega, \ \operatorname{div} \hat{y} = 0. \end{cases} \tag{24}$$

Considering equation (24), we see that if the flow of y is such that any point in $\overline{\Omega}$ at time T "comes from" the control zone Σ, then one can assign easily ω through a method of characteristics. Hence one has to find a solution $\overline{y}$ of the system, starting and ending at 0, and such that in its flow, all points in $\overline{\Omega}$ at time T, come from Σ at some stage between times 0 and T. Then a simple Gronwall-type argument shows that this property holds for y in a neighborhood of $\overline{y}$, hence equation (24) is controllable in a neighborhood of $\overline{y}$. Then a fixed-point scheme allows to prove a controllability result locally around 0.

But the Euler equation has some time-scale invariance:

$$y(t,x) \text{ is a solution on } [0,T] \Rightarrow y^{\lambda}(t,x) := \lambda^{-1} y(\lambda^{-1}t, x) \text{ is a solution on } [0, \lambda T].$$

Hence given y_0 and y_1, one can solve the problem of driving λy_0 to λy_1 for λ small enough. Changing the variables, one see that it is possible to drive y_0 to y_1 in time λT, that is, in *smaller* times. Hence one deduces a global controllability result from the above local result.

As a consequence, the central part of the proof is to find the function $\overline{y}$. This is done by considering a special type of solutions of the Euler equation, namely the potential solutions: any $\overline{y} := \nabla\theta(t,x)$ with θ regular satisfying

$$\Delta_x \theta(t,x) = 0 \text{ in } \Omega, \ \forall t \in [0,T], \partial_n \theta = 0 \text{ on } \partial\Omega, \ \forall t \in [0,T],$$

satisfies (22). In [13] is is proven that there exists some θ satisfying the above equation and whose flow makes all points at time T come from Σ. This concludes the argument.

This method has been used in many various situations; see [C] and references therein. Let us underline that this method can be of great interest even in the cases where the linearized equation is actually controllable. An important example obtained by Coron concerns the Navier-Stokes equation and is given in [14] (see also Coron and Fursikov [15]): here the return method is used to prove some global approximate result, while the linearized equation is actually controllable but yields in general local exact controllability result (see in particular Fursikov and Imanuvilov [D], Imanuvilov [36] and Fernández-Cara, Guerrero, Imanuvilov and Puel [25]).

5.3 Some other methods

Let us finally briefly mention that linearizing the equation (whether using the standard approach or the return method) is not systematically the only approach to the controllability of a nonlinear system. Sometimes, one can "work at the nonlinear level". An important example is the control of one-dimensional hyperbolic systems:

$$v_t + A(v)v_x = F(v), \ v : [0,T] \times [0,1] \to \mathbb{R}^n, \tag{25}$$

via the boundary controls. In (25), A satisfies the hyperbolicity property that it possesses at every point n real distinct eigenvalues; these are moreover supposed to be strictly separated from 0.

In the case of regular C^1 solutions, this was approached by a method of characteristics to give general local results, see in particular the works by Cirinà [12] and Li and Rao [40]. Interestingly enough, the linear tool of duality between observability and controllability has some counterpart in this particular nonlinear setting, see Li [45]. In the context of weak entropy solutions, some results for particular systems have been obtained via the ad hoc method of front-tracking, see in particular Ancona, Bressan and Coclite [2], the author [32] and references therein.

Other nonlinear tools can be found in Coron's book [C]. Let us mention two of them. The first one is power series expansion. It consists in considering, instead of the linearization of the system, the development to higher order of the nonlinearity. In such a way, one can hope to attain the directions which are unreachable for the linearized system. This has for instance applications for the Korteweg-de-Vries equation, see Coron and Crépeau [16] and the earlier work by Rosier [50]. The other one is quasi-static deformations. The general idea is to find an (explicit) "almost trajectory" $(\overline{y}(\varepsilon t), \overline{u}(\varepsilon t))$ during $[0, T/\varepsilon]$ of the control system $\dot{y} = f(y, u)$, in the sense that

$$\frac{d}{dt}\big[\overline{y}(\varepsilon t)\big] - f(\overline{y}(\varepsilon t), \overline{u}(\varepsilon t))$$

is of order ε (due to the "slow" motion). Typically, the trajectory $(\overline{y}, \overline{u})$ is composed of equilibrium states (that is $f(\overline{y}(\cdot), \overline{u}(\cdot)) = 0$). In such a way, one can hope to connect $\overline{y}(0)$ to a state close to $\overline{y}(T)$, and then to exactly $\overline{y}(T)$ via a local result. This was used for instance by Coron and Trélat in the context of a semilinear heat equation, see [17].

