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# On the controllability of the Vlasov–Poisson system

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**Abstract**

In this paper, we study the controllability of the Vlasov–Poisson system in a periodic domain, by means of an interior control located in an spatial subdomain.

The first result proves the local exact zero controllability in the two-dimensional torus between two small acceptable distribution functions, with an arbitrary control zone.

A second result establishes the global exact controllability in arbitrary dimension, provided the control zone satisfies the condition that it contains a hyperplane of the torus.


*MSC:* 93B05; 35B37; 82A45; 82D10

*Keywords:* Controllability; Vlasov–Poisson system

## 1. Introduction

In this paper, we study the exact controllability problem for the Vlasov–Poisson equation in the $n$-dimensional torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. The Vlasov–Poisson system reads, for two functions $f = f(t, x, \xi)$ and $\phi = \phi(t, x)$:

$$\partial_t f + \xi \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_\xi f = G(t, x, \xi), \quad \text{for } (t, x, \xi) \in [0, T] \times \mathbb{T}^n \times \mathbb{R}^n, \tag{1.1}$$

$$\begin{cases} -\Delta\phi(t,x) = \int_{\mathbb{T}^n \times \mathbb{R}^n} f(0, x, \xi)\, dx\, d\xi - \int_{\mathbb{R}^n} f(t, x, \xi)\, d\xi = \rho_0 - \int_{\mathbb{R}^n} f(t, x, \xi)\, d\xi, \\ \int_{\mathbb{T}^n} \phi(t, x)\, dx = 0. \end{cases} \tag{1.2}$$

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Here $n$ is the dimension; the variables are the time $t$, the position $x$ and the velocity $\xi$. The unknown functions are the distribution of particles $f$ and the potential $\phi$. The symbols $\nabla_x$ and $\nabla_\xi$ stand, respectively, for the gradient with respect to $x$ and to $\xi$. The term $\rho_0$ stands for a constant neutralizing density, equal to $\int_{\mathbb{T}^n \times \mathbb{R}^n} f_{|t=0}$. Finally, $G$ is a source term, used as a control in our problem.

The problem of exact controllability is the following: consider an arbitrary non-empty regular open set $\omega$ in $\mathbb{T}^n$, and fix $T>0$. Now consider two "reasonable" distributions $f_0$ and $f_1$. Is it possible to *steer* $f_0$ to $f_1$ when following (1.1)–(1.2) between times 0 and $T$, by choosing a relevant function $G = G(t,x,\xi)$ whose support according to the variable $x$ is localized in $\omega$? Precisely, does there exist a solution of (1.1)–(1.2), satisfying

$$f(0,x,\xi) = f_0(x,\xi) \quad \text{in } \mathbb{T}^n \times \mathbb{R}^n, \tag{1.3}$$

$$f(T,x,\xi) = f_1(x,\xi) \quad \text{in } \mathbb{T}^n \times \mathbb{R}^n, \tag{1.4}$$

and

$$G = 0 \quad \text{in } [0,T] \times [\mathbb{T}^n \backslash \omega] \times \mathbb{R}^n. \tag{1.5}$$

We answer this question in the affirmative in two cases. The first case concerns $n=2$ and requires $f_0$ and $f_1$ to be small enough. The second one is valid in any dimension, and does not require $f_0$ and $f_1$ to be small, but assumes that $\omega$ contains the image of a hyperplane of $\mathbb{R}^n$ by the canonical surjection (which we call a hyperplane of the torus). Precisely, we show the following results:

**Theorem 1.** *Set $n=2$. Consider $\gamma>2$ and $\kappa, \kappa' \geqslant 0$. Let $f_0$ and $f_1$ be two functions in $C^1(\mathbb{T}^2 \times \mathbb{R}^2) \cap W^{1,\infty}(\mathbb{T}^2 \times \mathbb{R}^2)$, satisfying the condition that for any $(x,\xi) \in \mathbb{T}^2 \times \mathbb{R}^2$ and $i \in \{0,1\}$,*

$$\begin{cases} |f_i(x,\xi)| \leqslant \kappa(1+|\xi|)^{-\gamma-1}, \\ |\nabla_x f_i| + |\nabla_\xi f_i| \leqslant \kappa'(1+|\xi|)^{-\gamma}, \end{cases} \tag{1.6}$$

*and*

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} f_0 = \int_{\mathbb{T}^n \times \mathbb{R}^n} f_1. \tag{1.7}$$

*Assume also that $\kappa$ and $\kappa'$ are small enough* (*in relation to $\omega$ and $T$*). *Then there exists $G \in C^0([0,T] \times \mathbb{T}^2 \times \mathbb{R}^2)$ satisfying* (1.5), *such that the solution of* (1.1)–(1.2) *and* (1.3) *exists*, *is unique*, *and satisfies* (1.4).

**Theorem 2.** *Set $n \geqslant 2$. Consider $\gamma>n$ and $\kappa, \kappa' \geqslant 0$. Suppose that the regular open set $\omega$ contains the image of a hyperplane in $\mathbb{R}^n$ by the canonical surjection. Let $f_0$ and $f_1$*

*be two functions in* $C^1(\mathbb{T}^n \times \mathbb{R}^n)$, *satisfying the conditions*

$$\begin{cases} |f_i(x,\xi)| \leqslant \kappa(1+|\xi|)^{-\gamma-2}, \\ |\nabla_x f_i| + |\nabla_\xi f_i| \leqslant \kappa'(1+|\xi|)^{-\gamma}, \end{cases} \tag{1.8}$$

*and* (1.7). *Then there exists* $G \in C^0([0,T] \times \mathbb{T}^n \times \mathbb{R}^n)$ *satisfying* (1.5), *such that the solution of* (1.1)–(1.2) *and* (1.3) *exists*, *is unique*, *and satisfies* (1.4).

**Remark 1.** Note that, because of the necessary global neutrality in the torus, one cannot ask for $G$ to be non-negative. On the contrary, it has to satisfy that for any $t \in [0,T]$,

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} G(t,x,\xi)\, d\xi = 0.$$

**Remark 2.** In general, the function $f_0$ and $f_1$ are non-negative. We do not need it for Theorems 1 and 2. However, the solution $f$ is also usually asked to be non-negative, which Theorems 1 and 2 do not ensure. But as will be clear during the proofs, one has

$$f_0, f_1 \geqslant 0 \;\; \Rightarrow \;\; \forall t \in [0,T],\; f(t,x,\xi) \geqslant 0 \quad \text{in } (\mathbb{T}^n \backslash \tilde{\omega}) \times \mathbb{R}^n,$$

for some open set $\tilde{\omega}$ satisfying $\overline{\tilde{\omega}} \subset \omega$, the non-negativeness being probably not satisfied inside $\tilde{\omega}$.

To be more consistent with the model, one can replace the source by the sum of two sources, corresponding to different species of particles (one of them never leaving $\omega$), of opposite charge. Consider indeed $f^+$, supported in $(0,T) \times \omega \times \mathbb{R}^n$, non-negative, with the same type of regularity as $f$, and such that $f + f^+ \geqslant 0$. Then $f^{\#} := f + f^+$ satisfies (1.1) with source

$$G^{\#} := G + (\partial_t f^+ + \xi \cdot \nabla_x f^+ + \nabla_x \phi \cdot \nabla_\xi f^+), \tag{1.9}$$

and $f^+$ satisfies (1.1) with source

$$G^+ := \partial_t f^+ + \xi \cdot \nabla_x f^+ + \nabla_x \phi \cdot \nabla_\xi f^+, \tag{1.10}$$

the corresponding potential being fixed by

$$-\Delta\phi(t,x) = \rho_0 - \int_{\mathbb{R}^n} f^{\#}(t,x,\xi)\, d\xi + \int_{\mathbb{R}^n} f^+(t,x,\xi)\, d\xi.$$

(One can even put a different mass for the new type of particle by putting a relevant multiplicative coefficient before $\nabla\phi \cdot \nabla_\xi f^+$ in (1.10).) In this setting, both $f^{\#}$ and $f^+$ are non-negative.

**Remark 3.** It is approximately equivalent to state that the control is given by the two following data:

- the value of the potential $\phi$ on $[0, T] \times \partial\omega$,
- the (non-negative) value of $f(t, x, \xi)$ at the points $(t, x, \xi)$ of $[0, T] \times \partial\omega \times \mathbb{R}^n$ where $\xi$ enters inside $\mathbb{T}^n \backslash \omega$, i.e. satisfies $\xi \cdot v(x) > 0$, if $v$ is the unit outward normal on $\partial\omega$.

**Remark 4.** In fact, if we do not require uniqueness for the solution, one can replace (1.6) or (1.8) by

$$\begin{cases} |f_i(x, \xi)| \leqslant \tilde{\kappa}(1 + |\xi|)^{-\gamma}, \\ |\nabla_x f_i| + |\nabla_\xi f_i| \leqslant \tilde{\kappa}', \end{cases} \tag{1.11}$$

in both cases. By the way, only the smallness of $(\tilde{\kappa}, \tilde{\kappa}')$ is required in Theorem 1.

The equivalent problem for the Euler system of incompressible inviscid fluids has been studied by Coron and the author (see [6,7,9,10]). The controllability of the Vlasov–Poisson equation shares certain properties with the Euler one. This is not so surprising, since the Vlasov–Poisson system has in some way a comparable structure with the Euler equation, and is even known to converge to it in a certain sense (see in particular [4]).

One of the major problems for the controllability of this system is that, as for the Euler equation, the linearized equation is not controllable in general. For instance, consider the linearized equation around the solution given by $f = \phi = 0$:

$$\partial_t f + \xi \cdot \nabla_x f = G. \tag{1.12}$$

This equation, which describes the free transport of particles, is not controllable unless $\omega = \mathbb{T}^n$ (take $f_0(x, 0) \neq f_1(x, 0)$ for some $x \in \mathbb{T}^n \backslash \omega$).

As for the Euler system, the main idea is to use the "return method" introduced by Coron (originally concerning a finite-dimensional stabilization problem, see [5]). This method has also been used by Coron to establish an approximate controllability result for the 2-D Navier–Stokes equations, see [8], and by Horsin to prove a controllability result for the Burgers equation, see [12]. The principle is to find a solution $(\bar{f}, \varphi)$ of the non-linear problem, which starts from 0 and goes back to 0 at time $T$, and around which the linearized equation is actually controllable. Then one looks for a solution of the non-linear problem "close" to $(\bar{f}, \varphi)$. In this paper, when writing "start from 0" or "go to 0", we will often refer to a configuration 0 *except in the control zone* $\omega$. Indeed, the distribution function $f(t, \cdot, \cdot)$ has a constant weight in our problem.

To use this method here, we will have to distinguish the high velocities from the low ones. The treatment of each one is given in separated sections. To simplify the notations, we call $T$ the time assigned for each part of the control process (instead of, for instance, $T/2$ or $T/3$).

The corresponding Cauchy problem (for strong solutions) has been solved in 2-D by Ukai-Okabe [17], and in 3-D by Lions-Perthame [14], Pfaffelmoser [15] and Schaeffer [16] (see also [2] in the periodic case). The construction of solutions to the non-linear problem that we use here is essentially the one of Ukai and Okabe but simplified because essentially the *local in time* existence is sufficient to our use, since, if one can control the system in any time, one can steer the configuration to a stable one (0 for example) before any possible blow up. Remark in particular that Theorem 2 guarantees the existence in the large of classical controlled solutions for $n \geqslant 4$, which is not necessarily the case for the uncontrolled one (see e.g. [13]).

In the next section, we give some notations and expose the principal tools for the construction of a solution of the non-linear problem. Sections 3 and 4 prove Theorem 1, and Sections 5 and 6 prove Theorem 2. Precisely, Section 3 is devoted to the treatment of the high velocities of $f_0$ and $f_1$. Then Section 4 gives an exposition of the treatment of "small" velocities (precisely, it studies the case when $f_0$ and $f_1$ have compact support in $\xi$), which allows to finish the proof of Theorem 1. Section 5 deals with the case corresponding to $f_0$ and $f_1$ with bounded velocities in the direction of the normal to the hyperplane in $\omega$. Section 6 shows how to restrict to the case of Section 5, which finishes the proof of Theorem 2. Finally, the appendix in Section 7 gives the proofs of some lemmas, useful to construct the solution $(\bar{f}, \varphi)$.

## 2. Notations and machinery

### 2.1. Notations

We will generally agree with [17] on the notations. For $T > 0$, we denote $Q_T \coloneqq [0, T] \times \mathbb{T}^n \times \mathbb{R}^n$, and $\Omega_T \coloneqq [0, T] \times \mathbb{T}^n$. For a domain $\Omega$, we write also $B^l(\Omega)$, for $l \in \mathbb{N}$, for the set $C^l(\Omega) \cap W^{l,\infty}(\Omega)$. All the same, $B^{l+\sigma}(\Omega)$ for $\sigma \in (0,1)$ stands for the set of $C^l$ functions with uniformly $\sigma$-Hölder $l$-th derivatives. Also, $B^{\sigma, l+\sigma'}(\Omega_T)$ (resp. $B^{\sigma, l+\sigma'}(Q_T)$), for $l \in \mathbb{N}$, $\sigma, \sigma' \in [0, 1)$ is the set of continuous functions in $\Omega_T$ (resp. $Q_T$), which are $C^l$ with respect to $x$ (resp. to $(x, \xi)$), and which $l$-th derivatives are all $C^\sigma$ with respect to $t$ and $C^{\sigma'}$ with respect to $x$ (resp. to $(x, \xi)$).

For $x$ in $\mathbb{T}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball with center $x$ and radius $r$, and by $S(x, r)$ the corresponding sphere. The radii will always be chosen small enough in order that $S(x, r)$ does not intersect itself (that is $r < 1/2$ in the standard torus). Let us agree to call $v(x)$ the outward unit normal vector on this sphere when there is no ambiguity. We will precise it by a subscript when we consider the ball in the whole space $\mathbb{R}^n$. The unit zero-centered sphere in $\mathbb{R}^n$ is also denoted by $\mathbb{S}$. We will denote the canonical surjection $\mathbb{R}^n \to \mathbb{T}^n$ by $\mathscr{S}$.

Finally, we will denote by $\Lambda$ the operator $g \mapsto \int_{\mathbb{R}^n} g(\xi)\, d\xi$, which maps a function in variable $(x, \xi)$ to a function in variable $x$. For the sake of simplicity and without losing generality, we suppose that $\mathbb{T}^n$ is of measure 1.

### 2.2. Characteristic equation

It is well known that, the function $\phi$ being fixed, solving (1.1) and (1.3) reduces to solve the characteristic equation

$$\frac{d}{dt}\begin{pmatrix} X \\ \Xi \end{pmatrix} = \begin{pmatrix} \Xi \\ \nabla\phi(t,X) \end{pmatrix}, \tag{2.1}$$

with the initial condition

$$\begin{pmatrix} X \\ \Xi \end{pmatrix}_{|t=s} = \begin{pmatrix} x \\ \xi \end{pmatrix}. \tag{2.2}$$

Let us denote by $(X(t,s,x,\xi), \Xi(t,s,x,\xi))$ the solution of (2.1) at time $t$ with initial conditions (2.2) at time $s$. One gets

$$\begin{cases} X(t,s,x,\xi) = x + \int_s^t \Xi(\tau,s,x,\xi)\,d\tau, \\ \Xi(t,s,x,\xi) = \xi + \int_s^t \nabla\phi(X(\tau,s,x,\xi))\,d\tau. \end{cases} \tag{2.3}$$

Then the classical solution of (1.1) and (1.3) is given by

$$f(t,x,\xi) = f_0(X(0,t,x,\xi), \Xi(0,t,x,\xi)) + \int_0^t G(s, X(s,t,x,\xi), \Xi(s,t,x,\xi))\,ds. \tag{2.4}$$

Let us recall that, the vector field $(\xi, \nabla\phi(x))$ being divergence-free with respect to $(x,\xi)$, this flow preserves measure. In the sequel, it may be useful to precise which field $\nabla\phi$ generates the flow: when necessary, we will precise this by an exponent.

### 2.3. A lemma for the characteristic equation

Here, we state a Gronwall-type result. In the spirit of Lemma [17, Lemma 5.2], we prove the following lemma:

**Lemma 1.** *Fix $\delta \in (0,1)$. For $\rho \in B^{0,\delta}(\Omega_T)$ such that*

$$\int_{\mathbb{T}^n} \rho(t,x)\,dx = 0 \quad \textit{for all } t \in [0,T],$$

*there exists $c(\|\rho\|_{B^{0,\delta}})$ such that, if $(X,\Xi)$ are the characteristics corresponding to the potential $\phi$ given by*

$$\begin{cases} -\Delta\phi(t,x) = \rho, \\ \int_{\mathbb{T}^n} \phi(t,x)\,dx = 0, \end{cases} \tag{2.5}$$

*then one has*: *for any* $(t,s,x,\xi)$ *and* $(t',s',x',\xi')$ *in* $[0,T]^2\times\mathbb{T}^n\times\mathbb{R}^n$ *with* $|\xi-\xi'|<1$,

$$\begin{aligned}|(X,\Xi)(t,s,x,\xi)-(X,\Xi)(t',s',x',\xi')|\\ \leqslant c(\|\rho\|_{B^{0,\delta}})(1+|\xi|)|(t,s,x,\xi)-(t',s',x',\xi')|,\end{aligned} \tag{2.6}$$

*and moreover*

$$|(X,\Xi)(t,s,x,\xi)-(X,\Xi)(t,s,x',\xi')|\leqslant c(\|\rho\|_{B^{0,\delta}})|(x,\xi)-(x',\xi')|. \tag{2.7}$$

**Proof of Lemma 1.** We follow [17, Lemma 5.2]. It is sufficient to study the case when $x=x'$, $\xi=\xi'$ and $s=s'$, the case when $s=s'$ and $t=t'$ and the case when $x=x'$, $\xi=\xi'$ and $t=t'$.

(i) The first case follows from

$$|X(t,s,x,\xi)-X(t',s,x,\xi)|\leqslant|t-t'|(|\xi|+T\|\nabla\phi^\rho\|_{L^\infty}),$$

$$|\Xi(t,s,x,\xi)-\Xi(t',s,x,\xi)|\leqslant|t-t'|\times\|\nabla\phi^\rho\|_{L^\infty(\Omega_T)},$$

and from usual elliptic estimates.

(ii) The second one is a consequence of Gronwall's inequality. Indeed, one has

$$\begin{aligned}&|\Xi(t,s,x,\xi)-\Xi(t,s,x',\xi')|\\ &\quad\leqslant|\xi-\xi'|+C\|\nabla^2\phi^\rho\|_{L^\infty(\Omega_T)}\int_s^t|X(\tau,s,x,\xi)-X(\tau,s,x',\xi')|\,d\tau,\end{aligned}$$

and

$$\begin{aligned}&|X(t,s,x,\xi)-X(t,s,x',\xi')|\\ &\quad\leqslant|x-x'|+T|\xi-\xi'|+CT\|\nabla^2\phi^\rho\|_{L^\infty(\Omega_T)}\int_s^t|X(\tau,s,x,\xi)-X(\tau,s,x',\xi')|\,d\tau,\end{aligned}$$

for which Gronwall's inequality gives first the $X$ part (again with elliptic estimates), then the $\Xi$ one, of (2.7).

(iii) The third case follows from

$$\begin{aligned}&(X,\Xi)(t,s,x,\xi)-(X,\Xi)(t,s',x,\xi)\\ &\quad=(X,\Xi)(t,s,x,\xi)-(X,\Xi)(t,s,X(s,s',x,\xi),\Xi(s,s',x,\xi)),\end{aligned}$$

and from the two previous cases.