Finally, let us cite a recent approach by Agrachev and Sarychev [1] (see also Shirikyan [54]), which use a generalization of the "Lie bracket" approach (a standard approach from finite-dimensional nonlinear systems), to get some global approximate results on the Navier-Stokes equation with a finite-dimensional (low modes) affine control.

6 Some other problems

As we wrote earlier, many aspects of the theory were not referred to in this brief note. Let us cite some of them. Concerning the connection between the problem of controllability and the problem of stabilization, we refer to Lions [41], Russell [I, 51, 52] and Lasiecka and Triggiani [G].

A very important problem which has raised a high interest recently is the problem of numerics and discretization of distributed systems (see in particular Zuazua [60, 61] and references therein). The main difficulty here comes from the fact that the operations of discretizing the equation and controlling it do not commute.

Also important questions considered in particular the second volume of Lions's book [H] are the problem of singular perturbations, homogenization, this domains in the context controllability problems. Here the questions are the following: considering a "perturbed" system, for which we have some controllability result, how does the solution of the controllability problem (for instance associated to the L^2-optimal control) behave as the system converges to its limit? This kind of question is the source of numerous new problems.

Another subject is the controllability of equations with some stochastic terms (see for instance Barbu, Răşcanu and Tessitore [6]). One can also consider systems with memory (see again [H] or Barbu and Iannelli [5]). Finally, let us mention that many partial differential equations widely studied from the point of view of Cauchy theory, still have not been studied from the point of view of controllability.

The reader looking for more discussion on the subject can consider the references below.

7 Future directions

There are many future challenging problems for the whole control theory. The controllability is one of the possible approaches to try to construct strategies for managing complex systems. On the road towards systems with increasing complexity, many additional difficulties have to be considered in the design of the control law: one should expect from the control to take into account the possible errors of modelization, of measurement of the state, of the control device, etc. All of this should be included in the model to expect some robustness of the control, and to make it closer to real world. Of course, numerics should play an important role in the direction of applications. Moreover one should expect more and more complex systems, such as for instance environmental or biological systems (how does are regulation mechanisms designed in a natural organism?), to be approached from this point of view. We refer for instance to [28, 63] for some discussions on some perspectives of the theory.

References

[A] S. A. Avdonin, S. A. Ivanov, *Families of exponentials. The method of moments in controllability problems for distributed parameter systems*, Cambridge Univ. Press, 1995.

[B] A. Bensoussan, G. Da Prato, M. C. Delfour, S. K. Mitter, *Representation and control of infinite-dimensional systems.* Vol. I & II, Systems & Control: Foundations & Applications, Birkhäuser Boston Inc., Boston, MA, 1993.

[C] J.-M. Coron, *Control and Nonlinearity*, Mathematical Surveys and Monographs 136, American Mathematical Society, Providence, RI, 2007.

[D] A. Fursikov, O. Yu. Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series, vol. 34, Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

[E] V. Komornik, *Exact controllability and stabilization, The multiplier method.* RAM: Research in Applied Mathematics, Masson, Paris, 1994.

[F] V. Komornik, P. Loreti, *Fourier series in control theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2005.

[G] I. Lasiecka, R. Triggiani, *Control theory for partial differential equations: continuous and approximation theories. I, Abstract parabolic systems & II, Abstract hyperbolic-like systems over a finite time horizon.* Encyclopedia of Mathematics and its Applications, vols. 74 & 75, Cambridge University Press, Cambridge, 2000.

[H] J.-L. Lions, *Contrôlabilité exacte, stabilisation et perturbations de systèmes distribués*, Tomes 1 & 2. Masson, RMA 8 & 9, Paris 1988.

[I] D. L. Russell, 1978, Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions, SIAM Review 20 (1978), pp. 639–739.

[J] E. Zuazua, *Controllability and observability of partial differential equations: some results and open problems*, Handbook of differential equations: evolutionary differential equations Vol. 3 (C. Dafermos and E. Feiresl, eds.), Elsevier/North-Holland, Amsterdam, 2006.

References

[1] A. Agrachev, A. Sarychev, Navier-Stokes equations: control lability by means of low modes forcing, J. Math. Fluid Mech. 7 (2005), no. 1, pp. 108–152.

[2] F. Ancona, A. Bressan, G.M. Coclite, Some results on the boundary control of systems of conservation laws. *Hyperbolic problems: theory, numerics, applications*, pp. 255–264, Springer, Berlin, 2003.

[3] F. Ancona, A. Marson, On the attainable set for scalar nonlinear conservation laws with boundary control. SIAM J. Control Optim. 36 (1998), no. 1, pp. 290–312.

[4] J.M. Ball, J.E. Marsden, M. Slemrod, Controllability for distributed bilinear systems, SIAM J. Control. Optim. 20 (1982), pp. 575–597.

[5] V. Barbu, M. Iannelli, Controllability of the heat equation with memory. Differential Integral Equations 13 (2000), no. 10-12, pp. 1393–1412.

[6] V. Barbu, A. Răşcanu, G. Tessitore, Null controllability of stochastic heat equations with a multiplicative noise, Appl. Math. Optim. 47 (2003), no. 2, pp. 97–120.

[7] C. Barbos, G. Lebeau, J. Rauch, Sharp sufficient conditions for the observation, control and stabilisation of waves from the boundary, SIAM J. Control Optim. 305 (1992) pp. 1024–1065.

[8] K. Beauchard, Local controllability of a 1D Schrödinger equation, J. Math. Pures Appl. 84 (2005), pp. 851–956.

[9] H. Brezis, Analyse fonctionnelle, Théorie et applications; Collection Mathématiques Appliquées pour la Maîtrise, Masson, Paris, 1983.

[10] N. Burq, Contrôle de l'équation des ondes dans des ouverts peu réguliers, Asymptot. Anal. 14 (1997), pp. 157–191.

[11] N. Burq, P. Gérard, Condition nécessaire et suffisante pour la contrôlabilité exacte des ondes, C. R. Acad. Sci. Paris Sér. I Math. 325 (1997), no. 7, pp. 749–752.

[12] M. Cirinà, Boundary controllability of nonlinear hyperbolic systems, SIAM J. Control 7 (1969), pp. 198–212.

[13] J.-M. Coron, On the controllability of 2-D incompressible perfect fluids, J. Math. Pures Appl., 75, 1996, pp. 155–188.

[14] J.-M. Coron, On the controllability of the 2-D incompressible Navier-Stokes equations with the Navier slip boundary conditions, ESAIM Control Optim. Calc. Var., 1 (1996), pp. 35–75.

[15] J.-M. Coron, A. V. Fursikov, Global exact controllability of the 2-D Navier-Stokes equations on a manifold without boundary, Russian J. Math. Phys., 4 (1996), pp. 1–19.

[16] J.-M. Coron, E. Crépeau, Exact boundary controllability of a nonlinear KdV equation with critical lengths, J. Eur. Math. Soc. (JEMS) 6 (2004), no. 3, pp. 367–398.

[17] J.-M. Coron, E. Trélat, Global steady-state controllability of one-dimensional semilinear heat equations, SIAM J. Control Optim. 43 (2004), no. 2, pp. 549–569.

[18] R. Courant, D. Hilbert, *Methods of mathematical physics* Vol. II, Partial differential equations, Wiley Classics Library, John Wiley & Sons Inc., New York, 1989.

[19] R. Dàger, E. Zuazua, *Wave propagation, observation and control in* 1*-d flexible multi-structures.* Mathématiques & Applications, 50. Springer-Verlag, Berlin, 2006.

[20] S. Dolecki, D. L. Russell, A general theory of observation and control, SIAM J. Control Optimization 15 (1977), no. 2, pp. 185–220.

[21] R. G. Douglas, On majorization, factorization, and range inclusion of operators on Hilbert space, Proc. Amer. Math. Soc. 17 (1966), pp. 413–415.