### 2.4. A remark concerning the scaling of the system

Let us first remark that system (1.1)–(1.2) is invariant by some change of scale, precisely, when $f$ is a solution of (1.1)–(1.2) in $[0, T] \times \mathbb{T}^n \times \mathbb{R}^n$, then for $\lambda \neq 0$, the function

$$f^\lambda(t, x, \xi) := |\lambda|^{2-n} f(\lambda t, x, \xi/\lambda), \tag{2.8}$$

is still a solution of (1.1)–(1.2), in $[0, T/\lambda] \times \mathbb{T}^n \times \mathbb{R}^n$ for the following potential:

$$\phi^\lambda(t, x) := \lambda^2 \phi(\lambda t, x). \tag{2.9}$$

Hence, using (2.8) with $\lambda = -1$, we see that it is sufficient, in order to prove Theorems 1 and 2, to restrict to the case where $f_1 = 0$ in $[\mathbb{T}^n \backslash \omega] \times \mathbb{R}^n$. Indeed, treat the cases:

- $f_0$ as initial value and 0 (in $(\mathbb{T}^n \backslash \omega) \times \mathbb{R}^n$) as the final one,
- $(x, \xi) \mapsto f_1(x, -\xi)$ as initial value and again 0 as the final one,

each in time $T/3$. We obtain two functions $\hat{f}_0$ and $\hat{f}_1$. Now consider the function $\hat{f}$ partially defined in $Q_T$ by

$$\begin{cases} \hat{f}(t, x, \xi) = \hat{f}_0(t, x, \xi), & \text{in } [0, T/3] \times \mathbb{T}^n \times \mathbb{R}^n, \\ \hat{f}(t, x, \xi) = 0, & \text{in } [T/3, 2T/3] \times [\mathbb{T}^n \backslash \omega] \times \mathbb{R}^n, \\ \hat{f}(t, x, \xi) = \hat{f}_1(T - t, x, -\xi) & \text{in } [2T/3, T] \times \mathbb{T}^n \times \mathbb{R}^n. \end{cases}$$

Then "fill" regularly $\hat{f}$ inside $[T/3, 2T/3] \times \omega \times \mathbb{R}^n$, taking care to preserve for any $t$ the value of $\int_{\mathbb{T}^n \times \mathbb{R}^n} \hat{f}(t, x, \xi)\, dx\, d\xi$. Then you get a relevant solution $f$. From now, we consider hence that $f_1 = 0$ (in $(\mathbb{T}^n \backslash \omega) \times \mathbb{R}^n$, of course).

Let us remark also that if we can solve the controllability problem with initial data $f_0^\lambda(x, \xi) = \lambda^{2-n} f_0(x, \xi/\lambda)$, for $\lambda \in (0, 1)$, then we can solve the problem with initial data $f_0$ (in smaller time).

### 2.5. Notations for Theorem 2

In Sections 5 and 6, the control zone $\omega$ is supposed to contain a hyperplane $H$ in the torus, that is, the image by the canonical surjection $\mathscr{S}$ of an Euclidean hyperplane $\mathscr{H}$ of $\mathbb{R}^n$. Now, up to a translation, $\mathscr{H}$ is a subgroup of $\mathbb{R}^n$, and there are two cases for $\overline{\mathscr{H} + \mathbb{Z}^n}$:

- either $\overline{\mathscr{H} + \mathbb{Z}^n} = \mathbb{R}^n$ and in that case, $H$ is dense in $\mathbb{T}^n$. Then, as $\omega$ is a regular open set, this implies $\omega = \mathbb{T}^n$. In that case, the problem is trivial since a "straight line" between $f_0$ and $f_1$ is a suitable solution of (1.1). From now, we suppose that we are in the following opposite case;

- or, $\overline{\mathscr{H}+\mathbb{Z}^n}$ is a strict closed subgroup of $\mathbb{R}^n$ (up to a translation). Then, see for instance Bourbaki [3, VII, Corollary 1], $\overline{\mathscr{H}+\mathbb{Z}^n}$ is composed of countably many parallel hyperplanes, whose intersection with any complementary linear subspace is a discrete group. For dimension reasons, see [3, VII, Corollary 1], such a discrete group is of the form $\mathbb{Z}\vec{u}$. This implies that $\mathscr{H}+\mathbb{Z}^n$ is of the form $\mathscr{H}+\mathbb{Z}\vec{u}$.

We call $v$ a unit vector, orthogonal to $\mathscr{H}$. By the previous argument, we can define $d\in\mathbb{R}^{+*}$ such that $H+[-2d,2d]v\subset\omega$ and such that $2d$ is less than the distance between two different hyperplanes in $\mathscr{S}^{-1}(H)$. We denote by $\bar{H}$ the linear vector space corresponding to the directions of $\mathscr{H}$.

It may happen that in Sections 3 and 4, we conserve the notation $n$ for the dimension (instead of 2), for objects useful in next sections.

## 3. Theorem 1: the problem of high velocities

In this section, we study the way to "suppress" the particles at high velocity, viz. to find a control such that, starting from an arbitrary $f_0$, the corresponding solution of the Vlasov–Poisson system reaches a configuration with compact support in $\xi$ at time $T$. Then, Section 4 shows how to reach exactly 0, which as we explained is sufficient to establish the general case.

The proof relies on a special solution $(\bar{f},\varphi)$ that we describe in the following paragraph. Then, we construct an operator $V_\varepsilon$, using this function $(\bar{f},\varphi)$. We show that this operator admits a fixed point for appropriate $\varepsilon$. Finally, we will show that, for $\varepsilon$ small enough, the fixed point that we found is relevant, that is, satisfies system (1.1)–(1.2) with (1.3) and (1.5), and actually "treats" the high velocities, in the sense that

$$f(T,\cdot,\cdot)\quad \text{is compactly supported.} \tag{3.1}$$

During the proof of the existence of solutions in Theorem 1, we will use only (1.11)—see Remark 4. In Section 3.5, we use the stronger assumption (1.6) to get uniqueness.

### 3.1. The function $\bar{f}$

We have the following proposition:

**Proposition 1.** *Given $x_0$ in $\mathbb{T}^2$, and $r_0$ a small positive number, there exist $\varphi\in C^\infty(\Omega_T;\mathbb{R})$ and $m\in\mathbb{R}^{+*}$ such that*

$$\Delta\varphi=0\quad in\ [0,T]\times\left[\mathbb{T}^2\backslash\bar{B}\left(x_0,\frac{1}{10}r_0\right)\right], \tag{3.2}$$

$$\operatorname{Supp}\varphi\subset(0,T)\times\mathbb{T}^2, \tag{3.3}$$

*and such that, if one considers the characteristics* $(\bar{X}, \bar{\Xi})$ *associated to* $\varphi$, *then*

$$\forall x \in \mathbb{T}^2, \forall \xi \in \mathbb{R}^2 \quad \textit{such that } |\xi| \geqslant m, \exists t \in (T/4, 3T/4), \ \bar{X}(t, 0, x, \xi) \in B\Big(x_0, \frac{r_0}{4}\Big). \tag{3.4}$$

We prove this proposition in the appendix.

Now let us describe the function $\bar{f}$. Consider a function $\mathscr{Z} \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$ satisfying the following constraints:

$$\begin{cases} \mathscr{Z} \geqslant 0 \quad \text{in } \mathbb{R}^n, \\ \operatorname{Supp} \mathscr{Z} \subset B_{\mathbb{R}^n}(0, 1), \\ \int_{\mathbb{R}^n} \mathscr{Z} = 1. \end{cases} \tag{3.5}$$

Consider $x_0$ in $\omega$ and $r_0$ a small positive number such that $B(x_0, 2r_0) \subset \omega$ (reducing $\omega$ if necessary, we now assume that $B(x_0, 2r_0) = \omega$). Define a function $\varphi$ as in Proposition 1. Then we introduce $\bar{f} = \bar{f}(t, x, \xi)$ as

$$\bar{f}(t, x, \xi) := \mathscr{Z}(\xi) \Delta\varphi(t, x). \tag{3.6}$$

Of course, $\bar{f}$ satisfies (1.1) in $[0, T] \times \mathbb{T}^2 \times \mathbb{R}^2$, with source term

$$\bar{G}(t, x, \xi) := \partial_t \bar{f} + \xi \cdot \nabla_x \bar{f} + \nabla\varphi \cdot \nabla_\xi \bar{f}, \tag{3.7}$$

which is supported in $[0, T] \times B(x_0, r_0) \times \mathbb{R}^2$. Up to an additive function of $t$, the function $\varphi$ satisfies Eq. (1.2) corresponding to $\bar{f}$ (with $\bar{f}(0, \cdot, \cdot) \equiv 0$). We denote $\bar{\rho}(t, x) := \Lambda \bar{f} = \Delta\varphi(t, x)$.

### 3.2. *The operator* $V_\varepsilon$

In this section, $\varepsilon$ is a positive real number such that $\varepsilon \leqslant 1$. We first define the domain $\mathscr{S}_\varepsilon$ of $V_\varepsilon$ by

$$\mathscr{S}_\varepsilon := \{ g \in B^{\delta_2}(Q_T) /$$

$$\begin{array}{ll} \text{(a)} & \|\Lambda(g - \bar{f})\|_{C^{\delta_1}(\Omega_T)} \leqslant \varepsilon, \\ \text{(b)} & \|(1 + |\xi|)^\gamma (g - \bar{f})\|_{L^\infty(Q_T)} \leqslant c_1 [\|f_0\|_{B^1(\mathbb{T}^2 \times \mathbb{R}^2)} + \|(1 + \xi)^\gamma f_0\|_{B^0(\mathbb{T}^2 \times \mathbb{R}^2)}], \\ \text{(c)} & \|g - \bar{f}\|_{B^{\delta_2}(Q_T)} \leqslant c_2 [\|f_0\|_{B^1(\mathbb{T}^2 \times \mathbb{R}^2)} + \|(1 + \xi)^\gamma f_0\|_{B^0(\mathbb{T}^2 \times \mathbb{R}^2)}], \\ \text{(d)} & \forall t \in [0, T], \int_{\mathbb{T}^2 \times \mathbb{R}^2} g(t, x, \xi)\, dx\, d\xi = \int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0(x, \xi)\, dx\, d\xi \}, \end{array} \tag{3.8}$$

with $c_1$, $c_2$ depending only on $\gamma$, $T$, and $\omega$ (and hence on $(\bar{f}, \varphi)$), but not on $\varepsilon$. The indices $\delta_1 < \delta_2$ in $(0, 1)$ are fixed as follows:

$$\delta_1 := \frac{\gamma - n}{2(\gamma+1)} \quad \text{and} \quad \delta_2 := \frac{\gamma}{\gamma+1}. \tag{3.9}$$

For fixed $c_1$ and $c_2$ large enough depending only on $(\bar{f}, \varphi)$, and $f_0$ small enough, one has $\mathscr{S}_\varepsilon \neq \emptyset$ (for instance, $f_0 + \bar{f} \in \mathscr{S}_\varepsilon$ in this case), and hence $|\rho_0| \leqslant \varepsilon$. From now, this is systematically supposed to be the case.

Now we introduce the following subsets of $S(x_0, r_0) \times \mathbb{R}^2$:

$$\gamma^- := \{(x,\xi) \in S(x_0, r_0) \times \mathbb{R}^2 / |\xi| > \tfrac{1}{2} \quad \text{and} \quad \xi \cdot v(x) < -\tfrac{1}{10}|\xi|\}, \tag{3.10}$$

$$\gamma^{2-} := \{(x,\xi) \in S(x_0, r_0) \times \mathbb{R}^2 / |\xi| \geqslant 1 \quad \text{and} \quad \xi \cdot v(x) \leqslant -\tfrac{1}{8}|\xi|\}, \tag{3.11}$$

$$\gamma^{3-} := \{(x,\xi) \in S(x_0, r_0) \times \mathbb{R}^2 / |\xi| \geqslant 2 \quad \text{and} \quad \xi \cdot v(x) \leqslant -\tfrac{1}{5}|\xi|\}, \tag{3.12}$$

$$\gamma^+ := \{(x,\xi) \in S(x_0, r_0) \times \mathbb{R}^2 / \xi \cdot v(x) \geqslant 0\}. \tag{3.13}$$

It can be easily seen that

$$\mathrm{dist}([S(x_0, r_0) \times \mathbb{R}^2] \backslash \gamma^{2-}; \gamma^{3-}) > 0.$$

We introduce a $C^\infty \cap B^1$ regular function $U : S(x_0, r_0) \times \mathbb{R}^2 \to \mathbb{R}$, satisfying

$$\begin{cases} 0 \leqslant U \leqslant 1, \\ U \equiv 1 \quad \text{in } [S(x_0, r_0) \times \mathbb{R}^2] \backslash \gamma^{2-}, \\ U \equiv 0 \quad \text{in } \gamma^{3-}. \end{cases} \tag{3.14}$$

We also introduce a function $\Upsilon : \mathbb{R}^+ \to \mathbb{R}^+$, of class $C^\infty$, such that

$$\begin{cases} \Upsilon \quad \text{is non-increasing}, \\ \Upsilon \equiv 1 \quad \text{in } [0, 1/4], \\ \Upsilon \equiv 0 \quad \text{in } [3/4, +\infty). \end{cases} \tag{3.15}$$

Now, given $g \in \mathscr{S}_\varepsilon$, we associate $\phi^g$ by

$$\begin{cases} -\Delta \phi^g(x,t) = \rho_0 - \Lambda g \quad \text{in } [0, T] \times \mathbb{T}^n, \\ \int_{\mathbb{T}^n} \phi^g(x,t)\, dx = 0 \quad \text{in } [0, T]. \end{cases} \tag{3.16}$$

Then, we define $\tilde{V}(g) := f$ to be the solution of the following system:

$$\begin{cases} f(0,x,\xi) = f_0 \quad \text{on } \mathbb{T}^2 \times \mathbb{R}^2, \\ \partial_t f + \xi \cdot \nabla_x f + \nabla \phi^g \cdot \nabla_\xi f = 0 \quad \text{in } [0,T] \times [(\mathbb{T}^2 \times \mathbb{R}^2) \backslash \gamma^-], \\ f(t,x,\xi) = \left[ \Upsilon\left(\frac{t}{\chi}\right) + \Upsilon\left(\frac{T-t}{\chi}\right) \right] f(t^-,x,\xi) \\ + \left[ 1 - \Upsilon\left(\frac{t}{\chi}\right) - \Upsilon\left(\frac{T-t}{\chi}\right) \right] U(x,\xi) f(t^-,x,\xi) \quad \text{on } [0,T] \times \gamma^-. \end{cases} \tag{3.17}$$

In the previous writing, $f(t^-,x,\xi)$ is the limit value of $f$ on the characteristic leading to $(x,\xi)$ as the time goes to $t^-$. (For times before $t$, but close to $t$, the corresponding characteristic is not in $\gamma^-$.) Of course, (3.17) has to be understood in a characteristics sense, that is, for each characteristic, $f$ is ruled by (2.4) as long as the characteristic curve does not meet $\gamma^-$. When the characteristic meets $\gamma^-$ at time $t$, then the value of $f$ at time $t^+$ is fixed according to the last equation in (3.17). Note that the set of different times a characteristic curve meets $\gamma^-$ is discrete as seen from the definition of $\gamma^-$.

We fix the parameter $\chi$ as the following positive number:

$$\chi = \frac{T}{8}. \tag{3.18}$$

We now consider a continuous linear extension operator $\bar{\pi} : C^0(\mathbb{T}^2 \backslash B(x_0, 2r_0); \mathbb{R}) \to C^0(\mathbb{T}^2; \mathbb{R})$, and which has the property that each $C^\alpha$-regular function is continuously mapped to a $C^\alpha$-regular function, for any $\alpha \in [0,1]$. Moreover, we manage in order that for any $f \in C^0(\mathbb{T}^2 \backslash B(x_0, 2r_0))$, one has

$$\int_{\mathbb{T}^n} \bar{\pi}(f) = \int_{\mathbb{T}^n \times \mathbb{R}^n} f_0(x,\xi)\, dx\, d\xi. \tag{3.19}$$

The last condition can easily be obtained by considering a regular non-negative function $u$ with integral 1 in $B(x_0, r_0)$, and adding to $\bar{\pi}(f)$ the function $(\int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 - \int_{\mathbb{T}^2} \bar{\pi}(f))u$.

We fix $c_\pi$ such that for any $f \in C^1(\mathbb{T}^2 \backslash B(x_0, 2r_0))$, one has

$$\begin{cases} \|\bar{\pi}(f)\|_{B^1} \leqslant c_\pi \|f\|_{B^1}, \\ \|\bar{\pi}(f)\|_{L^\infty} \leqslant c_\pi \|f\|_{L^\infty}. \end{cases} \tag{3.20}$$

From this operator, we deduce a new one $\pi : C^0((\mathbb{T}^2 \backslash B(x_0, 2r_0)) \times \mathbb{R}^2) \to C^0(\mathbb{T}^2 \times \mathbb{R}^2)$ according to the rule

$$(\pi f)(x,\xi) := [\bar{\pi} f(\cdot,\xi)](x). \tag{3.21}$$

Finally, we introduce the operator $\Pi : C^0(([0,T] \times [\mathbb{T}^2 \backslash B(x_0, 2r_0)] \times \mathbb{R}^2) \cup ([0,\chi/4] \times \mathbb{T}^2 \times \mathbb{R}^2)) \to C^0([0,T] \times \mathbb{T}^2 \times \mathbb{R}^2)$ given by

$$(\Pi f)(t,x,\xi) := \Upsilon\left(\frac{4t}{\chi}\right) f(t,x,\xi) + \left[1 - \Upsilon\left(\frac{4t}{\chi}\right)\right][\pi f(t,\cdot,\cdot)](x,\xi). \tag{3.22}$$

We now define $V[g]$ by

$$V[g] := \bar{f} + \Pi(f_{|([0,T]\times[\mathbb{T}^2\backslash B(x_0,2r_0)]\times\mathbb{R}^2)\cup([0,\chi/4]\times\mathbb{T}^2\times\mathbb{R}^2)}) \quad \text{in } [0,T] \times \mathbb{T}^2 \times \mathbb{R}^2. \tag{3.23}$$

It will be shown that the function $f$ is $C^1$ regular at the neighborhood of any point $(t,x,\xi) \in [0,T] \times \mathbb{T}^2 \times \mathbb{R}^2$ such that $(x,\xi) \notin \gamma^-$. This implies in particular that $f_{|[0,T]\times[\mathbb{T}^2\backslash B(x_0,2r_0)]\times\mathbb{R}^2}$ is $C^1$ regular. Another argument (see (3.26) below) proves the $C^1$ regularity of $f_{|[0,\chi/4]\times\mathbb{T}^2\times\mathbb{R}^2}$. This will imply, together with the construction of $\Pi$, that $V[g]$ is in $C^1([0,T] \times \mathbb{T}^2 \times \mathbb{R}^2)$.

Finally, we note $W$ the same operator as $\tilde{V}$ but without absorption, that is, where the function $U$ is replaced by 1. In other words, $W$ satisfies

$$W[g]_{|t=0} = f_0, \tag{3.24}$$

$$\partial_t W[g] + \xi \cdot \nabla_x W[g] + \nabla\phi^g \cdot \nabla_\xi W[g] = 0 \quad \text{in } Q_T. \tag{3.25}$$

(Hence, $W[g]$ is transported by the characteristics of $\phi^g$.) Let us remark that, due to (3.15) and (3.17), one has

$$\tilde{V}[g](t,x,\xi) = W[g](t,x,\xi) \quad \text{for } t \in [0,\chi/4]. \tag{3.26}$$

In the sequel, the goal is to find a fixed point to $V_\varepsilon$, for any $\varepsilon$ small enough, provided $f_0$ is chosen small enough in terms of $\varepsilon$. We may sometimes omit the index $\varepsilon$ in $V_\varepsilon$ and $\mathscr{S}_\varepsilon$.

### 3.3. *Finding a fixed point of $V_\varepsilon$*

Here, the goal is to apply the Leray–Schauder Theorem. Let us check that its assumptions are satisfied.

1. The first point, precisely that $\mathscr{S}_\varepsilon$ is a convex compact subset of $C^0([0,T] \times \mathbb{T}^2 \times \mathbb{R}^2)$ is a consequence of Ascoli's Theorem.

2. Let us check the continuity of $V_\varepsilon$ for the $C^0$ topology. Assume that a sequence $g_i$ of elements of $\mathscr{S}_\varepsilon$ converges to $g$ for the $C^0$ norm (set $g_\infty = g$). As $\mathscr{S}_\varepsilon$ is compact, one has that $g \in \mathscr{S}_\varepsilon$. Then, the corresponding sequence $\Delta\phi^{g_i}$ is bounded in $C^{\delta_1}([0,T] \times \mathbb{T}^2)$. It follows that $\nabla\phi^{g_i} \to \nabla\phi^g$ for the $C^0(\Omega_T)$ topology. Now consider the characteristics $(X^{g_i}, \Xi^{g_i})$ corresponding to the potential $\phi^{g_i}$, and $(X^g, \Xi^g)$ those

for $\phi^g$. We have for any $(t,s,x,\xi)\in[0,T]^2\times\mathbb{T}^2\times\mathbb{R}^2$, (in $L^\infty$ norm),

$$\frac{d}{dt^+}||(X^{g_i}(t,s,x,\xi),\Xi^{g_i}(t,s,x,\xi))-(X^g(t,s,x,\xi),\Xi^g(t,s,x,\xi))||$$
$$\leqslant||\Xi^{g_i}(t,s,x,\xi)-\Xi^g(t,s,x,\xi))||+||\nabla\phi^{g_i}(t,X^{g_i})-\nabla\phi^g(t,X^g)||.$$

But one has

$$\begin{aligned}&||\nabla\phi^{g_i}(t,X^{g_i})-\nabla\phi^g(t,X^g)||\\&\quad\leqslant||\nabla\phi^{g_i}(t,X^{g_i})-\nabla\phi^g(t,X^{g_i})||+||\nabla\phi^g(t,X^{g_i})-\nabla\phi^g(t,X^g)||\\&\quad\leqslant||\nabla\phi^{g_i}(t,X^{g_i})-\nabla\phi^g(t,X^{g_i})||+||\nabla\phi^g||_{B^{0,1}(\Omega_T)}||X^{g_i}-X^g||,\\&\quad\leqslant||\nabla\phi^{g_i}-\nabla\phi^g||_{B^0(\Omega_T)}+||\nabla\phi^g||_{B^{0,1}(\Omega_T)}||X^{g_i}-X^g||.\end{aligned}$$

It follows from Gronwall's lemma and elliptic estimates for $||\nabla\phi^g||_{B^{0,1}(\Omega_T)}$, that for $i\in\{1,\dots,\infty\}$,

$$||(X^{g_i},\Xi^{g_i})-(X^g,\Xi^g)||_{B^0([0,T]\times Q_T)}\leqslant C||\nabla\phi^{g_i}-\nabla\phi^g||_{B^0(\Omega_T)},\tag{3.27}$$

for $C$ independent from $i$. Hence one has

$$||(X^{g_i},\Xi^{g_i})-(X^g,\Xi^g)||_{B^0([0,T]\times Q_T)}\to 0\quad\text{as } i\to+\infty.\tag{3.28}$$

Let us prove that this involves that for $(t,x,\xi)\in Q_T\backslash([0,T]\times\gamma^-)$, one has

$$\tilde{V}[g_i](t,x,\xi)\to\tilde{V}[g](t,x,\xi).\tag{3.29}$$

Fix $(t,x,\xi)\in Q_T\backslash([0,T]\times\gamma^-)$. In a first case: if the trajectory from $(X^g,\Xi^g)(0,t,x,\xi)$ to $(x,\xi)$ does not meet $\gamma^{2-}$ for a time $s\geqslant\chi/4$, then it is a clear consequence of (3.28) that $\tilde{V}[g_i](t,x,\xi)\to\tilde{V}[g](t,x,\xi)$. Indeed, for $i$ large enough, the trajectory $(X^{g_i},\Xi^{g_i})(\cdot,t,x,\xi)$ does not meet $\gamma^{2-}$ either for $s\geqslant\chi/4$, and we consequently have

$$\tilde{V}[g](t,x,\xi)=f_0[(X^g,\Xi^g)(0,t,x,\xi)]\quad\text{and}$$

$$\tilde{V}[g_i](t,x,\xi)=f_0[(X^{g_i},\Xi^{g_i})(0,t,x,\xi)].\tag{3.30}$$

In particular this is valid for $(t,x,\xi)\in[0,\chi/4]\times\mathbb{T}^2\times\mathbb{R}^2$.

Now, let us study the case when the trajectory from $(X^g,\Xi^g)(0,t,x,\xi)$ to $(x,\xi)$ meets $\gamma^{2-}$ for a time $s\geqslant\chi/4$. Using (2.3), the value of $\xi\cdot\nu(x)$ on $\gamma^-$, $\text{dist}(\gamma^-,\gamma^+)>0$ and the uniform bound for $||\nabla\phi^h||_{L^\infty}$, for all $h\in\mathscr{S}_\varepsilon$, one can see that there exists $m=m(|\xi|)$ such that, if $s\in[0,t)$ satisfies $X^{g_i}(s,t,x,\xi)\in\gamma^-$, then $X^{g_i}([s-m,s),t,x,\xi)\subset\mathbb{T}^2\backslash\overline{B(x_0,r_0)}$ and $X^{g_i}((s,s+m],t,x,\xi)\subset B(x_0,r_0)$ (for $i\in\{1,\dots,\infty\}$). It follows then that two different times for which $(X^{g_i},\Xi^{g_i})(\cdot,t,x,\xi)$ meets $\gamma^-$

(a fortiori $\gamma^{2-}$), are distant at least from $m(|\xi|)>0$ (independently from $i$). We denote by $t_1, \dots, t_n \in [\chi/4, t)$ the different times for which $(X^g, \Xi^g)(\cdot, t, x, \xi)$ belongs to $\gamma^{2-}$, in increasing order.

Now we claim that, for $i$ large enough, one has

$$\exists (s_{i,1}, \dots, s_{i,n}) \in [0,t)^n \quad \text{and} \quad \exists \tau > 0,\ \exists \varrho > 0 \text{ such that}$$

$$\begin{aligned}
&\text{(i)} \quad (X^{g_i}, \Xi^{g_i})([s_{i,j}-\tau, s_{i,j}+\tau], t, x, \xi) \cap \gamma^- = \{(X^{g_i}, \Xi^{g_i})(s_{i,j}, t, x, \xi)\},\\
&\text{(ii)} \quad \text{for } s \in [0,t),\ \operatorname{dist}(s, \{s_{i,1}, \dots, s_{i,n}\}) \geqslant \tau \Rightarrow \operatorname{dist}((X^{g_i}, \Xi^{g_i})(s,t,x,\xi), \gamma^{2-}) \geqslant \varrho,\\
&\text{(iii)} \quad s_{i,j} \to t_j \quad \text{as } i \to +\infty, \quad \text{for all } j = 1, \dots, n.
\end{aligned} \tag{3.31}$$

Indeed, consider two points $(X^g, \Xi^g)(t_j+\tau, t, x, \xi)$ and $(X^g, \Xi^g)(t_j-\tau, t, x, \xi)$, with $\tau < m(|\xi|)$, the first one just after the point has left the circle $S(x_0, r_0)$ and the second one just before. For $i$ large enough, by (3.28), $X^{g_i}(t_i+\tau, t, x, \xi)$ is inside $B(x_0, r_0)$ and $X^{g_i}(t_i-\tau, t, x, \xi)$ is outside. Consequently, there is a point $X^{g_i}(s_{i,j}, t, x, \xi)$ (close to the one for $g$) which cuts the circle at time $s_{i,j} \in [t_j-\tau, t_j+\tau]$. Then, again by (3.28), the corresponding point $(X^{g_i}, \Xi^{g_i})(s_{i,j}, t, x, \xi)$ is in $\gamma^-$ for $i$ large enough, and by the definition of $m(|\xi|)$ satisfies (3.31)(i).

Now consider $(X^g, \Xi^g)(\cdot, t, x, \xi)$ for times in $[0,T] \setminus \bigcup_{i=1}^n [t_i-\tau, t_i+\tau]$: the distance of these points to $\gamma^{2-}$ is positive. Hence with (3.28), one gets (3.31)(ii) for $i$ large enough.

To get (3.31) (iii), it suffices to consider $(X^g, \Xi^g)(t_j+\tau', t, x, \xi)$ and $(X^g, \Xi^g)$ $(t_j-\tau', t, x, \xi)$ for $\tau' < \tau$, and arguing as previously, we obtain $s_{i,j} \in [t_j-\tau', t_j+\tau']$ for $i$ large enough.

This involves (3.29) in the general case. Indeed, if $n=0$, this is the first case that we have already treated. If $n \geqslant 1$, one gets for $i$ large enough

$$\begin{aligned}
\tilde{V}[g_i](s,x,\xi) = {} & f_0[(X^{g_i}, \Xi^{g_i})(0,t,x,\xi)] \times \prod_{j=1}^n \left\{ \left[ \Upsilon\left(\frac{s_{i,j}}{\chi}\right) + \Upsilon\left(\frac{T-s_{i,j}}{\chi}\right) \right] \right.\\
& \left. + \left[ 1 - \Upsilon\left(\frac{s_{i,j}}{\chi}\right) - \Upsilon\left(\frac{T-s_{i,j}}{\chi}\right) \right] U\big[(X^{g_i}, \Xi^{g_i})(s_{i,j}, t, x, \xi)\big] \right\}
\end{aligned}$$

(including for $i=\infty$ if we fix $s_{i,\infty} = t_i$) which together with (3.28) and (3.31), leads to (3.29).

Now we conclude in a standard way, using the property (yet to be proven) that $V(\mathscr{S}_\varepsilon) \subset \mathscr{S}_\varepsilon$. Indeed, if we admit this for the moment, we get that the $V[g_i]_{|[0,T]\times(\mathbb{T}^2\setminus\omega)\times\mathbb{R}^2}$ are in a compact subset of $C^0([0,T]\times(\mathbb{T}^2\setminus\omega)\times\mathbb{R}^2)$. So the previous pointwise convergence is in fact valid in $C^0([0,T]\times(\mathbb{T}^2\setminus\omega)\times\mathbb{R}^2)$. It is also valid in $C^0([0,\chi/4]\times\mathbb{T}^2\times\mathbb{R}^2)$ (as (3.30) applies in $[0,\chi/4]\times\mathbb{T}^2\times\mathbb{R}^2$). With the continuity of $\Pi$, this involves the convergence of $V[g_i]$ to $V[g]$ in $C^0(Q_T)$.

3. Now we have to show that $V_\varepsilon(\mathscr{S}_\varepsilon) \subset \mathscr{S}_\varepsilon$ if $f_0$ is small enough.

The fourth point in the definition of $\mathscr{S}_\varepsilon$ is observed as a consequence of the construction of $\Pi$. So there are essentially three points to check. We establish points b and c, before returning to the central point a. We may omit the indices $g$ in the sequel.

(b) We want to establish the second condition in $V_\varepsilon(\mathscr{S}_\varepsilon) \subset \mathscr{S}_\varepsilon$. We recall $f := \tilde{V}[g]$. One has

$$|f(t,x,\xi)| \leqslant |f_0[(X,\Xi)(0,t,x,\xi)]|.$$

This is due to the fact that, by (3.17), $f$ is constant along a characteristic as long as it does not meet $\gamma^-$, and when it does, then one gets $|f(t^+,x,\xi)| \leqslant |f(t^-,x,\xi)|$, thanks to the choices of $\Upsilon$ and $U$. Consequently, one gets

$$\begin{aligned}|f(t,x,\xi)| &\leqslant ||(1+|\xi|)^\gamma f_0||_{L^\infty}(1+|\Xi(0,t,x,\xi)|)^{-\gamma},\\ &\leqslant ||(1+|\xi|)^\gamma f_0||_{L^\infty}(1+|\xi-\xi+\Xi(0,t,x,\xi)|)^{-\gamma}.\end{aligned}$$

Now we remark that for all $t \in [0,T]$,

$$|\xi - \Xi(0,t,x,\xi)| \leqslant T||\nabla\phi^g||_{L^\infty} \leqslant CT(||\Delta\varphi||_{L^\infty}+1). \tag{3.32}$$

Using

$$(1+|x-x'|)^{-1} \leqslant \frac{1+|x|}{1+|x'|}, \tag{3.33}$$

we get for $(t,x,\xi) \in Q_T$,

$$|(1+|\xi|)^\gamma f(t,x,\xi)| \leqslant ||(1+|\xi|)^\gamma f_0||_{L^\infty}[1+CT(||\Delta\varphi||_{L^\infty}+1)]^\gamma.$$

Now using the construction of $\Pi$, we deduce

$$|(1+|\xi|)^\gamma V[g](t,x,\xi)| \leqslant c_\pi ||(1+|\xi|)^\gamma f_0||_{L^\infty}[1+CT(||\Delta\varphi||_{L^\infty}+1)]^\gamma. \tag{3.34}$$

We choose $c_1 = c_\pi[1+CT(||\Delta\varphi||_{L^\infty}+1)]^\gamma$. Then the second condition in $V(\mathscr{S}) \subset \mathscr{S}$ is established.

(c) Now we wish to have estimates on the derivatives of $V[g]$. First, we prove the following lemma.

**Lemma 2.** *For $g \in \mathscr{S}_\varepsilon$, one has $\tilde{V}[g] \in C^1(Q_T \backslash \Sigma_T)$, with $\Sigma_T := [0,T] \times \gamma^-$. Moreover, for any $(t,x,\xi)$ and $(t',x',\xi')$ in $[0,T] \times [\mathbb{T}^2 \backslash \omega] \times \mathbb{R}^2$, with $|\xi - \xi'| \leqslant 1$, one has,*

$$\begin{aligned}|\tilde{V}[g](t,x,\xi) - \tilde{V}[g](t',x',\xi')| \leqslant{}& C[||f_0||_{B^1(\mathbb{T}^2\times\mathbb{R}^2)} + ||(1+|\xi|)^\gamma f_0||_{L^\infty(\mathbb{T}^2\times\mathbb{R}^2)}]\\ &\times (1+|\xi|)|(t,x,\xi)-(t',x',\xi')|,\end{aligned} \tag{3.35}$$

*and also*

$$|\tilde{V}[g](t,x,\xi)-\tilde{V}[g](t,x',\xi')| \leqslant C[||f_0||_{B^1(\mathbb{T}^2\times\mathbb{R}^2)}+||(1+|\xi|)^\gamma f_0||_{L^\infty(\mathbb{T}^2\times\mathbb{R}^2)}]|(x,\xi)-(x',\xi')|, \tag{3.36}$$

*the constant C being independent from* $f_0$.

**Proof.** We begin with a remark concerning the number of times a particle can meet $\gamma^-$. We fix $(t,x,\xi)$ in $[0,T]\times\mathbb{T}^2\times\mathbb{R}^2$. Here, we denote $t_1,\dots,t_n$ the different times in $[\chi/4,t]$ for which $(X,\Xi)(t_i,t,x,\xi)$ belongs to $\gamma^-$ (sorted increasingly). Of course, $n$ depends on $(x,\xi)$. As $g\in\mathscr{S}_\varepsilon$, we get, at least if we have chosen $\varepsilon$ small enough, that

$$|\Xi(t,0,x,\xi)|\leqslant|\xi|+T||\nabla\varphi||_{C^0(Q_T)}+1 \quad \text{for } t\in[0,T]. \tag{3.37}$$

On another side, we remark that

$$\operatorname{dist}(\gamma^+,\gamma^-)>0. \tag{3.38}$$

From (3.37), (3.38) and the boundedness of the acceleration for $g\in\mathscr{S}_\varepsilon$, we deduce that for a certain $c>0$,

$$n(x,\xi)\leqslant c(1+|\xi|). \tag{3.39}$$

Now, let us briefly explain the continuity of $f:=\tilde{V}[g]$ in $Q_T\backslash\Sigma_T$ (this is proven approximately as (3.29)). Locally around $(t,x,\xi)\in Q_T\backslash\Sigma_T$, $f$ is constant along characteristics, hence (with Lemma 1) it is sufficient to prove the continuity of $f(t,\cdot,\cdot)$ close to $(x,\xi)$.

Given $(t,x,\xi)\in Q_T\backslash\Sigma_T$, we consider the characteristic curve from $(X(0,t,x,\xi),\Xi(0,t,x,\xi))$ to $(x,\xi)$. If this characteristic curve has not met $\gamma^{2-}$ after time $\chi/4$ during the process, then for $(x',\xi')$ close to $(x,\xi)$, the characteristic from $(X(0,t,x',\xi'),\Xi(0,t,x',\xi'))$ to $(x',\xi')$ is close (for the $B^0(\mathbb{T}^2\times\mathbb{R}^2)$ norm) to the one for $(x,\xi)$ (see (2.7)), and as a consequence, if $|(x,\xi)-(x'\xi')|$ is small enough, this characteristic does not meet $\gamma^{2-}$ after $\chi/4$. Consequently one has

$$\tilde{V}[g](t,x,\xi)=f_0[(X^g,\Xi^g)(0,t,x,\xi)] \quad \text{and} \quad \tilde{V}[g](t,x',\xi')=f_0[(X^g,\Xi^g)(0,t,x',\xi')].$$

Then the continuity at point $(x,\xi)$ follows from the regularity of the flow—see Lemma 1.

If the characteristic curve for $(x,\xi)$ has met $\gamma^{2-}$ after $\chi/4$ (i.e. $n(x,\xi)\geqslant1$), then for $(x',\xi')$ close to $(x,\xi)$, we consider the characteristic curve from $(X(0,t,x',\xi'),\Xi(0,t,x',\xi'))$ to $(x',\xi')$. For $\tau$ small (in particular such that $\tau<\chi/4$ and $\tau<t-t_n$), one has $X(t_i+\tau,t,x,\xi)\in B(x_0,r_0)$ and $X(t_i-\tau,t,x,\xi)\in\mathbb{T}^2\backslash B(x_0,r_0)$. We suppose that $\tau$ is small enough in order that any characteristic corresponding to a velocity $\xi'$ such that $|\xi'|\leqslant|\xi|+1$ cuts $\gamma^-$ at most once by time interval of length $2\tau$ (as seen previously).

As for (3.31) we get that $(X(0,t,x',\xi'),\Xi(0,t,x',\xi'))$ cuts $\gamma^-$ once in each $[t_i-\tau,t_i+\tau]$ and that the characteristic corresponding to $(x',\xi')$ cannot meet $\gamma^{2-}$ outside these intervals, if it is close enough to the one for $(x,\xi)$. As $(x',\xi')$ tends to $(x,\xi)$, the place in $\gamma^-$ where $(X(s,t,x',\xi'),\Xi(s,t,x',\xi'))$ cuts $\gamma^-$ during $[t_i-\tau,t_i+\tau]$ tends to the one for the $(x,\xi)$-characteristic. This gives again that $\tilde{V}(t,x',\xi')\to\tilde{V}(t,x,\xi)$ as $(x',\xi')$ tends to $(x,\xi)$, as for (3.29).