[22] C. Fabre, J.-P. Puel, and E. Zuazua, Approximate controllability of the semilinear heat equation, Proc. Roy. Soc. Edinburgh Sect. A 125 (1995), no. 1, pp. 31–61.

[23] H. O. Fattorini, D. L. Russell, Exact controllability theorems for linear parabolic equation in one space dimension, Arch. Rat. Mech. Anal., 43 (1971), pp. 272–292.

[24] E. Fernández-Cara, S. Guerrero, Global Carleman inequalities for parabolic systems and applications to controllability, SIAM Rev.

[25] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, J.-P. Puel, Local exact controllability to the trajectories of the Navier-Stokes equations, J. Math. Pures Appl., 83 (2004), pp. 1501–1542.

[26] E. Fernández-Cara, E. Zuazua, The cost of approximate controllability for heat equations: The linear case, Adv. Differential Equations, 5 (2000), pp. 465–514.

[27] E. Fernández-Cara, E. Zuazua, Null and approximate controllability for weakly blowing up semilinear heat equations. Ann. Inst. H. Poincaré Anal. Non Linaire 17 (2000), no. 5, pp. 583–616.

[28] E. Fernández-Cara, E. Zuazua, *Control Theory: History, mathematical achievements and perspectives.* Boletin SEMA, 26, 2003, pp. 79–140.

[29] M. Fliess, J. Lévine, P. Martin, P. Rouchon, Flatness and defect of non-linear systems: introductory theory and examples, Internat. J. Control 61 (1995), no. 6, pp. 1327–1361.

[30] A. Fursikov, O. Yu. Imanuvilov, On controllability of certain systems simulating a fluid flow, *Flow control* (Minneapolis, MN, 1992), IMA Vol. Math. Appl., 68, Springer, New York, 1995, pp. 149–184.

[31] O. Glass, On the controllability of the Vlasov-Poisson system. J. Differential Equations 195 (2003), no. 2, pp. 332–379.

[32] O. Glass, On the controllability of the 1-D isentropic Euler equation, J. Eur. Math. Soc. 9 (2007), pp. 427–486.

[33] J. Henry, Controllability of some non linear parabolic equations, in *Distributed Parameter Systems: Modelling and Identification*, Proceedings of the IFIP working conference, Rome, Italy, June 21-24, 1976, (A. Ruberti, ed.), Lecture Notes in Control and Information Sciences 1, Springer Verlag, Berlin, 1978.

[34] J. Henry, Étude de la contrôlabilité de certaines équations paraboliques non linéaires, Thèse, Paris VI, 1977.

[35] L. Hörmander, The analysis of linear partial differential operators, I, Grundlehren der mathematischen Wissenschaften 256, Springer-Verlag, Berlin, 1983.

[36] O. Yu. Imanuvilov, Remarks on exact controllability for the Navier-Stokes equations, ESAIM Control Optim. Calc. Var., 6 (2001), pp. 39–72.

[37] J. E. Lagnese, ; G. Leugering, E. J. P. G. Schmidt, *Modeling, analysis and control of dynamic elastic multi-link structures.* Systems & Control: Foundations & Applications. Birkhuser Boston, Inc., Boston, MA, 1994.

[38] B. Laroche, P. Martin, P. Rouchon, Motion planning for the heat equation, Internat. J. Robust Nonlinear Control 10 (2000), no. 8, pp. 629643.

[39] G. Lebeau, L. Robbiano, Contrôle exact de l'équation de la chaleur, Comm. P.D.E., 20 (1995), pp. 335–356.

[40] Li T.-T., Rao B.-P., Exact boundary controllability for quasi-linear hyperbolic systems, SIAM J. Control Optim. 41 (2003), no. 6, pp. 1748–1755.

[41] J.-L. Lions, Exact controllability, stabilizability and perturbations for distributed systems, SIAM Review, 30 (1988), pp. 1–68.

[42] J.-L. Lions, Are there connections between turbulence and controllability?, *Analysis and Optimization of Systems*, A. Bensoussan and J.-L. Lions, eds., Lecture Notes Control and Inform. Sci. 144, Springer-Verlag, Berlin, 1990.