We now consider the derivatives of $\tilde{V}[g]$. We remark that if the characteristic curve starting at $(x,\xi)$ at time $s$, does not meet $\gamma^-$ during the interval $[s,t]$ then the derivatives at time $t$ and $s$ are linked by

$$d_{x,\xi}f(t,x,\xi)=d_{x,\xi}f(s,X(t,s,x,\xi),\Xi(t,s,x,\xi))\circ d_{x,\xi}(X,\Xi). \tag{3.40}$$

(Note that $\nabla_{x,\xi}(X,\Xi)(t,s,\cdot,\cdot)$ is uniformly bounded for $g\in\mathscr{S}_\varepsilon$ by Lemma 1.) Given $t_i\in(0,T)$, we define on $\gamma^-$ the functions $f^-(t_i,x,\xi)$ as

$$f^-(t_i,x,\xi)=\lim_{t\to t_i^-}f(t,(X,\Xi)(t,t_i,x,\xi)),$$

and $f^+$ given similarly with the right limit. The functions $f^-$ and $f^+$ are related by (3.17). We wish to compare $|\nabla f^-(t_i,x,\xi)|$ and $|\nabla f^+(t_i,x,\xi)|$, for $(x,\xi)\in\gamma^-$. Observe that the surface $\gamma^-$ is not characteristic in our problem. Consider $\tau$ one of tangential unit vector field on the circle $S(x_0,r_0)$. The partial derivatives $\partial_t f^+$, $\partial_\tau f^+$, $\nabla_\xi f^+$ and $\xi\cdot\nabla f^+$ can be easily derived on $\gamma^-$, and are continuous. Indeed, let us begin with $\partial_\tau$:

$$\begin{aligned}&\partial_\tau f^+(t_i,x,\xi)=\\&\left[\Upsilon\left(\frac{t_i}{\chi}\right)+\Upsilon\left(\frac{T-t_i}{\chi}\right)+\left(1-\Upsilon\left(\frac{t_i}{\chi}\right)-\Upsilon\left(\frac{T-t_i}{\chi}\right)\right)U(x,\xi)\right]\nabla_x f^-(t_i,x,\xi)\cdot\tau\\&+\left(1-\Upsilon\left(\frac{t_i}{\chi}\right)-\Upsilon\left(\frac{T-t_i}{\chi}\right)\right)f^-(t_i,x,\xi)\nabla_x U\cdot\tau.\end{aligned} \tag{3.41}$$

Now we have the same for the $\xi$ derivatives (consider $h\in\mathbb{R}^2$):

$$\begin{aligned}&\nabla_\xi f^+(t_i,x,\xi)\cdot h=\\&\left[\Upsilon\left(\frac{t_i}{\chi}\right)+\Upsilon\left(\frac{T-t_i}{\chi}\right)+\left(1-\Upsilon\left(\frac{t_i}{\chi}\right)-\Upsilon\left(\frac{T-t_i}{\chi}\right)\right)U(x,\xi)\right]\nabla_\xi f^-(t_i,x,\xi)\cdot h\\&+\left(1-\Upsilon\left(\frac{t_i}{\chi}\right)-\Upsilon\left(\frac{T-t_i}{\chi}\right)\right)f^-(t_i,x,\xi)\nabla_\xi U(x,\xi)\cdot h.\end{aligned} \tag{3.42}$$

Concerning the time direction, we have

$$\begin{aligned}\partial_t f^+(t_i,x,\xi)\\ &= \left[\Upsilon\left(\frac{t_i}{\chi}\right)+\Upsilon\left(\frac{T-t_i}{\chi}\right)+\left(1-\Upsilon\left(\frac{t_i}{\chi}\right)-\Upsilon\left(\frac{T-t_i}{\chi}\right)\right)U(x,\xi)\right]\partial_t f^-(t_i,x,\xi)\\ &\quad -\frac{1}{\chi}\left[\Upsilon'\left(\frac{t_i}{\chi}\right)-\Upsilon'\left(\frac{T-t_i}{\chi}\right)\right](1-U(x,\xi))f^-(t_i,x,\xi). \end{aligned} \tag{3.43}$$

From (3.17), we finally get

$$\begin{aligned}&\nabla_x f^+(t_i,x,\xi)\cdot\xi =\\ &\left[\Upsilon\left(\frac{t_i}{\chi}\right)+\Upsilon\left(\frac{T-t_i}{\chi}\right)+\left(1-\Upsilon\left(\frac{t_i}{\chi}\right)-\Upsilon\left(\frac{T-t_i}{\chi}\right)\right)U(x,\xi)\right]\nabla_x f^-(t_i,x,\xi)\cdot\xi\\ &-\mathscr{C}(t_i,x,\xi)f^-(t_i,x,\xi), \end{aligned} \tag{3.44}$$

with

$$\begin{aligned}\mathscr{C}(t_i,x,\xi) \coloneqq{}& \frac{1}{\chi}\left[\Upsilon'\left(\frac{t_i}{\chi}\right)-\Upsilon'\left(\frac{T-t_i}{\chi}\right)\right](1-U(x,\xi))\\ &+\left[1-\Upsilon\left(\frac{t_i}{\chi}\right)-\Upsilon\left(\frac{T-t_i}{\chi}\right)\right]\nabla_\xi U(x,\xi)\cdot\nabla\phi^g(t_i,x).\end{aligned}$$

Consequently, at each time a characteristic curve meets $\gamma^-$, one has

$$\nabla_{t,x,\xi}f^+(t_i,x,\xi)=\alpha\nabla_{t,x,\xi}f^-(t_i,x,\xi)+B_i, \tag{3.45}$$

with $0\leqslant\alpha\leqslant 1$ and $B_i$ which coordinates in the basis $(\tau,\xi)$ are bounded by $\beta|f^-(t_i,x,\xi)|$, with $\beta$ independent from $f$. Note that when $(x,\xi)$ belongs to $\gamma^-$, the matrix of the transformation from the base $(\tau,\xi)$ to the base $(\tau,v)$ is bounded, hence $B_i$ itself is bounded by $\beta|f^-(t_i,x,\xi)|$, with $\beta$ independent from $f$. With (3.40), this leads to the relation valid for times $t\in[t_i^+,t_{i+1}^-]$:

$$\nabla_{x,\xi}\tilde{V}[g](t,x,\xi)=\tilde{\alpha}_i\nabla_{x,\xi}W[g](t,x,\xi)+\sum_{j=1}^{i}\mathscr{B}_j, \tag{3.46}$$

with $0\leqslant\tilde{\alpha}_i\leqslant 1$ and $\mathscr{B}_j$ bounded (in $L^\infty$ norm) by $\beta|f_j|\,||\nabla_{x,\xi}(X,\Xi)||$, where $f_j$ is given by $f_j=f(t_j,X(t_j,t,x,\xi),\Xi(t_j,t,x,\xi))$. Note that, thanks to Lemma 1, $||\nabla_{x,\xi}(X,\Xi)||$ is uniformly bounded for $g\in\mathscr{S}_\varepsilon$. Now using again (3.34), we get

$$|f_j|\leqslant K(1+|\xi|)^{-\gamma}||(1+|\xi|)^\gamma f_0||_{B^0(\mathbb{T}^2\times\mathbb{T}^2)}.$$

With (3.39), one gets, for $(t,x,\xi)\in Q_T\backslash\Sigma_T$,

$$|\nabla_{x,\xi}\tilde{V}[g](t,x,\xi)|\leqslant|\nabla_{x,\xi}W[g](t,x,\xi)| + c(||\Lambda g||_{B^{0,\delta_1}(\Omega_T)})|||(1+|\xi|)^{\gamma}f_0||_{B^0(\mathbb{T}^2\times\mathbb{R}^2)}. \tag{3.47}$$

Now, using Lemma 1, we have the estimate on $W[g]$, for $|\xi-\xi'|\leqslant 1$ and $t\in[0,T]$:

$$|\nabla_{x,\xi}W[g](t,x,\xi)|\leqslant c(||\Lambda g||_{B^{0,\delta_1}(\Omega_T)})||\nabla f_0||_{C^0(\mathbb{T}^2\times\mathbb{R}^2)}, \tag{3.48}$$

where in fact $c(||\Lambda g||_{B^{0,\delta_1}(\Omega_T)})$ depends only on $\mathscr{S}_\varepsilon$. We consequently get that for $(t,x,\xi)\in Q_T\backslash\Sigma_T$,

$$|\nabla_{x,\xi}\tilde{V}[g](t,x,\xi)|\leqslant c(||\Lambda g||_{B^{0,\delta_1}(\Omega_T)})(||\nabla f_0||_{C^0(\mathbb{T}^2\times\mathbb{R}^2)}+||(1+|\xi|)^{\gamma}f_0||_{B^0(\mathbb{T}^2\times\mathbb{R}^2)}).$$

Using (3.17), we deduce

$$|\partial_t\tilde{V}[g](t,x,\xi)|\leqslant c(||\Lambda g||_{B^{0,\delta_1}(\Omega_T)})(1+|\xi|) \times[||\nabla f_0||_{C^0(\mathbb{T}^2\times\mathbb{R}^2)}+||(1+|\xi|)^{\gamma}f_0||_{B^0(\mathbb{T}^2\times\mathbb{R}^2)}].$$

Now for $x,x'\in\mathbb{T}^2\backslash\omega$ such that $|x-x'|\leqslant r_0$ (in order that $[x,y]$ does not cut $\bar{B}(x_0,r_0)$), we deduce that one has

$$|\tilde{V}[g](t,x,\xi)-\tilde{V}[g](t',x',\xi')|\leqslant C(1+|\xi|) \times(||\nabla f_0||_{B^0(\mathbb{T}^2\times\mathbb{R}^2)}+||(1+|\xi|)^{\gamma}f_0||_{B^0(\mathbb{T}^2\times\mathbb{T}^2)})|(t,x,\xi)-(t',x',\xi')|$$

and

$$|\tilde{V}[g](t,x,\xi)-\tilde{V}[g](t,x',\xi')| \leqslant C(||\nabla f_0||_{B^0(\mathbb{T}^2\times\mathbb{R}^2)}+||(1+|\xi|)^{\gamma}f_0||_{B^0(\mathbb{T}^2\times\mathbb{T}^2)})|(x,\xi)-(x',\xi')|.$$

Now for $x,x'\in\mathbb{T}^2\backslash\omega$ such that $|x-x'|\geqslant r_0$, one gets the same type of inequality, using the $L^\infty$ estimate. This ends the proof of Lemma 2. $\square$

**Remark 5.** Note that if we use (1.6), we get all the same as for (3.34) that

$$|f(t,x,\xi)|\leqslant K||(1+|\xi|)^{\gamma+1}f_0||_{B^0(\mathbb{T}^2\times\mathbb{T}^2)}(1+|\xi|)^{-\gamma-1}.$$

Then one can precise (3.47) by

$$|\nabla_{x,\xi}\tilde{V}[g](t,x,\xi)|\leqslant|\nabla_{x,\xi}W[g](t,x,\xi)|+c||(1+|\xi|)^{\gamma+1}f_0||_{B^0(\mathbb{T}^2\times\mathbb{R}^2)}(1+|\xi|)^{-\gamma}.$$

Using the boundedness of $\nabla_{x,\xi}(X,\Xi)$ and (3.32)–(3.33), one can also precise (3.48) by

$$|(1+|\xi|)^{\gamma}\nabla_{x,\xi}W[g](t,x,\xi)|\leqslant c(||\Lambda g||_{B^{0,\delta_1}(\Omega_T)})||(1+|\xi|)^{\gamma}\nabla f_0||_{B^0(\mathbb{T}^2\times\mathbb{R}^2)},$$

and one gets finally

$$|(1+|\xi|)^{\gamma}\nabla_{x,\xi}\tilde{V}[g](t,x,\xi)|\leqslant c(||\Lambda g||_{B^{0,\delta_1}(\Omega_T)}) \times (||(1+|\xi|)^{\gamma}\nabla f_0||_{B^0(\mathbb{T}^2\times\mathbb{R}^2)} + ||(1+|\xi|)^{\gamma+1}f_0||_{B^0(\mathbb{T}^2\times\mathbb{R}^2)}). \tag{3.49}$$

This is important for uniqueness.

It follows from the construction of $\Pi$ that the same estimates as in Lemma 2 hold for any $(x,x')\in(\mathbb{T}^2)^2$ for $V[g]$. Now to conclude about point c, it suffices to interpolate (3.34) and (3.35) to get the desired estimate in $[0,T]\times(\mathbb{T}^2\backslash\omega)\times\mathbb{R}^2$. Using the construction of $\Pi$, we get that for the index $\delta_2$ defined in (3.9), one has

$$||V[g]-\bar{f}||_{B^{\delta_2}(Q_T)}\leqslant c_2[||f_0||_{B^1(\mathbb{T}^2\times\mathbb{R}^2)}+||(1+\xi)^{\gamma}f_0||_{B^0(\mathbb{T}^2\times\mathbb{R}^2)}]. \tag{3.50}$$

(a) The last point is to prove that, if $f_0$ is small enough, one gets

$$||\Lambda(V[g]-\bar{f})||_{C^{\delta_1}(\Omega_T)}\leqslant\varepsilon. \tag{3.51}$$

Let us start with the $L^{\infty}$-norm before the $C^{\delta_1}$ one. One has

$$|f(t,x,\xi)|\leqslant|f_0[(X^g,\Xi^g)(0,t,x,\xi)]|.$$

It follows from (3.34) that

$$\left|\int_{\mathbb{R}^2}f(t,x,\xi)\,d\xi\right|\leqslant c_1||f_0(1+|\xi|)^{\gamma}||_{L^{\infty}(\mathbb{T}^2\times\mathbb{R}^2)}\int_{\mathbb{R}^2}(1+|\xi|)^{-\gamma}\,d\xi.$$

Hence

$$\left|\int_{\mathbb{R}^2}f(t,x,\xi)\,d\xi\right|\leqslant c_1'||f_0(1+|\xi|)^{\gamma}||_{L^{\infty}(\mathbb{T}^2\times\mathbb{R}^2)}.$$

One deduces from (3.23) that

$$||\Lambda(V[g]-\bar{f})||_{L^{\infty}(\Omega_T)}\leqslant C||f_0(1+|\xi|)^{\gamma}||_{L^{\infty}(\mathbb{T}^2\times\mathbb{R}^2)}. \tag{3.52}$$

Now let us deal with the $C^{\delta_1}$-norm. We introduce the following notation:

$$|g|_{\alpha}^{\beta}:=\sup_{\substack{x\neq x'\\ \xi,\xi'\ \text{s.t.}\ |\xi-\xi'|\leqslant 1}}\left[(1+|\xi|)^{\beta}\frac{|g(t,x,\xi)-g(t',x',\xi')|}{|(t,x,\xi)-(t',x',\xi')|^{\alpha}}\right].$$

Now it follows by interpolation between (3.34) and (3.50), that for a certain constant $C$ independent from $f_0$, for $\tilde{\gamma} = \frac{n+\gamma}{2}$ and $\delta_1$ defined in (3.9), one has

$$|V[g] - \bar{f}|^{\tilde{\gamma}}_{\delta_1} \leqslant C||f_0||^{\delta_1}_{B^1(\mathbb{T}^2\times\mathbb{R}^2)}[||f_0||_{B^1(\mathbb{T}^2\times\mathbb{R}^2)} + ||(1+|\xi|)^{\gamma} f_0||_{C^0(\mathbb{T}^2\times\mathbb{R}^2)}]^{1-\delta_1}.$$

We deduce that, for $f_0$ small enough,

$$||\Lambda(V_\varepsilon[g] - \bar{f})||_{C^{\delta_1}(\Omega_T)} \leqslant \varepsilon.$$

This finally proves $V(\mathscr{S}) \subset \mathscr{S}$, for $f_0$ small enough (depending on $\varepsilon$).

We then conclude by Schauder's Theorem, that there exists a fixed point $g^\star \in \mathscr{S}_\varepsilon$ of the operator $V_\varepsilon$. We now have to prove that such a $g^\star$ answers to the problem (at least for $\varepsilon$ small enough).

### 3.4. *The final state*

From the construction, Eqs. (1.1)–(1.2) and (1.3) are clearly satisfied by $g^\star$ for a certain $G$ supported in $[0, T] \times \omega \times \mathbb{R}^2$:

$$\begin{aligned} \partial g^\star + \xi \cdot \nabla_x g^\star + \nabla_x \phi^{g^\star} \cdot \nabla_\xi g^\star &= \bar{G} + (\nabla_x \phi^{g^\star} - \nabla_x \varphi) \cdot \nabla_\xi \bar{f} \\ &\quad + [\partial_t + \xi \cdot \nabla_x + \nabla_x \phi^{g^\star} \cdot \nabla_\xi] \Pi(\tilde{V}[g^\star]). \end{aligned}$$

What we have left to establish is (3.1) (at least for small $\varepsilon$). This is done in two steps: first, we show that property (3.4) is still true for the perturbed system, that is when the action of $\phi^{g^\star} - \varphi$ is taken into account, at least if $\varepsilon$ is small enough (and with a slightly larger radius for the ball). Then in a second step, we show that particles in $B(x_0, r_0/2)$, if fast enough, do meet $\gamma^{3-}$ during the process, which allows to conclude.

*Step* 1: The first step is a consequence of the following Gronwall's inequality (as for (3.27)): for any $s, t \in [0, T]$,

$$\begin{aligned} &||(\bar{X}, \bar{\Xi})(t, s, x, \xi) - (X^{g^\star}, \Xi^{g^\star})(t, s, x, \xi)||_{B^0(\mathbb{T}^2\times\mathbb{R}^2)} \\ &\quad \leqslant C||\nabla\varphi - \nabla\phi^{g^\star}||_{B^0(Q_T)} e^{T||\nabla\varphi||_{B^0([0,T],B^1(\mathbb{T}^2\times\mathbb{R}^2))}}. \end{aligned} \tag{3.53}$$

This involves that for some $c_3 > 0$,

$$||X^{\bar{f}} - X^{g^\star}||_{B^0([0,T]\times[0,T]\times\mathbb{T}^2\times\mathbb{R}^2)} < c_3\varepsilon. \tag{3.54}$$

It follows from (3.4) and (3.54) that, for relevant $\varepsilon$,

$$\exists M>0, \forall x\in\mathbb{T}^2, \forall\xi\in\mathbb{R}^2\backslash\{0\} \text{ such that } |\xi|\geqslant M, \exists t\in(0,T),$$
$$X^{g^\star}(t,0,x,\xi)\in B(x_0,r_0/2). \tag{3.55}$$

*Step*: 2 For $x\in B(x_0,r_0)$ and $\xi\in\mathbb{R}^2\backslash\{0\}$, we introduce $P_S(x,\xi)$ as the point in $S(x_0,r_0)$ last met by $x+t\xi$, with $t<0$. Let us prove that

$$\forall\eta>0,\ \exists\tilde{M}, \forall t\in[T/4,3T/4],\ \forall x\in B(x_0,r_0),\ \forall\xi\in\mathbb{R}^2 \quad \text{s.t. } |\xi|\geqslant\tilde{M}, \exists\tilde{t}\in(0,t) \quad \text{s.t.}$$
$$X^{g^\star}(\tilde{t},t,x,\xi)\in S(x_0,r_0), |X^{g^\star}(\tilde{t},t,x,\xi)-P_S(x,\xi)|<\eta,$$
$$\text{and } \forall s\in[\tilde{t},t],\ |\Xi^{g^\star}(s,t,x,\xi)-\xi|<\eta. \tag{3.56}$$

Indeed, as follows from (2.3), one has

$$|X^{g^\star}(s,t,x,\xi)-x-(s-t)\xi|+|\Xi^{g^\star}(s,t,x,\xi)-\xi|\leqslant c(||\Lambda g^\star||_\infty)|s-t|. \tag{3.57}$$

Fix $\eta'>0$. For $|\xi|$ large enough (say $|\xi|\geqslant\tilde{M}$), one has $d(x+(s-t)\xi,x_0)\geqslant 2r_0$, for a certain $s<t$, with $|s-t|<\eta'$. Reducing $\eta'$ if necessary (hence, enlarging $\tilde{M}$), we see that (3.57) involves that $|X^{g^\star}(s,t,x,\xi)-x_0|\geqslant\frac{3}{2}r_0$. Hence, one gets the existence of $\tilde{t}$. Finally, we get $|X^{g^\star}(\tilde{t},t,x,\xi)-P_S(x,\xi)|<\eta$ and $|\Xi^{g^\star}(s,t,x,\xi)-\xi|<\eta$ in $[\tilde{t},t]$ from (3.57), reducing again $\eta'$ (and again enlarging $\tilde{M}$) if necessary.

Now, it is easy to show that a straight line arising from $B(x_0,r_0/2)$ cuts $S(x_0,r_0)$ with an angle to the normal at the circle of value at most $\pi/6$. Consequently, using again (3.57), one gets that, at least for small $\varepsilon$,

$$\exists M'>0, \forall x\in\mathbb{T}^2, \forall\xi\in\mathbb{R}^2\backslash\{0\} \quad \text{such that } |\xi|\geqslant M', \exists t\in(0,3T/4),$$
$$(X^{g^\star},\Xi^{g^\star})(t,0,x,\xi)\in\gamma^{3-}. \tag{3.58}$$

It follows then that for $\xi$ large enough, one gets

$$g^\star(T,\cdot,\xi)=0,$$

which is what we wanted for this section. Besides, it follows from Lemma 2 and from the construction that one has

$$||g^\star(T,\cdot,\cdot)||_{B^1(\mathbb{T}^2\times\mathbb{R}^2)}+||(1+|\xi|)^\gamma g^\star(T,\cdot,\cdot)||_{L^\infty(\mathbb{T}^2\times\mathbb{R}^2)}$$
$$\leqslant C[||f_0||_{B^1(\mathbb{T}^2\times\mathbb{R}^2)}+||(1+|\xi|)^\gamma f_0||_{L^\infty(\mathbb{T}^2\times\mathbb{R}^2)}]. \tag{3.59}$$

### 3.5. Uniqueness

The uniqueness of the solution for the system with interior control is proven approximately the same way as for the "homogeneous" (uncontrolled) case. Indeed, the source term disappears when doing the difference of two Eqs. (1.1) computed for two different potential solutions (see [17, Section 8]).

The main point is that under assumption (1.6), the solution described above satisfies

$$|\nabla_{x,\xi} g^{\star}(t,x,\xi)| \leqslant C(f_0)(1+|\xi|)^{-\gamma}.$$

This follows from (3.49) and the construction of $\Pi$. Now $g^{\star}$ is unique among the solutions that satisfy

$$g \in C^1(Q_T),$$

$$|g(t,x\xi)| + |\nabla_{x,\xi} g(t,x,\xi)| \leqslant C(1+|\xi|)^{-\gamma},$$

$$\nabla\phi \in B^{0,1}(\Omega_T),$$

as obtained in [17].

## 4. Theorem 1: the problem with compact velocities

In this section, we consider $f_0 \in B^1(\mathbb{T}^2 \times \mathbb{R}^2)$, which moreover satisfies for a certain $M > 0$,

$$\forall (x,\xi) \in \mathbb{T}^2 \times \mathbb{R}^2, |\xi| \geqslant M \Rightarrow f_0(x,\xi) = 0, \tag{4.1}$$

and moreover (1.6). Let us remark that (1.6) is not really a consequence of (4.1) since $\kappa$ and $\kappa'$ in (1.6) are supposed to be small; but it is satisfied by the final value of $g^{\star}$ in Section 3 if again $f_0$ is small enough—see (3.59).

The previous $M$ can be given only in terms of $\omega$ and $T$, and is independent from the choice of $f_0$ (in its neighborhood of 0)—see Section 3.

### 4.1. The idea of the treatment of low velocities

The principal idea is to reduce to the case studied in Section 3.

Indeed, by a control localized in $\omega$, one can "accelerate" all particles at velocity $\xi$ with $|\xi| \leqslant M$, in such a way that at the end of the process, $f(t,x,\cdot)$ is supported in $\{\xi/|\xi| \geqslant M\}$. Then, one can apply again the control described in the previous section; at the end one gets a distribution function supported in $\omega$ in variable $x$.

In the next paragraph, we state a proposition that shows that such a process is possible. Then we sketch the proof of the construction of $f$.

## 4.2. The central proposition

**Proposition 2.** *There exist* $\tau > 0$ *and* $\varphi \in C^\infty([0,\tau] \times \mathbb{T}^2; \mathbb{R})$ *satisfying*

$$\mathrm{Supp}(\varphi) \subset (0,\tau) \times \mathbb{T}^2, \tag{4.2}$$

$$\Delta\varphi = 0 \quad \text{in } [0,\tau] \times (\mathbb{T}^2 \backslash \omega), \tag{4.3}$$

*such that for any* $f_0 \in B^1(\mathbb{T}^2 \times \mathbb{R}^2)$ *satisfying* (4.1), *one has, if we denote by* $(X, \Xi)$ *the characteristics corresponding to the potential* $\varphi$:

$$f_0'(x,\xi) \coloneqq f_0(X(0,\tau,x,\xi), \Xi(0,\tau,x,\xi)) = 0$$
$$\text{for all } (x,\xi) \quad \text{in } (\mathbb{T}^2 \backslash \omega) \times (\mathbb{R}^2 \backslash [B(0,\tilde{M}) \backslash B(0, M+1)]), \tag{4.4}$$

*for a certain* $\tilde{M} > 0$ *independent from* $f_0$. *For some* $K > 0$ *independent from* $f_0$ *one has also*:

$$\begin{cases} \|(1+|\xi|)^{\gamma+1} f_0'\|_{L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)} \leqslant K \|(1+|\xi|)^{\gamma+1} f_0\|_{L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)}, \\ \|f_0'\|_{B^1(\mathbb{T}^2 \times \mathbb{R}^2)} \leqslant K \|f_0\|_{B^1(\mathbb{T}^2 \times \mathbb{R}^2)}. \end{cases} \tag{4.5}$$

This proposition relies on the following lemma.

**Lemma 3.** *For any nonempty open set* $\mathcal{O}$ *in the two-dimensional torus, there exists* $\theta \in C^\infty(\mathbb{T}^2; \mathbb{R})$ *satisfying*

$$\Delta\theta = 0 \quad \text{in } \mathbb{T}^2 \backslash \mathcal{O}, \tag{4.6}$$

$$|\nabla\theta(x)| > 0 \quad \text{for any } x \text{ in } \mathbb{T}^2 \backslash \mathcal{O}. \tag{4.7}$$

We prove this lemma in the appendix.

**Proof of Proposition 2.** We introduce a $\theta$ as in Lemma 3, with $\mathcal{O}$ chosen as a ball $B(x_0, r_0)$ such that $B(x_0, 2r_0) \subset \omega$. Define $m$ as the lower bound of $|\nabla\theta(x)|$ in $\mathbb{T}^2 \backslash \mathcal{O}$.

Then, the idea is to fix $\varphi(t,x) \coloneqq a\Lambda(bt)\theta(x)$, defined in $[0, 1/b] \times \mathbb{T}^2$, where $\Lambda \in C_0^\infty(]0,1[)$ satisfies

$$\begin{cases} \Lambda \geqslant 0, \\ \int_{[0,1]} \Lambda = 1, \end{cases}$$

and where $a$ and $b \geqslant 1$ are to be chosen large enough, for fixed $c = a/b$.

With such a $\varphi$ one gets that for any $(x,\xi)\in\mathbb{T}^2\times B_{\mathbb{R}^2}(0,M)$, one has during $[0,\frac{1}{b}]$ that

$$|\Xi(t,0,x,\xi)-\xi|\leqslant\frac{a}{b}||\nabla\theta||_\infty,$$

and it follows that, during $[0,\frac{1}{b}]$,

$$|X(t,0,x,\xi)-x|\leqslant\frac{1}{b}(M+c||\nabla\theta||_\infty).$$

Hence for $b$ large enough, one has $X(t,0,x,\xi)\in\mathbb{T}^2\backslash B(x_0,r_0)$ for any $(t,x,\xi)\in[0,1/b]\times(\mathbb{T}^2\backslash\omega)\times B_{\mathbb{R}^2}(0,M)$. Then one deduces that

$$\left|\Xi\left(\frac{1}{b},0,x,\xi\right)-\xi+c\nabla\theta(x)\right|\leqslant C(\theta)\frac{1}{b}(M+c||\nabla\theta||_\infty), \tag{4.8}$$

where $C(\theta)$ depends on the derivatives of $\nabla\theta$. Now we choose $c=\frac{2M+2}{m}$ and let $b$ become large in order that the right-hand side of (4.8) is less than $1/2$. This gives (4.4). One just has to check (4.5). The first part in (4.5) can be seen as a consequence of (3.33). The second one is a consequence of the Lipschitzian character of the characteristics (see Lemma 1). $\square$

### 4.3. *Sketch of the construction of $f$*

Now we can describe the control used to "treat" $f_0$ such as described at the beginning of the section. We define as previously $\bar{f}$ in $[0,\tau]\times\mathbb{T}^2\times\mathbb{R}^2$ by (3.6) (with here $\varphi$ defined as in Proposition 2).

From (4.7), we deduce that

$$\mathrm{Ind}_{S(x_0,r_0)}(\nabla\theta)=0.$$

Hence, $\nabla\theta$ can be extended inside $B(x_0,r_0)$ in such a way that it does not vanish (call $V:\mathbb{T}^2\to\mathbb{R}^2$ such an extension). Call again $m$ the lower bound for $|V(x)|$. Now arguing as previously, one gets the following for the flow of $(\xi,\mathscr{V})$ (i.e. the characteristics for the field $\mathscr{V}$), with $\mathscr{V}$ defined by $\mathscr{V}(t,x)=a\Lambda(bt)V(x)$: for suitable $\tau>0$,

$$f_0'(x,\xi):=f_0(X^{\mathscr{V}}(0,\tau,x,\xi),\Xi^{\mathscr{V}}(0,\tau,x,\xi))=0$$
$$\text{for all } (x,\xi) \text{ in } \mathbb{T}^2\times(\mathbb{R}^2\backslash[B(0,\tilde{M})\backslash B(0,M+1)]).$$

We consider $\mathscr{J}\in C^\infty(\mathbb{T}^2)$ satisfying

$$\begin{cases}0\leqslant\mathscr{J}\leqslant 1,\\ \mathscr{J}=1 \quad \text{in } \mathbb{T}^2\backslash B(x_0,\frac{3}{2}r_0),\\ \mathscr{J}=0 \quad \text{in } \bar{B}(x_0,r_0).\end{cases}$$

We introduce $\pi$ as in Section 3, and $\Pi : C^0(([0,\tau] \times [\mathbb{T}^2 \backslash B(x_0, 2r_0)] \times \mathbb{R}^2) \to C^0([0,\tau] \times \mathbb{T}^2 \times \mathbb{R}^2)$ with a slight simplification:

$$(\Pi f)(t,x,\xi) := \pi[f(t,\cdot,\cdot)](x,\xi). \tag{4.9}$$

To $g \in \mathscr{S}_\varepsilon$, we associate $\phi^g$ by (3.16), and then $J^g$ by

$$J^g = [\mathscr{J}(x)\nabla\phi^g + (1 - \mathscr{J}(x))\mathscr{V}].$$

We introduce the following operator $W$ which maps $g \in \mathscr{S}_\varepsilon$ to $f$ satisfying

$$f_{|t=0} = f_0,$$

$$\partial_t f + \xi \cdot \nabla_x f + J^g \cdot \nabla_\xi f = 0 \quad \text{in } [0,\tau] \times \mathbb{T}^2 \times \mathbb{R}^2.$$

We finally define $V$ as the operator given by

$$V[g] = \bar{f} + \Pi(W[g]).$$

It can be proven to have a fixed point $f^\star$ close to $\bar{f}$, solution of the system, at least for small $f_0$, exactly as in the previous section (or as in [17]). We omit the details; let us just underline that $\Pi$ is employed here not to ensure the regularity of the solution inside $\omega$, but in order that the resulting solution found here can be smoothly glued with the one of Section 3.

Let us just check that, at least for $\varepsilon$ small enough, the final value of this solution is small and "well" supported in $\xi$. The first point is done as previously, using Lemma 1 and the point b in Section 3.3. The second one is a consequence of Gronwall's lemma which leads to

$$||(X^\star, \Xi^\star) - (\bar{X}, \bar{\Xi})||_{B^0([0,\tau]\times\mathbb{T}^2\times\mathbb{R}^2)} \leqslant C\varepsilon,$$

where $C$ can be described in terms of $\bar{f}$, and where $(X^\star, \Xi^\star)$ and $(\bar{X}, \bar{\Xi})$ stand, respectively, for the flows of $(\xi, J^{f^\star})$ and $(\xi, J^{\bar{f}}) = (\xi, \mathscr{V})$.

Once applied the control of this section, we apply again the control of Section 3 with $f^\star(\tau,\cdot,\cdot)$ as initial condition to get a complete solution (as we explained, $f^\star(\tau,\cdot,\cdot)$ is small provided the original $f_0$ was small enough). In this second use of the control of Section 3, one can use $\Pi$ as in (4.9), since at the end of the control process of Section 4 the distribution function is already of the form $\pi(\tilde{f}_0)$ for a certain $\tilde{f}_0$.

Uniqueness is proven as in Section 3.5. This concludes the proof of Theorem 1.

## 5. Theorem 2: the problem with compact velocities according to $v$

We now turn to the proof of Theorem 2. In this section, we consider $f_0 \in B^1(\mathbb{T}^n \times \mathbb{R}^n)$ satisfying (1.8), which moreover satisfies for a certain $M > 0$,

$$\forall (x,\xi) \in \mathbb{T}^n \times \mathbb{R}^n, |\xi \cdot v| \geqslant M \Rightarrow f_0(x,\xi) = 0. \tag{5.1}$$

Again, most of the time, we will use only the decreasing of $f_0$ in $|\xi|^{-\gamma}$ and the boundedness of $\nabla_{x,\xi} f_0$. We will also use a kind of compatibility condition on $f_0$, satisfied by any distribution function at the end of the process of Section 6, and which we describe later.

Remark that one has

$$\forall \lambda \in (0,1), \forall (x,\xi) \in \mathbb{T}^n \times \mathbb{R}^n, |\xi \cdot v| \geqslant M \Rightarrow f_0^{\lambda}(x,\xi) = 0, \tag{5.2}$$

as $|\xi \cdot v| \geqslant \lambda M$ is sufficient. (We put $\lambda$ as an exponent for the scaled distribution function, see Section 2.4.)

We first introduce a function $\bar{f}$ which is central in the proof.

### 5.1. The function $\bar{f}$

Consider a function $\mathscr{Y} \in C_0^{\infty}(]0,T[)$ satisfying

$$\begin{cases} \operatorname{Supp} \mathscr{Y} \subset (0, T/4), \\ \mathscr{Y} \geqslant 0, \\ \int_{[0,T]} \mathscr{Y} = 1. \end{cases} \tag{5.3}$$

Before describing precisely the function $\bar{f}$, we begin with a remark.

**Remark 6.** One can decompose $\mathbb{T}^n$ into "slices", each slice being obtained by thickening hyperplanes parallel to $H$ with an arbitrary length. In particular, one can cut $\mathbb{T}^n$ in the following way:

$$\mathbb{T}^n = \bigcup_{i=1}^{N} \mathscr{H}_i, \tag{5.4}$$

with $\mathscr{H}_i := x_i + \bar{H} + ]-r, r[v$, for some $x_i \in \mathbb{T}^n$ and with $r \in \mathbb{R}^{+*}$ small enough to get that

$$\forall i \in \{1, \dots, N\}, \ X^{\mathscr{Y}(t)\tilde{v}}(t_i, 0, \mathscr{H}_i, 0) \subset H + [-d/2, d/2]v, \tag{5.5}$$

with $\tilde{v}$ given by $\tilde{v} := Av$ for a certain $A \in \mathbb{R}^{+*}$ depending only on $T$, and with $t_i := T/4 + iT/2N$.

**Remark 7.** Consequently, there exists $\mu > 0$ such that

$$X^{\mathscr{Y}(t)\tilde{v}}(t_i, 0, \mathscr{H}_i, \xi) \subset H + [-2d/3, 2d/3]v, \quad \forall i \in \{1, \dots, N\}, \tag{5.6}$$

whenever $|\xi \cdot v| \leqslant \mu$.

**Remark 8.** Remark that $X^{\mathscr{Y}(t)\tilde{v}}$ has also the following property: there exists $c \in \mathbb{R}^{+*}$ such that, for any $x, y \in \mathbb{T}^n$, one has

$$c^{-1}|x - y| \leqslant |X^{\mathscr{Y}(t)\tilde{v}}(t, 0, x, 0) - X^{\mathscr{Y}(t)\tilde{v}}(t, 0, y, 0)| \leqslant c|x - y|. \tag{5.7}$$

Let us now describe the function $\bar{f}$. In the domain $\mathbb{T}^n \backslash (H + [-d, d]v)$, $x \mapsto \tilde{v}$ coincides with the gradient of a harmonic function. Call $\varphi$ a function in $C^\infty(\mathbb{T}^n; \mathbb{R})$, whose gradient coincides in $\mathbb{T}^n \backslash (H + [-d, d]v)$ with $\tilde{v}$; this function is harmonic in $\mathbb{T}^n \backslash \omega$. Note that Remark 8 is no longer necessarily true when considering $\nabla \varphi$ instead of $\tilde{v}$.

Now consider a function $\mathscr{Z} \in C^\infty(\mathbb{R}^n; \mathbb{R})$ as in (3.5). Define $\bar{f} = \bar{f}(t, x, \xi)$ as

$$\bar{f}(t, x, \xi) := \mathscr{Y}(t)\mathscr{Z}(\xi)\Delta\varphi(x). \tag{5.8}$$

Of course, $\bar{f}$ satisfies (1.1) in $[0, T] \times \mathbb{T}^n \times \mathbb{R}^n$ with a source term given by $\bar{G} := \partial_t \bar{f} + \xi \cdot \nabla_x \bar{f} + \mathscr{Y}(t)\nabla\varphi(x) \cdot \nabla_\xi \bar{f}$ supported in $[0, T] \times \omega \times \mathbb{R}^n$ and $\bar{\phi}(t, x) := \mathscr{Y}(t)\varphi(x)$ satisfies Eq. (1.2) corresponding to $\bar{f}$ (and $\bar{f}(0, \cdot, \cdot) = 0$), up to a function of $t$.

### 5.2. *The operator $V_\varepsilon^\lambda$*

Here we introduce a certain operator $V_\varepsilon^\lambda$, which depends on two parameters $\lambda$ and $\varepsilon$ (intended both to be small, and both systematically supposed to be in $(0, 1)$). We will show that, for any $\varepsilon > 0$, $V_\varepsilon^\lambda$ has a fixed point if $\lambda$ is small enough (in terms of $\varepsilon$). Then we show that for $\varepsilon$ small enough, such a fixed point gives a solution to the problem.

We introduce a function $\mathfrak{H} \in C^\infty(\mathbb{T}^n; \mathbb{R})$ such that

$$\begin{cases} 0 \leqslant \mathfrak{H} \leqslant 1 & \text{in } \mathbb{T}^n, \\ \mathfrak{H} \equiv 0 & \text{in } H + [-d, d]v, \\ \mathfrak{H} \equiv 1 & \text{in } \mathbb{T}^n \backslash (H + [-\frac{3}{2}d, \frac{3}{2}d]v). \end{cases} \tag{5.9}$$

We define again a subset $\mathscr{S}_\varepsilon^\lambda$ of $B^{\delta_2}(Q_T)$ on which we will define the operator $V_\varepsilon^\lambda$:

$$\mathscr{S}_\varepsilon^\lambda := \{g \in B^{\delta_2}(Q_T) /$$

$$
\begin{aligned}
&\text{(a)} \quad \|\Lambda(g-\bar{f})\|_{C^{\delta_1}(\Omega_T)} \leqslant \varepsilon, \\
&\text{(b)} \quad \|(1+|\xi|)^{\gamma}(g-\bar{f})\|_{L^{\infty}(Q_T)} \leqslant c_1[\|f_0^{\lambda}\|_{B^1(\mathbb{T}^n\times\mathbb{R}^n)} + \|(1+\xi)^{\gamma} f_0^{\lambda}\|_{B^0(\mathbb{T}^n\times\mathbb{R}^n)}], \\
&\text{(c)} \quad \|g-\bar{f}\|_{B^{\delta_2}(Q_T)} \leqslant c_2\Big[\|f_0^{\lambda}\|_{B^1(\mathbb{T}^n\times\mathbb{R}^n)} + \|(1+\xi)^{\gamma} f_0^{\lambda}\|_{B^0(\mathbb{T}^n\times\mathbb{R}^n)}\Big], \\
&\text{(d)} \quad \forall t\in[0,T], \textstyle\int_{\mathbb{T}^n\times\mathbb{R}^n} g(t,x,\xi)\,dx\,d\xi = \int_{\mathbb{T}^n\times\mathbb{R}^n} f_0^{\lambda}(x,\xi)\,dx\,d\xi\},
\end{aligned} \tag{5.10}
$$

with $c_1$, $c_2$ to be fixed later depending only on $\gamma$, $T$ and $\omega$ (and hence on $(\bar{f},\varphi)$), but not on $\lambda$; here, $\delta_1$ and $\delta_2$ are fixed as follows

$$
\delta_1 = \frac{\gamma - n}{2(n+1)(\gamma+1)} \quad \text{and} \quad \delta_2 = \frac{\gamma}{\gamma+1}.
$$

For fixed $c_1$ and $c_2$ large enough depending only on $(\bar{f},\varphi)$, one has $\mathscr{S}_\varepsilon^\lambda \neq \emptyset$ for $\lambda$ small enough depending on $\varepsilon$ (for instance $f(t,x,\xi) = f_0^\lambda(x,\xi) + \bar{f}(t,x,\xi)$ belongs to $\mathscr{S}_\varepsilon^\lambda$ for $\lambda < \mu(\varepsilon)$—see (2.9)). From now, we suppose that this is the case; in particular, one gets $|\rho_0^\lambda| \leqslant \varepsilon$.

Consider $g \in \mathscr{S}_\varepsilon^\lambda$. To $g$, we first associate the corresponding solution $\phi^g$ of the Poisson equation, viz. (3.16) (with $\rho_0$ replaced by $\rho_0^\lambda$).