[43] J.-L. Lions, Remarks on approximate controllability, J. Analyse Math. 59 (1992), pp. 103–116.

[44] C. Le Bris, Control theory applied to quantum chemistry: some tracks, Contrôle des systèmes gouvernés par des équations aux dérivées partielles (Nancy, 1999), ESAIM Proc., vol. 8 (2000), pp. 77–94.

[45] T.-T. Li, Exact boundary observability for quasilinear hyperbolic systems, ESAIM Control Optim. Calc. Var., 2007, to appear.

[46] J.-L. Lions, *Optimal control of systems governed by partial differential equations*, Grundlehren der mathematischen Wissenschaften 170, Springer-Verlag, New-York Berlin, 1971.

[47] A. López, E. Zuazua, Uniform null controllability for the one dimensional heat equation with rapidly oscillating periodic density, Annales IHP Analyse non linéaire 19 (5) (2002), pp. 543–580.

[48] L. Miller On exponential observability estimates for the heat semigroup with explicit rates. Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 17 (2006), no. 4, pp. 351–366.

[49] A. Osses, A rotated multiplier applied to the controllability of waves, elasticity, and tangential Stokes control. SIAM J. Control Optim. 40 (2001), no. 3, pp. 777–800.

[50] L. Rosier, Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. ESAIM Control Optim. Calc. Var. 2 (1997), pp. 33–55.

[51] D. L. Russell, A unified boundary controllability theory for hyperbolic and parabolic partial differential equations, Int. Math. Research Notices 52 (1973), pp. 189–221.

[52] D. L. Russell, Exact boundary value controllability theorems for wave and heat processes in star-complemented regions, *Differential games and control theory* (Proc. NSFCBMS Regional Res. Conf., Univ. Rhode Island, Kingston, R.I., 1973), Lecture Notes in Pure Appl. Math. 10, Dekker, New York, 1974, pp. 291–319.

[53] D. L. Russell, On boundary-value controllability of linear symmetric hyperbolic systems. Mathematical Theory of Control (Proc. Conf., Los Angeles, Calif., 1967) pp. 312–321.

[54] A. Shirikyan, Approximate control lability of three-dimensional navier-stokes equations, Comm. Math. Phys. 266 (2006), no. 1, pp. 123–151.

[55] D. R. Smart, *Fixed point theorems*, Cambridge Tracts in Mathematics, No. 66. Cambridge University Press, London-New York, 1974.

[56] X. Zhang, Explicit observability inequalities for the wave equation with lower order terms by means of Carleman inequalities, SIAM J. Cont. Optim. 39 (2001), pp. 812–834.

[57] E. Zuazua, Exact controllability for the semilinear wave equation. J. Math. Pures Appl. 69 (1990), no. 1, pp. 1–31.

[58] E. Zuazua, Exact controllability for semilinear wave equations in one space dimension, Ann. Inst. H. Poincaré Anal. Non Linéaire 10 (1993), no. 1, pp. 109-129.

[59] E. Zuazua, Some problems and results on the controllability of Partial Differential Equations, Proceedings of the Second European Conference of Mathematics, Budapest, July 1996, Progress in Mathematics, 169, 1998, Birkhäuser Verlag Basel, pp. 276–311.

[60] E. Zuazua, Controllability of Partial Differential Equations and its Semi-Discrete Approximations, Discrete and Continuous Dynamical Systems, 8 no. 2 (2002), pp. 469–513.

[61] E. Zuazua, Propagation, observation, and control of waves approximated by finite difference methods, SIAM Rev. 47 (2005), no. 2, pp. 197–243.

[62] *Quantum control: mathematical and numerical challenges.* Papers from the CRM Workshop held at the Université de Montréal, Montréal, QC, October 6–11, 2002. Edited by André D. Bandrauk, Michel C. Delfour and Claude Le Bris. CRM Proceedings & Lecture Notes, 33. American Mathematical Society, Providence, RI, 2003.

[63] *Control in an information rich world.* Report of the Panel on Future Directions in Control, Dynamics, and Systems. Papers from the meeting held in College Park, MD, July 16–17, 2000. Edited by Richard M. Murray. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2003.