To $\phi^g$, we associate $V_g : [0,T] \times \mathbb{T}^n \to \mathbb{R}^n$ by

$$
V_g(t,x) \coloneqq \mathfrak{H}(x)\nabla\phi^g(t,x) + [1-\mathfrak{H}(x)]\mathscr{Y}(t)\tilde{v}(x). \tag{5.11}
$$

We then define the functions $f^k$ for $k = 0,1,\dots,N$ defined recursively as follows:

$$
\begin{cases}
f^0(0,x,\xi) = f_0^\lambda \quad \text{on } \mathbb{T}^n \times \mathbb{R}^n, \\
\partial_t f^0 + \xi\cdot\nabla_x f^0 + V_g\cdot\nabla_\xi f^0 = 0 \quad \text{in } [0,t_1]\times\mathbb{T}^n\times\mathbb{R}^n,
\end{cases} \tag{5.12}
$$

and then, for any $k \in \{1,\dots,N\}$,

$$
\begin{cases}
f^k(t_k,x,\xi) = \mathfrak{H}(x) f^{k-1}(t_k,x,\xi) \quad \text{on } \mathbb{T}^n \times \mathbb{R}^n, \\
\partial_t f^k + \xi\cdot\nabla_x f^k + V_g\cdot\nabla_\xi f^k = 0 \quad \text{in } [t_k,t_{k+1}]\times\mathbb{T}^n\times\mathbb{R}^n.
\end{cases} \tag{5.13}
$$

(We set $t_{N+1} \coloneqq T$.) We now consider, as in Section 3, a continuous linear extension operator $\bar{\pi}$ from $C^0(\mathbb{T}^n \backslash (H + ]-2d,2d[v);\mathbb{R})$ to $C^0(\mathbb{T}^n;\mathbb{R})$, and which has the same property that each $C^\alpha$-regular function is continuously mapped to a $C^\alpha$-regular function, for any $\alpha\in[0,1]$. Moreover, we manage again in order that for any $f \in C^0(\mathbb{T}^n\backslash(H+(-2d,2d)v))$, (3.19) occurs. From this operator, we deduce a new one $\pi : C^0([\mathbb{T}^n\backslash(H+]-2d,2d[v)]\times\mathbb{R}^n) \to C^0(\mathbb{T}^n\times\mathbb{R}^n)$ the same way as in Section 3, and we fix $c_\pi$ correspondingly. Note that $\pi$ depends on $\lambda$ because of (3.19). But the constant $c_\pi$ can be made independent from $\lambda \in (0,1)$, because

$$\int_{\mathbb{T}^n \times \mathbb{R}^n} f_0^\lambda(x,\xi)\,dx\,d\xi = \lambda^2 \int_{\mathbb{T}^n \times \mathbb{R}^n} f_0^1(x,\xi)\,dx\,d\xi.$$

Finally, we introduce the operator $\Pi$ as in (4.9).

We now define the functions

$$\tilde{V}_\varepsilon^\lambda[g] := \begin{cases} f^0 & \text{in } [0,t_1] \times \mathbb{T}^n \times \mathbb{R}^n, \\ f^k & \text{in } ]t_k, t_{k+1}] \times \mathbb{T}^n \times \mathbb{R}^n, \quad \text{for } k \in \{1,\dots,N\}, \end{cases} \tag{5.14}$$

$$V_\varepsilon^\lambda[g] := \bar{f} + \Pi(\tilde{V}_\varepsilon^\lambda[g]_{|[0,T]\times[\mathbb{T}^n \backslash (H+[-2d+2d]v)]\times\mathbb{R}^n}). \tag{5.15}$$

Again, $\tilde{V}$ is not necessarily continuous. As previously, we define $W$ to be the same operator as $\tilde{V}$, but where we replaced $\mathfrak{H}$ by 1, that is, $W[g]$ is transported by the flow of $(\xi, V_g)$ (this makes $W[g]$ continuous in $Q_T$). In this section, we denote by $(X^g, \Xi^g)$ and $(\bar{X}, \bar{\Xi})$ the flows of $(\xi, V_g)$ and $(\xi, V_{\bar{f}}) = (\xi, \mathscr{Y}(t)\tilde{v})$.

## 5.3. Regularity of $V_\varepsilon^\lambda[g]$

We have that, except at times $t_i$,

$$\partial_t \tilde{V}[g] + \xi \cdot \nabla_x \tilde{V}[g] + V_g \cdot \nabla_\xi \tilde{V}[g] = 0.$$

Consequently, the function $\tilde{V}_\varepsilon^\lambda[g]$ is given by the characteristic Eq. (2.4) during each interval $[t_i, t_{i+1})$. From the fact that $\Lambda g \in B^{\delta_1}(\Omega_T)$, one deduces, together with (2.1), (2.2), (2.4) and Lemma 1 that $f^0$ is of class $C^1(Q_{t_1})$.

Of course, it follows from the construction that $f^0(t_1,\cdot,\cdot)$ coincides with $f^1(t_1,\cdot,\cdot)$ on $\mathbb{T}^n \backslash (H + [-\frac{3}{2}, \frac{3}{2}]v) \times \mathbb{R}^n$. We have also that the support of $\tilde{V}[g]$ in $\xi$ stays bounded in the direction $v$. This follows from (2.3), (5.2) and from the fact that $V_g$ is uniformly bounded for $g$ in $\mathscr{S}_\varepsilon^\lambda$. Hence, the function $\tilde{V}[g]$ is of class $C^1$ in the domain $[t_1 - \alpha, t_1 + \alpha] \times \mathbb{T}^n \backslash (H+] - 2d, 2d[v) \times \mathbb{R}^n$, for a certain $\alpha > 0$.

By arguing similarly for times $t_2, \dots, t_N$, we get that $\tilde{V}[g]$ is of class $C^1$ in the domain $[0,T] \times [\mathbb{T}^n \backslash (H+] - 2d, 2d[v)] \times \mathbb{R}^n$, and hence, with (5.15) that $V_\varepsilon^\lambda[g] \in C^1(Q_T)$.

## 5.4. Finding a fixed point of $V_\varepsilon^\lambda$

Now our goal is to check that, for any $\varepsilon > 0$ (small), there exists $\mu(\varepsilon) > 0$ such that for any positive $\lambda < \mu(\varepsilon)$, the operator $V_\varepsilon^\lambda$ satisfies the assumptions for the Leray–Schauder fixed point Theorem on the domain $\mathscr{S}_\varepsilon^\lambda$. In order to avoid too heavy notations, we will sometimes forget the indices and exponents $\varepsilon$ and $\lambda$.

1. Again, $\mathscr{S}$ is a convex compact subset of $C^0(Q_T)$.

2. The continuity of $V$ can be proven in the same way as in Section 3. Consider a sequence $g_i$ of $\mathscr{S}$ converging to $g \in \mathscr{S}$, for the $C^0$ topology (we write $g_\infty = g$). Here, we denote by $(X^{g_i}, \Xi^{g_i})$ the characteristics corresponding to the flow of $(\xi, V_{g_i})$. As for (3.27), we have

$$\begin{aligned} ||(X^{g_i}, \Xi^{g_i}) - (X^g, \Xi^g)||_{C^0([0,T]^2 \times \mathbb{T}^n \times \mathbb{R}^n)} &\leqslant C||V_{g_i} - V_g||_{C^0(\Omega_T)} \\ &\leqslant C'||\nabla \phi^{g_i} - \nabla \phi^g||_{C^0(\Omega_T)}, \end{aligned}$$

where $C$ depends only on $\bar{f}$. Then it follows from the construction that $V[g_i] \to V[g]$ for the $C^0$ topology (first it converges pointwise in $[0,T] \times [\mathbb{T}^n \backslash (H + (-2d, 2d)v)] \times \mathbb{R}^n$, then, using again the compactness of $\mathscr{S}^\lambda_\varepsilon$, uniformly, and finally one uses the construction of $\Pi$).

3. Let us now verify the most problematic condition, viz. $V(\mathscr{S}^\lambda_\varepsilon) \subset \mathscr{S}^\lambda_\varepsilon$. We have to check the three first points in the definition of $\mathscr{S}^\lambda_\varepsilon$ (the last one is again consequence of the construction of $\Pi$).

It follows from the same Gronwall's inequality as (3.27) that for a constant $C$ depending on $\mathscr{Y}(t)\tilde{v}$, one has

$$||(X^g, \Xi^g) - (\bar{X}, \bar{\Xi})||_{B^0([0,T]^2 \times \mathbb{T}^n \times \mathbb{R}^n)} \leqslant C\varepsilon,$$

where $C$ does not depend on $g \in \mathscr{S}$. One can get a more precise inequality in the following way (when it is not explicit, the norm considered is the $L^\infty$ one):

$$\begin{aligned} &\frac{d}{dt^+}||\nabla(X^g, \Xi^g)(t,s,x,\xi) - \nabla(\bar{X}, \bar{\Xi})(t,s,x,\xi)|| \\ &\quad \leqslant ||\nabla \Xi^g(t,s,x,\xi) - \nabla \bar{\Xi}(t,s,x,\xi)|| \\ &\qquad + ||\nabla_x V_g(t, X^g(t,s,x,\xi))\nabla X^g(t,s,x,\xi) - \nabla_x V_{\bar{f}}(t, \bar{X}(t,s,x,\xi))\nabla \bar{X}(t,s,x,\xi)||, \end{aligned}$$

where $\nabla$ stands either for $\nabla_x$ or for $\nabla_\xi$. Now the last term is bounded as follows

$$\begin{aligned} &||\nabla_x V_g(t, X^g(t,s,x,\xi))\nabla X^g(t,s,x,\xi) - \nabla_x V_{\bar{f}}(t, \bar{X}(t,s,x,\xi))\nabla \bar{X}(t,s,x,\xi)|| \\ &\qquad \leqslant A + B + C, \end{aligned}$$

with

$$\begin{cases} A = & ||\nabla_x V_g(t, X^g(t,s,x,\xi))\nabla X^g(t,s,x,\xi) \\ & -\nabla_x V_g(t, X^g(t,s,x,\xi))\nabla \bar{X}(t,s,x,\xi)||, \\ B = & ||\nabla_x V_g(t, X^g(t,s,x,\xi))\nabla \bar{X}(t,s,x,\xi) \\ & -\nabla_x V_{\bar{f}}(t, X^g(t,s,x,\xi))\nabla \bar{X}(t,s,x,\xi)||, \\ C = & ||\nabla_x V_{\bar{f}}(t, X^g(t,s,x,\xi))\nabla \bar{X}(t,s,x,\xi) \\ & -\nabla_x V_{\bar{f}}(t, \bar{X}(t,s,x,\xi))\nabla \bar{X}(t,s,x,\xi)||. \end{cases}$$

Now

$$A \leqslant \|\nabla_x V_g\|_{B^0(\Omega_T)} \|\nabla X^g(t,s,x,\xi) - \nabla \bar{X}(t,s,x,\xi)\|_{B^0([0,T]^2 \times \mathbb{T}^n \times \mathbb{R}^n)},$$

$$B \leqslant \|\nabla_x V_g - \nabla_x V_{\bar{f}}\|_{B^0(\Omega_T)} \|\nabla \bar{X}\|_{B^0([0,T]^2 \times \mathbb{T}^n \times \mathbb{R}^n)},$$

$$C = 0.$$

It follows then by Gronwall's lemma that for a certain constant $C$, one has

$$\|(X^g, \Xi^g) - (\bar{X}, \bar{\Xi})\|_{L^\infty([0,T];B^1(\mathbb{T}^n \times \mathbb{R}^n))} \leqslant C \|V_g - V_{\bar{f}}\|_{B^{0,1}(\Omega_T)}$$
$$\leqslant C \|\nabla \phi^g - \nabla \bar{\phi}\|_{B^{0,1}(\Omega_T)}.$$

Hence, if $\varepsilon$ is small enough, then (5.7) is still valid when replacing $(\bar{X}, \bar{\Xi})$ by $(X^g, \Xi^g)$, precisely: $\exists c' > 0$ such that for any $\lambda \in (0, \mu(\varepsilon))$, for any $g \in \mathscr{S}^\lambda_\varepsilon$, one has

$$\forall (x,y) \in (\mathbb{T}^n)^2, \forall t \in [0,T],$$
$$(c')^{-1}|x-y| \leqslant |X^g(t,0,x,0) - X^g(t,0,y,0)| \leqslant c'|x-y|. \tag{5.16}$$

(Indeed, $V_g$ is close to $\mathscr{Y}(t)\tilde{v}$ in $C^1$-norm, which is not the case of $\nabla \phi^g(t,x)$.) From now, we will systematically suppose $\varepsilon \leqslant 1$ small enough in order that (5.16) occurs.

We first check the points b and c and then we treat point a.

(b) In this point, we shall not use (5.1) but only (1.8) and the fact that $f := \tilde{V}[g]$ is decreasing along characteristics. In the sequel, we omit in the writing the dependence of the flow $(X, \Xi)$ on $g$. We have

$$|f(t,x,\xi)| \leqslant |f_0^\lambda[(X,\Xi)(0,t,x,\xi)]| \leqslant \|(1+|\xi|)^\gamma f_0^\lambda\|_{L^\infty}(1+|\Xi(0,t,x,\xi)|)^{-\gamma}.$$

Now we have

$$|\xi - \Xi(0,t,x,\xi)| \leqslant T\|V_g\|_{L^\infty} \leqslant CT(|\tilde{v}|+1). \tag{5.17}$$

Using (3.33) we get

$$|(1+|\xi|)^\gamma f(t,x,\xi)| \leqslant \|(1+|\xi|)^\gamma f_0^\lambda\|_{L^\infty}[1+CT(|\tilde{v}|+1)]^\gamma.$$

Now using (5.15), we get the same estimate for $V[g]$ in $Q_T$. We choose $c_1 = c_\pi[1+CT(|\tilde{v}|+1)]^\gamma$. Then the second condition in $V(\mathscr{S}) \subset \mathscr{S}$ is established.

(c) We use Lemma 1, and deduce that for any $(t,x,\xi)$ and $(t',x',\xi')$ in $Q_{t_1}$,

$$|f^0(t,x,\xi) - f^0(t',x',\xi')| \leqslant C(1+|\xi|)|(t,x,\xi)-(t',x',\xi')| \|f_0^\lambda\|_{B^1(\mathbb{T}^n \times \mathbb{R}^n)},$$

and that

$$||f^0(t_1,\cdot,\cdot)||_{B^1(\mathbb{T}^n\times\mathbb{R}^n)} \leqslant C||f_0^\lambda||_{B^1(\mathbb{T}^n\times\mathbb{R}^n)},$$

where $C$ does not depend on $\lambda$ (but depends on $\varphi$). It follows from the construction then that

$$||f^1(t_1,\cdot,\cdot)||_{B^1(\mathbb{T}^n\times\mathbb{R}^n)} \leqslant c_\pi C||f_0^\lambda||_{B^1(\mathbb{T}^n\times\mathbb{R}^n)}.$$

Iterating the procedure $N$ times, we get that, for a constant $C$ independent from $\lambda$, one has, for any $i = 0, \dots, N$,

$$||f^i(t_i,\cdot,\cdot)||_{B^1(\mathbb{T}^n\times\mathbb{R}^n)} \leqslant C||f_0^\lambda||_{B^1(\mathbb{T}^n\times\mathbb{R}^n)},$$

and moreover, when $(t,t')\in[t_i,t_{i+1}]^2$,

$$|f(t,x,\xi)-f(t',x',\xi')| \leqslant C(1+|\xi|)|(t,x,\xi)-(t',x',\xi')|||f_0^\lambda||_{B^1(\mathbb{T}^n\times\mathbb{R}^n)}. \tag{5.18}$$

It remains only to consider $t$ and $t'$ in different intervals $[t_i, t_{i+1}]$. But we know that for any $(x,\xi)$ in the support of $f(t,\cdot,\cdot)$, one has

$$|\xi\cdot v| \leqslant M + CT(|\tilde{v}|+1)$$

(independently from $\lambda\in(0,1)$ thanks to (5.2)). This implies that $f_i$ and $f_{i+1}$ coincide during some time interval $[t_i-\alpha, t_i+\alpha]$ independent from $g$ in $\mathscr{S}_\varepsilon^\lambda$, on the domain $(\mathbb{T}^n\backslash(H+[-\frac{5}{3}d,\frac{5}{3}d]v))\times\mathbb{R}^n$. This is sufficient to establish (5.18) for all times.

Now it remains only to interpolate (5.18) and point b to get point c. This fixes the value of $c_2$ only in terms of $(\bar{f},\varphi)$.

(a) One has to check that for appropriate $\lambda$,

$$||A(V_\varepsilon^\lambda[g]-\bar{f})||_{C^{\delta_1}(\Omega_T)} \leqslant \varepsilon. \tag{5.19}$$

In this point a, we will not use the fact that $f_0$ is compactly supported in velocity in direction $v$, but only points b and c, and the decreasing of $\tilde{V}$ along characteristics. As a consequence, the proof of that point will still be valid in the case treated in the next section.

Let us treat the $L^\infty$-norm before the $C^{\delta_1}$ one. From (5.16), we deduce that $X^g(t,0,\cdot,0):\mathbb{T}^n\to\mathbb{T}^n$ is invertible; call $\mathfrak{X}_t^g$ its inverse, and define the function $\mathfrak{M}^g:[0,T]\times\mathbb{T}^n\to\mathbb{R}^n$ by

$$\mathfrak{M}(t,x) := \Xi^g(t,0,\mathfrak{X}_t^g(x),0).$$

Let us prove the following statement: for some $k>0$ independent from $\lambda$, one has, for any $\lambda$ and any $g\in\mathscr{S}_\varepsilon^\lambda$,

$$\forall(t,x,\xi)\in[0,T]\times\mathbb{T}^n\times\mathbb{R}^n, \quad |\xi-\mathfrak{M}(t,x)| \leqslant k|\Xi^g(0,t,x,\xi)|. \tag{5.20}$$

(i) It follows from $x = X^g(t, 0, \mathfrak{X}_t^g(x), 0)$ and $\mathfrak{X}_t^g(x) = X^g(0, t, x, \mathfrak{M}(t, x))$, (5.16) that

$$(c')^{-1}|X^g(0, t, x, \mathfrak{M}(t, x)) - X^g(0, t, x, \xi)| \leqslant |x - X^g(t, 0, X^g(0, t, x, \xi), 0)|.$$

(ii) Besides, it follows from $x = X^g(t, 0, X^g(0, t, x, \xi), \Xi(0, t, x, \xi))$ and Lemma 1 that

$$|x - X^g(t, 0, X^g(0, t, x, \xi), 0)| \leqslant K|\Xi^g(0, t, x, \xi)|.$$

(iii) From the two previous steps, we deduce that

$$|X^g(0, t, x, \mathfrak{M}(t, x)) - X^g(0, t, x, \xi)| \leqslant K'|\Xi^g(0, t, x, \xi)|.$$

Now by applying the "$\Xi$ part" of Lemma 1, we get

$$\begin{aligned} |\xi - \mathfrak{M}(t, x)| &= |\Xi^g[t, 0, X^g(0, t, x, \xi), \Xi^g(0, t, x, \xi)] - \Xi^g[t, 0, \mathfrak{X}_t^g(x), 0]| \\ &\leqslant C\big[|X^g(0, t, x, \xi) - \mathfrak{X}_t^g(x)| + |\Xi^g(0, t, x, \xi)|\big] \\ &\leqslant k|\Xi^g(0, t, x, \xi)|. \end{aligned}$$

Hence we deduce (5.20). Now, one has

$$\begin{aligned} |f(t, x, \xi)| &\leqslant |f_0^\lambda[(X^g, \Xi^g)(0, t, x, \xi)]| \\ &\leqslant \lambda^{2-n}||f_0(1 + |\xi|)^\gamma||_{L^\infty(\mathbb{T}^n \times \mathbb{R}^n)}\left(1 + \frac{1}{\lambda}|\Xi^g(0, t, x, \xi)|\right)^{-\gamma}. \end{aligned}$$

Using (5.20), we get that

$$|f(t, x, \xi)| \leqslant \lambda^{2-n}||f_0(1 + |\xi|)^\gamma||_{L^\infty(\mathbb{T}^n \times \mathbb{R}^n)}\left(1 + \frac{1}{k\lambda}|\xi - \mathfrak{M}(t, x)|\right)^{-\gamma}.$$

It follows that

$$\begin{aligned} &\left|\int_{\mathbb{R}^n} f(t, x, \xi)\, d\xi\right| \\ &\quad\leqslant \lambda^{2-n}||f_0(1 + |\xi|)^\gamma||_{L^\infty(\mathbb{T}^n \times \mathbb{R}^n)} \int_{\mathbb{R}^n}\left(1 + \frac{1}{k\lambda}|\xi - \mathfrak{M}(t, x)|\right)^{-\gamma} d\xi. \end{aligned}$$

We deduce that

$$\left|\int_{\mathbb{R}^n} \tilde{V}[g](t, x, \xi)\, d\xi\right| \leqslant \kappa\lambda^{2-n}||f_0(1 + |\xi|)^\gamma||_{L^\infty(\mathbb{T}^n \times \mathbb{R}^n)} k^n \lambda^n.$$

One deduces from the construction of $V$ that

$$||\Lambda(V[g]-\bar{f})||_{L^\infty(\Omega_T)} \leqslant C\lambda^{2-n}||f_0(1+|\xi|)^\gamma||_{L^\infty(\mathbb{T}^n\times\mathbb{R}^n)}\lambda^n \leqslant C(f_0)\lambda^2. \quad (5.21)$$

Now we turn to the Hölder estimate. It follows by interpolation between points b and c, that for a certain constant $C$ independent from $\lambda$, and for $\tilde{\gamma}=\frac{n+\gamma}{2}$ and $\delta=\gamma/(\gamma+1)$ one has

$$|V[g]-\bar{f}|_\delta^{\tilde{\gamma}} \leqslant C[||f_0^\lambda||_{B^1(\mathbb{T}^n\times\mathbb{R}^n)} + ||(1+|\xi|)^\gamma f_0^\lambda||_{C^0(\mathbb{T}^n\times\mathbb{R}^n)}].$$

We deduce that, for $\lambda\leqslant 1$ and another constant $C$ (depending on $f_0$ but not on $\lambda$),

$$||\Lambda(V_\varepsilon^\lambda[g]-\bar{f})||_{C^\delta(\Omega_T)} \leqslant C\lambda^{1-n}.$$

Now we interpolate again this inequality with (5.21). We get that for $\delta_1$, one has

$$||\Lambda(V_\varepsilon^\lambda[g]-\bar{f})||_{C^{\delta_1}(\Omega_T)} \leqslant K\lambda,$$

which concludes point a, for it is sufficient to find a proper $\lambda$. This finally proves $V(\mathscr{S})\subset\mathscr{S}$.

### 5.5. *The final state*

Using the previous section and the Leray–Schauder fixed point theorem, we hence find a fixed point, say $g^\bigstar_{\lambda,\varepsilon}$ of the operator $V_\varepsilon^\lambda$ in the domain $\mathscr{S}_\varepsilon^\lambda$. It is now to show that, for $(\lambda,\varepsilon)$ small enough, $g^\bigstar$ is a suitable solution, precisely that it satisfies (1.1)–(1.5).

It follows directly from the construction that Eqs. (1.1)–(1.2) and (1.5) are satisfied; in particular

$$\begin{aligned}
&\partial_t g^\bigstar + \xi\cdot\nabla_x g^\bigstar + \nabla\phi^{g^\bigstar}\cdot\nabla_\xi g^\bigstar \\
&\quad = \partial_t\bar{f} + \xi\cdot\nabla_x\bar{f} + \nabla\phi^{g^\bigstar}\cdot\nabla_\xi\bar{f} \\
&\qquad + \left[\partial_t + \xi\cdot\nabla_x + V^{g^\bigstar}\cdot\nabla_\xi\right]\Pi(\tilde{V}[g^\bigstar]) + \left[\nabla_x\phi^{g^\bigstar} - V^{g^\bigstar}\right]\cdot\nabla_\xi\Pi(\tilde{V}[g^\bigstar]),
\end{aligned}$$

which is supported in $[0,T]\times\omega\times\mathbb{R}^n$. Eq. (1.3) is satisfied provided that we suppose that $f_0$ satisfies $f_0=\pi(f_0)$ or equivalently that it is of the form $\pi(\tilde{f_0})$; this is satisfied for any final value of the control process of Section 6. We have to check (1.4). We use again Gronwall's lemma: for any $t \in [0, T]$,

$$\begin{aligned} &\|(\bar{X}, \bar{\Xi})(t,0,x,\xi) - (X^{g^\star}, \Xi^{g^\star})(t,0,x,\xi)\|_{B^0(\mathbb{T}^n \times \mathbb{R}^n)} \\ &\leqslant C\|V_{\bar{f}} - V_{g^\star}\|_{B^0(\Omega_T)} e^{T\|\mathscr{Y}(t)\tilde{v}\|_{B^{0,1}(Q_T)}} \\ &\leqslant \mathscr{K}\|\nabla\bar{\phi} - \nabla\phi^{g^\star}\|_{B^0(\Omega_T)}, \end{aligned} \tag{5.22}$$

for some constant $\mathscr{K}$ independent from $\varepsilon$ and $\lambda$. We can estimate $\nabla\bar{\phi} - \nabla\phi^{g^\star}$ by $\|\nabla\bar{\phi} - \nabla\phi^{g^\star}\|_{C^0} \leqslant C(\|\Lambda(g^\star - \bar{f})\|_{L^\infty} + |\rho_0^\lambda|)$. Now, if $\varepsilon$ is small enough, then one gets for any $(t, x, \xi) \in Q_T$

$$\|X^{g^\star}(t,0,x,\xi) - \bar{X}(t,0,x,\xi)\| < \tfrac{1}{8}d. \tag{5.23}$$

Now because of (5.1) and of (5.6), using Remark 7, one gets

$$\bar{X}(t_i, 0, x, \xi) \in H + [-\tfrac{7}{8}d, \tfrac{7}{8}d]v,$$

for any $(x, \xi) \in \mathscr{H}_i \times \mathbb{R}^n$ with $|\xi \cdot v| \leqslant \mu$. Reducing $\varepsilon$ and $\lambda$ again if necessary, we can ask that $\lambda M \leqslant \mu$. Using this scaling we see that each point in $\mathscr{H}_i$ at time 0 corresponding to a non-zero value of $f_0^\lambda$ is transported in $H + [-d, d]v$ at time $t_i$, by the flow $(X^{g^\star}, \Xi^{g^\star})$.

At time $t_i$, the construction makes $f$ to be 0 in $H + [-d, d]v$. Hence it follows from (5.4),(5.23) and (5.12)–(5.13) that one has

$$g^\star(T, \cdot, \cdot) = 0 \quad \text{in } (\mathbb{T}^n \backslash \omega) \times \mathbb{R}^n. \tag{5.24}$$

This gives a solution to the problem of controllability when $f_0$ is compactly supported in velocity in direction $v$.

## 6. Theorem 2: the problem of high velocities (according to $v$)

In this section, we show how to "get rid" of particles at high velocity in the direction $v$. Precisely, we find a control such that, starting from an arbitrary $f_0$, the corresponding solution of the Vlasov–Poisson system reaches a configuration which satisfies

$$\mathrm{Supp}(f(T, \cdot, \cdot)) \subset \mathbb{T}^n \times \{\xi \in \mathbb{R}^n / |\xi \cdot v| \leqslant M\}, \tag{6.1}$$

at time $T$, for a certain $M > 0$. Then, the previous section proves that one can steer any such configuration to 0, which completes the proof of Theorem 2.

As in the previous section, the proof relies on a special solution $(\bar{f}, \varphi)$: here it is the trivial one:

$$(\bar{f}, \varphi) = (0, 0).$$

Again, we construct an operator $V_\varepsilon^\lambda$. We show that this operator admits a fixed point for appropriate $(\varepsilon, \lambda)$ (still chosen in $(0,1)^2$). Finally, we show that the fixed point that we find is relevant, that is, satisfies system (1.1)–(1.2) with (1.3) and (1.5), and that its final value satisfies (6.1) (and is of the form $\pi(\tilde{f_1})$ for a certain $\tilde{f_1}$).

### 6.1. The operator $V_\varepsilon^\lambda$

As in the previous section, we first define the domain $\mathscr{S}_\varepsilon^\lambda$ of $V_\varepsilon^\lambda$ by (3.8) (where $\bar{f} = 0$), with the same constants $\delta_1$ and $\delta_2$ as in Section 5, and $c_1$ and $c_2$ to be redefined. Again we suppose that $\lambda < \mu(\varepsilon)$ in order that $\mathscr{S}_\varepsilon \neq \emptyset$ and $|\rho_0^\lambda| \leqslant \varepsilon$.

We write $\mathscr{T} := H + \{-dv, dv\}$ ($d$ is defined in Section 2.5). We introduce the following subsets of $\mathscr{T} \times \mathbb{R}^n$:

$$\gamma^- := \{(x,\xi) \in \mathscr{T} \times \mathbb{R}^n / \xi \cdot v(x) < -1\}, \tag{6.2}$$

where $v$ is the unit outward (that is, pointing outside $\omega$) normal on $\mathscr{T}$

$$\gamma^{2-} := \{(x,\xi) \in \mathscr{T} \times \mathbb{R}^n / \xi \cdot v(x) \leqslant -3/2\}, \tag{6.3}$$

$$\gamma^{3-} := \{(x,\xi) \in \mathscr{T} \times \mathbb{R}^n / \xi \cdot v(x) \leqslant -2\}, \tag{6.4}$$

$$\gamma^+ := \{(x,\xi) \in \mathscr{T} \times \mathbb{R}^n / \xi \cdot v(x) \geqslant 0\}. \tag{6.5}$$

Again, we observe that

$$\operatorname{dist}(\mathscr{T} \times \mathbb{R}^n \backslash \gamma^{2-}; \gamma^{3-}) > 0.$$

We introduce a $C^\infty \cap B^1$ regular function $U$ from $\mathscr{T} \times \mathbb{R}^n$ to $\mathbb{R}$ the same way as previously, by

$$\begin{cases} 0 \leqslant U \leqslant 1, \\ U \equiv 1 \quad \text{in } \mathscr{T} \times \mathbb{R}^n \backslash \gamma^{2-}, \\ U \equiv 0 \quad \text{in } \gamma^{3-}. \end{cases} \tag{6.6}$$

The function $\Upsilon$ is again introduced by (3.15), and $\chi$ by (3.18). We define $\pi$ as in Section 5; then the operator $\Pi$ is given by (3.22).

Now, given $g \in \mathscr{S}_\varepsilon^\lambda$, we introduce $f = \tilde{V}[g]$ as the solution of the following system:

$$\begin{cases} f(0,x,\xi) = f_0^\lambda \text{ on } \mathbb{T}^n \times \mathbb{R}^n, \\ \partial_t f + \xi \cdot \nabla_x f + \nabla \phi^g \cdot \nabla_\xi f = 0 \quad \text{in } [0,T] \times [(\mathbb{T}^n \times \mathbb{R}^n) \backslash \gamma^-], \\ f(t,x,\xi) = \left[\Upsilon\left(\frac{t}{\chi}\right) + \Upsilon\left(\frac{T-t}{\chi}\right)\right] f(t^-,x,\xi) \\ \qquad\qquad + \left[1 - \Upsilon\left(\frac{t}{\chi}\right) - \Upsilon\left(\frac{T-t}{\chi}\right)\right] \times U(x,\xi) f(t^-,x,\xi) \text{ on } [0,T] \times \gamma^-. \end{cases} \tag{6.7}$$

(The meaning of this equation is the same one as in Section 3). Then, as for Section 3, we define $V[g]$ by

$$V[g] := \Pi(f_{|\{[0,T]\times[\mathbb{T}^n\backslash(H+]-2d,2d[v)]\times\mathbb{R}^n\}\cup\{[0,\chi/4]\times\mathbb{T}^n\times\mathbb{R}^n\}}) \quad \text{in } [0,T]\times\mathbb{T}^n\times\mathbb{R}^n. \tag{6.8}$$

Again, $f_{|\{[0,T]\times[\mathbb{T}^n\backslash(H+]-2d,2d[v)]\times\mathbb{R}^n\}\cup\{[0,\chi/4]\times\mathbb{T}^n\times\mathbb{R}^n\}}$ is $C^1$ regular, and, together with the construction of $\Pi$, it will follow that $V[g]$ is in $C^1([0,T]\times\mathbb{T}^n\times\mathbb{R}^n)$.

Finally, we note $W$ the same operator as $\tilde{V}$ without absorption, as in the previous sections. In the sequel, the goal is to find a fixed point to $V_\varepsilon^\lambda$, for any $\varepsilon$ small enough, provided $\lambda$ is chosen small enough in terms of $\varepsilon$.

### 6.2. *Finding a fixed point of* $V_\varepsilon^\lambda$

The goal here is again to apply the Leray–Schauder Theorem. Let us check that its assumptions are satisfied.

1. $\mathscr{S}_\varepsilon^\lambda$ is all the same a convex compact subset of $C^0([0,T]\times\mathbb{T}^n\times\mathbb{R}^n)$.

2. Continuity of $V_\varepsilon^\lambda$ for the $C^0$ topology: let $g_i$ be a sequence of $\mathscr{S}_\varepsilon^\lambda$ converging to $g$ for the $C^0$ norm. Again, the sequence $\Lambda g_i$ is bounded in some Hölder space, and one gets

$$||(X^{g_i},\Xi^{g_i})-(X^g,\Xi^g)||_{B^0([0,T]^2\times\mathbb{T}^n\times\mathbb{R}^n)}\to 0 \quad \text{as } i\to+\infty,$$

and one concludes as in Section 3.

3. Now we have to show that one has that $V_\varepsilon^\lambda(\mathscr{S}_\varepsilon^\lambda)\subset\mathscr{S}_\varepsilon^\lambda$ for suitable $\lambda<\mu(\varepsilon)$. Again, the fourth point in the definition of $\mathscr{S}_\varepsilon^\lambda$ is guaranteed by the construction of $\Pi$. Let us check the three others. Again, we establish points b and c, before returning to a.

(b) For the second condition in $V_\varepsilon^\lambda(\mathscr{S}_\varepsilon^\lambda)\subset\mathscr{S}_\varepsilon^\lambda$, the proof is Section 5 is again valid, since we did not take into account the compactness of the support of $f_0$, nor the particular form of $V$, but only the decrease of $\tilde{V}[g]$ along characteristics.

(c) Concerning Hölder estimates on $V[g]$, we have the following lemma:

**Lemma 4.** *One has* $\tilde{V}[g]\in C^1(Q_T\backslash\Sigma_T)$, *with* $\Sigma_T := \{(t,x,\xi)\in Q_T/(x,\xi)\in\gamma^-\}$. *One has, for any* $(t,x,\xi)$ *and* $(t',x',\xi')$ *in* $[0,T]\times[\mathbb{T}^n\backslash(H+(-d,d)v)]\times\mathbb{R}^n$ *satisfying* $|x-x'|\leqslant d$ *and* $|\xi-\xi'|\leqslant 1$,

$$\begin{aligned}|\tilde{V}[g](t,x,\xi)&-\tilde{V}[g](t',x',\xi')|\\ &\leqslant C(1+|\xi|)[||f_0^\lambda||_{B^1(\mathbb{T}^n\times\mathbb{R}^n)}+||(1+|\xi|)^\gamma f_0^\lambda||_{L^\infty(\mathbb{T}^n\times\mathbb{R}^n)}]\\ &\qquad\times|(t,x,\xi)-(t',x',\xi')|,\end{aligned} \tag{6.9}$$

$$\begin{aligned}|\tilde{V}[g]&(t,x,\xi)-\tilde{V}[g](t,x',\xi')|\\ &\leqslant C[||f_0^\lambda||_{B^1(\mathbb{T}^n\times\mathbb{R}^n)}+||(1+|\xi|)^\gamma f_0^\lambda||_{L^\infty(\mathbb{T}^n\times\mathbb{R}^n)}]|(x,\xi)-(x',\xi')|.\end{aligned} \tag{6.10}$$

The proof of Lemma 4 is approximately the same as the one of Lemma 2 up to minor changes; in particular, we can derive the same calculus to Eq. (3.45). Let us just underline the changing point: here, the matrix of the transformation from $(\tau_1, \dots, \tau_{n-1}, \xi)$ to $(\tau_1, \dots, \tau_{n-1}, v)$ is no longer bounded, but of order $|\xi|$. Since here we have the estimate

$$|f(t,x,\xi)| \leqslant K \| (1+|\xi|)^\gamma f_0 \|_{L^\infty(\mathbb{T}^n \times \mathbb{R}^n)} \ (1+|\xi|)^{-\gamma},$$

we can estimate at each step $\mathscr{B}$ (see (3.46)) all the same by

$$\|\mathscr{B}\|_{L^\infty} \leqslant K \|\nabla(X,\Xi)\|_{L^\infty} \|(1+|\xi|)^\gamma f_0\|_{L^\infty(\mathbb{T}^n \times \mathbb{R}^n)} (1+|\xi|)^{-\gamma+1},$$

or, when making use of (1.8), by

$$\|\mathscr{B}\|_{L^\infty} \leqslant K \|\nabla(X,\Xi)\|_{L^\infty} \|(1+|\xi|)^{\gamma+2} f_0\|_{L^\infty(\mathbb{T}^n \times \mathbb{R}^n)} (1+|\xi|)^{-\gamma-1}.$$

The lemma follows, with Remark 5 still true. (Remark that $|x-x'| \leqslant d$ and $x, x' \notin H + (-d,d)v$ imply that $[x,x']$ does not cut $H + (-d,d)v$.)

Again, it follows from the construction of $\Pi$ that estimates (6.9) and (6.10) are also valid for $V$ instead of $\tilde{V}$, on the whole domain $Q_T$. Then we obtain again point c by interpolation between (6.9) and b, that is

$$\|(1+|\xi|)^\gamma V[g]\|_{L^\infty(Q_T)} \leqslant c_1 [\|f_0^\lambda\|_{B^1(\mathbb{T}^n \times \mathbb{R}^n)} + \|(1+\xi)^\gamma f_0^\lambda\|_{C^0(\mathbb{T}^n \times \mathbb{R}^n)}].$$

We get the desired estimate in $[0,T] \times (\mathbb{T}^n \backslash \omega) \times \mathbb{R}^n$. Using the construction of $\Pi$, we get that

$$\|V[g]\|_{B^{\delta_2}(Q_T)} \leqslant c_2 [\|f_0^\lambda\|_{B^1(\mathbb{T}^n \times \mathbb{R}^n)} + \|(1+\xi)^\gamma f_0^\lambda\|_{C^0(\mathbb{T}^n \times \mathbb{R}^n)}].$$

(a) The last point is to prove that

$$\|\Lambda V[g]\|_{C^{\delta_1}(\Omega_T)} \leqslant \varepsilon. \tag{6.11}$$

The proof is exactly the same as its equivalent in Section 5, since we took care not to use the compactness of the velocities in direction $v$ at that time, but only points b and c, and the following remark:

**Remark 9.** For some $c > 0$ (here $c = 1$), one has

$$c^{-1}|x-y| \leqslant |X^0(t,0,x,0) - X^0(t,0,y,0)| \leqslant c|x-y|. \tag{6.12}$$

We now have to prove that a fixed point $g^\star$ answers to the problem (at least for $(\varepsilon, \lambda)$ small enough).

*6.3. The final state*

From the construction, Eqs. (1.1)–(1.2),(1.3) and (1.5) are clearly satisfied by $g^\star$. What we have to establish is (6.1) (at least for small $\varepsilon$).

We begin with a remark.

**Remark 10.** One has the following property: there exists $m>0$ such that

$$\forall x\in\mathbb{T}^n, \forall\xi\in\mathbb{R}^n\backslash\{0\} \quad \text{such that } |\xi\cdot v|\geqslant m,$$
$$\exists t\in(T/4,3T/4),\ X^0(t,0,x,\xi)\in H+[-d/2,d/2]v. \tag{6.13}$$

Now we proceed as in Section 3: first, we show that property (6.13) is still true when $\nabla\phi^g$ is taken into account, at least if $\varepsilon$ is small enough and if we widen a little the "strip" $H+[-d/2,d/2]v$. Then, we show that particles in $H+[-3d/4,3d/4]v$, if the $v$ component of their velocity is large enough, have met $\gamma^{3-}$ during the process, which allows to conclude.

*Step* 1: The first step is again a consequence of Gronwall's inequality (see (3.53)):

$$||X^{\bar f}-X^{g^\star}||_{C^0([0,T]\times[0,T]\times\mathbb{T}^n\times\mathbb{R}^n)}<c_3\varepsilon.$$

It follows that, for relevant $\varepsilon$, we get

$$\exists M>0, \forall x\in\mathbb{T}^n, \forall\xi\in\mathbb{R}^n\backslash\{0\} \quad \text{such that } |\xi\cdot v|\geqslant m+1,$$
$$\exists t\in(0,T),\ X^{g^\star}(t,0,x,\xi)\in H+[-\tfrac{3}{4}d,\tfrac{3}{4}d]v. \tag{6.14}$$

Enlarging $m$ if necessary, we can ask that $|\Xi^{g^\star}(t,0,x,\xi)\cdot v|$ is large too (using (2.3)).

*Step* 2: We introduce, for $x\in H+[-\frac{3}{4}d,\frac{3}{4}d]v$ and $\xi\in\mathbb{R}^n\backslash\bar H$, $P_{\mathscr{T}}(x,\xi)$ as the point in $\mathscr{T}$ last met by $x+t\xi$, with $t<0$. We get similarly as in Section 3 that

$$\forall\eta>0,\ \exists\tilde M,\ \forall t\in[0,3T/4], \forall x\in H+[-\tfrac{3}{4}d,\tfrac{3}{4}d]v,$$
$$\forall\xi\in\mathbb{R}^n$$
$$\text{s.t. } |\xi\cdot v|\geqslant\tilde M, \exists\tilde t\in(0,t) \quad \text{s.t.}:$$

$$X^{g^\star}(\tilde t,t,x,\xi)\in\mathscr{T}, |X^{g^\star}(\tilde t,t,x,\xi)-P_{\mathscr{T}}(x,\xi)|<\eta, \quad \text{and}$$
$$\forall s\in[\tilde t,t],\ |\Xi^{g^\star}(s,t,x,\xi)-\xi|<\eta. \tag{6.15}$$

Indeed, (3.57) is still true here. But for $|\xi\cdot v|$ large enough, one has, for a certain $s<t$ with $|s-t|<\eta$, that $d(x+(s-t)\xi,H)\geqslant 3d$. Consequently, if $\varepsilon$ is small enough, one has $d(X^{g^\star}(s,t,x,\xi),H)\geqslant 2d$. Hence, one gets again the existence of $\tilde t$. Finally, we get $|X^{g^\star}(\tilde t,t,x,\xi)-P_{\mathscr{T}}(x,\xi)|<\eta$ and the estimate on $\Xi$ from (3.57), enlarging again $\tilde M$

if necessary. Consequently, using again (3.57), one gets that

$$\exists M' > 0, \forall x \in \mathbb{T}^n, \ \forall \xi \in \mathbb{R}^n \backslash \{0\} \quad \text{such that } |\xi \cdot v| \geqslant M', \exists t \in (0, 3T/4),$$
$$(X^{g^\star}, \Xi^{g^\star})(t, 0, x, \xi) \in \gamma^{3-}. \tag{6.16}$$

It follows then that for $|\xi \cdot v|$ large enough, one gets

$$f(T, \cdot, \xi) = 0 \quad \text{in } \mathbb{T}^n.$$

### 6.4. *Uniqueness*

Concerning uniqueness in both Sections 5 and 6, we refer to Section 3.5. This ends the proof of Theorem 2.

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## Appendix A

### A.1. *Proof of Proposition 1*

Let us first construct the function $\varphi$ and then show that it is convenient.

Given $(x_0, r_0) \in \mathbb{T}^2 \times (0, 1)$, we remark that there exist only a finite number of directions $v \in \mathbb{S}$ such that there exists a half-line in the torus with this direction, say $\{\mathscr{S}(x + tv), \ t \in \mathbb{R}^+\}$, that does not cut $B(x_0, r_0/4)$. Indeed, consider $v \in \mathbb{S}$, not horizontal nor vertical (there are only four such vectors in $\mathbb{S}$). Then, either $v$ corresponds to an irrational line (that is, the ratio of its coordinates is irrational), then the corresponding half-line is dense in $\mathbb{T}^2$ and meets $B(x_0, r_0/4)$. Either it is rational, say $v$ is proportional to $(p, q)$ with $p \in \mathbb{Z}^*$ and $q \in \mathbb{N}^*$ coprime integers. Then the trajectory $\mathscr{S}(x + tv)$ is periodic and hence $\{\mathscr{S}(x + tv), \ t \in \mathbb{R}^+\} = \{\mathscr{S}(x + tv), \ t \in \mathbb{R}\} =: L$. When pulling up this straight line to $\mathbb{R}^2$ by the canonical surjection, we get a countable quantity of parallel straight lines in $\mathbb{R}^2$. Then one has

$$\text{the distance between two parallel lines is inferior to } \min\left(\frac{1}{|p|}, \frac{1}{q}\right). \tag{A.1}$$

Indeed, when $(x, y)$ belongs to one of these lines, $(x + (kp + k'q)/q, y + (kp + k'q)/p)$ belongs also to it, for any $(k, k') \in \mathbb{Z}^2$, and then $(x + kp/q, y + k'q/p)$ belongs to another line in $\mathscr{S}^{-1}(L)$. Then by Bézout's Theorem, $(x + 1/q, y)$ and $(x, y + 1/p)$ belong to other lines in $\mathscr{S}^{-1}(L)$. This gives (A.1).

When $\min(\frac{1}{|p|}, \frac{1}{q}) < r_0/4$, (A.1) implies that the half-line in the torus must meet the ball $B(x_0, r_0/4)$; of course, there are only a finite number of $(p, q)$ satisfying $\min(\frac{1}{|p|}, \frac{1}{q}) \geqslant r_0/4$. Let us remark that these $v$ are symmetric in $\mathbb{S}$ (if $v$ corresponds to an irrational line, $x + \mathbb{R}^+ v$ is already dense, and if $v$ corresponds to a rational line the curve is periodic).

Call $v_1, \ldots, v_N$ these points in $\mathbb{S}$ (corresponding to a rational slope) for which there exists a half-line of direction $v_i$, which does not meet $B(x_0, r_0/4)$. Then, for each $i$ in $1, \ldots, N$, one can approximate $x \mapsto v_i^\perp$ in $\mathbb{T}^2 \backslash [B(x_0, \frac{r_0}{10}) + \mathbb{R} v_i]$—note that this set cannot be empty—for the $C^1$ norm, by the gradient of a harmonic function:

**Lemma A.1.** *For any $i \in \{1, \ldots, N\}$, for all $\varepsilon > 0$, there exists $\theta^i \in C^\infty(\mathbb{T}^2; \mathbb{R})$ such that*

$$\Delta \theta^i = 0 \quad \text{in } \mathbb{T}^2 \backslash B\left(x_0, \frac{r_0}{10}\right), \tag{A.2}$$

$$||\nabla \theta^i(x) - v_i^\perp||_{C^1(\mathbb{T}^2 \backslash [B(x_0, \frac{r_0}{10}) + \mathbb{R} v_i])} \leqslant \varepsilon. \tag{A.3}$$

This lemma follows from a harmonic approximation result of Bagby and Blanchet (see [1, Theorem 9.2]), which is close to Runge's Theorem of rational approximation for holomorphic functions:

**Theorem A.1** (Bagby-Blanchet). *Let $F$ be a closed subset of an orientable compact Riemannian manifold $\Omega$, and $U$ an open subset of $\Omega \backslash F$. Suppose that $U$ meets every connected component of $\Omega \backslash F$. For $f$ harmonic in a neighborhood of $F$ and $\varepsilon > 0$, there is a Newtonian function $u$ on $\Omega$, which poles all lie in $U$, and such that*

$$\sup_F |u - f| < \varepsilon.$$

In the previous theorem, a Newtonian function $u$ is a function harmonic everywhere but on its poles, and such that, for $p$ one of its poles, there exists $c \in \mathbb{R}$ and a regular neighborhood $R$ of $p$, such that $u - c G_R(\cdot, p)$—where $G_R(\cdot, \cdot)$ is the Green function defined in $R$—is regularly defined and harmonic in $R$.

When regularizing $u$ in $U$, we get a function harmonic in $\Omega \backslash U$, regular on $\Omega$.

In our case, on $\mathbb{T}^2 \backslash (B(x_0, \frac{r_0}{20}) + \mathbb{R} v_i)$, the vector field $v_i^\perp$ can easily be seen as the gradient of a harmonic function. It follows then that it can be approximated, in $C^0$ norm, on the domain $\mathbb{T}^2 \backslash [B(x_0, \frac{r_0}{20}) + \mathbb{R} v_i]$, by the gradient a of function harmonic everywhere but in $B(x_0, r_0/20)$. Then the $C^0$ convergence determines the $C^k$ convergence in smaller sets for harmonic functions.

Once constructed the $\theta^i$ (we will fix $\varepsilon$ later), we can describe the shape of $\varphi$. We put $t_i = \frac{T}{4} + \frac{iT}{2(N+1)}$ for $i \in \{0, \ldots, 4(N+1)\}/4$. We introduce a function $\eta \in C_0^\infty(]0, 1[)$ such that

$$\begin{cases} \eta \geqslant 0, \\ \int_{[0,1]} \eta = 1. \end{cases}$$

Then $\varphi$ is defined as

$$\begin{cases} \varphi(t,x) = \dfrac{A}{v}\eta\left(\dfrac{t-t_{i+\frac{1}{4}}}{v}\right)\theta^i(x) \quad \text{in } [t_{i+\frac{1}{4}}, t_{i+\frac{1}{2}}] \times \mathbb{T}^2, \\ \varphi \equiv 0 \quad \text{elsewhere in } [0,T] \times \mathbb{T}^2, \end{cases} \tag{A.4}$$

where $v$ and $A$ are fixed in order to satisfy $v < T/8(N+1)$ and $A > [12(N+1)/T] + 1$. Hence, these constants depend only on $\omega$ and $T$.

Now we have to show that this $\varphi$ is convenient. Call $(X, \Xi)$ the characteristics for $\varphi$. The only delicate point is to prove (3.4). We make a discussion according to the direction of $\xi$.

If $\frac{\xi}{|\xi|} \notin \mathbb{S}\backslash\{v_1, \ldots, v_N\}$, then there exists $m = m(\xi/|\xi|)$ such that if $|\xi| \geqslant m(\xi/|\xi|)$ then for any $x \in \mathbb{T}^2$, $x + t\xi$ cuts $B(x_0, r_0/4)$ for a certain $t \in [t_0, t_1]$ (as seen by using a simple compactness argument). Now let us suppose that $\frac{\xi}{|\xi|}$ is close to $v_i$. We are then interested to what happens during the interval $[t_i, t_{i+1}]$.

Let us prove that for any $\xi$ such that $\xi/|\xi|$ is in a neighborhood of $v_i$ in $\mathbb{S}$ and $|\xi|$ is large enough, then for any $x \in \mathbb{T}^2$, $X(t, 0, x, \xi)$ meets $B(x_0, r_0/4)$ during the interval $[t_i, t_{i+1}]$. This is done in three steps.

(a) The first point is to observe that for any $t \in [t_i, t_{i+\frac{3}{4}}]$ and any $x \in B(x_0, r_0/5) + \mathbb{R}v_i$, there exists $s \in [t, t_{i+1}]$ such that $X(s, t, x, \xi) \in B(x_0, r_0/4)$, at least if $|\xi|$ is large enough, and $\xi/|\xi|$ belongs to a neighborhood $\mathscr{V}_1$ of $v_i$ in $\mathbb{S}$. Indeed, for $\xi/|\xi|$ in a neighborhood $\mathscr{V}_1$ of $v_i$, if $x \in B(x_0, r_0/5) + \mathbb{R}v_i$ then $x + t\xi$ cuts $B(x_0, r_0/\frac{9}{40})$ for $t < C/|\xi|$. It follows then from

$$\begin{cases} |X(s,t,x,\xi) - x - (s-t)\xi| \leqslant C(\varphi)|s-t|, \\ |\Xi(s,t,x,\xi) - \xi| \leqslant C(\varphi)|s-t|, \end{cases} \tag{A.5}$$

that for $|\xi|$ large (say $|\xi| \geqslant m_1$), $X(s, t, x, \xi)$ meets $B(x_0, r_0/4)$ for some $s \in [t_i, t_{i+1}]$.

(b) The second point consists in proving that for a $(x, \xi)$ with $x \in \mathbb{T}^2$, $|\xi| \geqslant m_2$ and $\xi/|\xi|$ in a neighborhood $\mathscr{V}_2$ of $v_i$ in $\mathbb{S}$, there exists $t \in [t_i, t_{i+\frac{3}{4}}]$ such that $X(t, t_i, x, \xi) \in B(x_0, r_0/5) + \mathbb{R}v_i$. We discuss according to $\mathscr{P}_{v_i^\perp}(\xi)$, where $\mathscr{P}_{v_i^\perp}$ is the linear orthogonal projection on the direction $v_i^\perp$. If $|\mathscr{P}_{v_i^\perp}(\xi)| \geqslant 6(N+1)/T$ then during $[t_i, t_{i+\frac{1}{4}}]$, $X(t, t_i, x, \xi) = x + (t - t_i)\xi$ meets $B(x_0, r_0/5) + \mathbb{R}v_i$.

If $|\mathscr{P}_{v_i^\perp}(\xi)| < 6(N+1)/T$, suppose that $X(t, t_i, x, \xi)$ does not meet $B(x_0, r_0/5) + \mathbb{R}v_i$ during $[t_{i+\frac{1}{4}}, t_{i+\frac{1}{2}}]$. Then

$$||\nabla\varphi(t, X(t, t_i, x, \xi)) - \tfrac{A}{v}\eta(\tfrac{t-t_{i+\frac{1}{4}}}{v})v_i^\perp|| \leqslant A\varepsilon/v.$$

If $\varepsilon$ is small enough in order that $A\varepsilon < 1$, then in that case one has

$$|\mathscr{P}_{v_i^\perp}(\Xi(t_{i+\frac{1}{2}}, t_i, x, \xi))| \geqslant 6(N+1)/T,$$

and then for some $t$ in the time interval $[t_{i+\frac{1}{2}}, t_{i+\frac{3}{4}}]$ one has $X(t, t_i, x, \xi) \in B(x_0, r_0/5) + \mathbb{R}v_i$.

Moreover, by (A.5), if $\mathscr{V}_2$ is small enough and $m_2$ is large enough, then for any $x \in \mathbb{T}^2$ and any $\xi \in \mathbb{R}^2$ satisfying $\xi/|\xi| \in \mathscr{V}_2$ and $|\xi| \geqslant m_2$, then one has $|\Xi(t, t_i, x, \xi)| \geqslant m_1$ and $\Xi(t, t_i, x, \xi)/|\Xi(t, t_i, x, \xi)| \in \mathscr{V}_1$ at the time $t$ when the particle belongs to the domain $B(x_0, r_0/5) + \mathbb{R}v_i$.

(c) The last step is to prove that for some neighborhood $\mathscr{V}_3$ of $v_i$ in $\mathbb{S}$ and some $m_3 > 0$, if $|\xi| \geqslant m_3$ and $\xi/|\xi| \in \mathscr{V}_3$, then for any $x$ in $\mathbb{T}^2$, $\Xi(t_i, 0, x, \xi)/|\Xi(t_i, 0, x, \xi)| \in \mathscr{V}_2$, with $|\Xi(t_i, 0, x, \xi)| \geqslant m_2$. This follows again from

$$\Xi(t, s, x, \xi) - \xi = \int_s^t \nabla\varphi(\tau, X(\tau, s, x, \xi))\, d\tau.$$

So $m_3$ and $\mathscr{V}_3$ can easily be found in terms of $m_2$, $\mathscr{V}_2$ and $\varphi$.

Now, using the compactness of $\mathbb{S}$, one gets easily (3.4).

### *A.2. Proof of Lemma 3*

The proof is close to [11, Proposition 1]. Let $D$ be a horizontal line in $\mathbb{T}^2$ that does not cut $\mathcal{O}$, and $D^\perp$ be a vertical line in $\mathbb{T}^2$ that does not cut $\mathcal{O}$ (reduce $\mathcal{O}$ if this is not possible). To prove Lemma 3, it suffices to find $v \in C^\infty(\mathbb{T}^2; \mathbb{R}^2)$ satisfying:

$$\operatorname{curl} v = 0 \quad \text{in } \mathbb{T}^2 \backslash \mathcal{O}, \tag{A.6}$$

$$\operatorname{div} v = 0 \quad \text{in } \mathbb{T}^2 \backslash \mathcal{O}, \tag{A.7}$$

$$|v(x)| > 0 \quad \text{for any } x \text{ in } \mathbb{T}^2 \backslash \mathcal{O}, \tag{A.8}$$

$$\int_D v \cdot \mathrm{v}\, dx = \int_D v \cdot\, d\tau = 0, \tag{A.9}$$

where $\mathrm{v}$ stands for one of the two normal continuous unit vectors on $D$, and $\tau$ for one of the two tangent continuous unit vectors on $D$. Indeed, let us suppose that we have found such a $v$. Denote $\rho_\varphi$ the rotation of center 0 and angle $\varphi$ in $\mathbb{R}^2$. Then it follows from (A.9) that for any $\varphi$,

$$\int_D \rho_\varphi(v) \cdot\, d\tau = 0, \tag{A.10}$$

and it follows from (A.6) and (A.7) that $\rho_\varphi(v)$ still satisfies (A.6) and (A.7). Then for a certain $\varphi \in [0, \pi]$, one has also

$$\int_{D^\perp} \rho_\varphi(v) \cdot d\tau = 0, \tag{A.11}$$

But a $\rho_\varphi(v)$ satisfying (A.6), (A.7),(A.10) and (A.11) is certainly of the form $\nabla\theta$ in $\mathbb{T}^2 \backslash \mathcal{O}$. We extend it arbitrarily in $\mathcal{O}$; then $\theta$ satisfies the conclusions of Lemma 3.

From now, we look for $v$ satisfying (A.6)–(A.9). The central point is that if $v = (v^1, v^2)$ satisfies (A.6)–(A.7), then $\phi = v^1 - iv^2$ is a holomorphic function in $\mathbb{T}^2 \backslash \mathcal{O}$, and conversely. By the way, $v$ satisfies (A.9) if and only if the corresponding $\phi$ satisfies

$$\int_D \phi(z)\,dz = 0. \tag{A.12}$$

So we have left to find a holomorphic function $\phi$ defined on $\mathbb{T}^2\backslash\mathcal{O}$, satisfying (A.12) and $\phi(x)\neq 0$ for any $x$ in $\mathbb{T}^2\backslash\mathcal{O}$. For that, we introduce two nonempty open intervals $I_1$ and $I_2$ in $D$, with disjoints closures, and a function $h\in C^\infty(D,\mathbb{R}^2)$ satisfying

$$||h(x)||\geqslant 1 \quad \text{on } D, \tag{A.13}$$

$$\mathrm{Ind}_D(h) = 0, \tag{A.14}$$

$$h \equiv \tau \quad \text{on } I_1, \tag{A.15}$$

$$h \equiv v \quad \text{on } I_2, \tag{A.16}$$

$$\int_D h\cdot v\,dx = \int_D h\cdot\, d\tau = 0. \tag{A.17}$$

It is elementary to construct such a $h$. Now we use the following fact: given $\tilde{h}\in C^\infty(D,\mathbb{R}^2)$, for any $\varepsilon>0$, there exists $f\in\mathscr{H}(\mathscr{V}(D))$ (where $\mathscr{V}(D)$ is a neighborhood of $D$) such that

$$||f-\tilde{h}||_0\leqslant\varepsilon.$$

Indeed, it follows from Dirichlet's Theorem for Fourier series, that for some constants $c_n$, one has

$$\left|\left|\tilde{h}(x) - \sum_{i=-N}^{N} c_i e^{2i\pi x}\right|\right|_{L^\infty(D)} \leqslant\varepsilon.$$

Hence, $z\mapsto\sum_{i=-N}^{N} c_i e^{2i\pi z}$, defined in a neighborhood of $D$, is relevant (for instance, let us agree that $D$ is given by the equation $y=0$). Then it follows from the variant Theorem 3 of Runge's Theorem, that for a certain $\hat{\phi}\in\mathscr{H}(\mathbb{T}^2\backslash\mathcal{O})$ one has

$$||\hat{\phi}-\tilde{h}||_0\leqslant 2\varepsilon.$$

So $\tilde{h}$ can be approximated on $D$ in $C^0$ norm by the restriction of a holomorphic function of $\mathbb{T}^2\backslash\mathcal{O}$.

The idea it to apply this remark to $\tilde{h}:=\log h$ (for a certain $\varepsilon\in(0,1/2)$ small enough), which is well defined thanks to (A.13) and to (A.14). We obtain this way a $f_\varepsilon$. Then $e^{f_\varepsilon}$ is relevant, except that it does not necessarily satisfy (A.12) exactly; however this integral satisfies

$$\left|\int_D e^{f_\varepsilon}(z)dz\right|\leqslant K\varepsilon. \tag{A.18}$$

To ensure that we have this integral exactly nil, we introduce two regular functions $w_1$ and $w_2$ from $D$ to $\mathbb{C}$ in the following way: for $i \in \{1, 2\}$,

$$\mathfrak{Im}(w_i) = 0 \quad \text{on } D, \tag{A.19}$$

$$\mathfrak{Re}(w_i) = 0 \quad \text{on } D \backslash I_i, \tag{A.20}$$

$$\mathfrak{Re}(w_i) \geqslant 0 \text{ on } D, \tag{A.21}$$

$$\int_D \mathfrak{Re}(w_i) = 1. \tag{A.22}$$

We approximate $w_1$ and $w_2$ by the restriction of a holomorphic function of $\mathbb{T}^2 \backslash \mathcal{O}$ in $C^0$-norm, with error at most $\varepsilon'$, obtaining this way $W_1^{\varepsilon'}$ and $W_2^{\varepsilon'}$. Then, by considering

$$\Phi(\lambda, \mu) := e^{f_\varepsilon} e^{\lambda W_1^{\varepsilon'}} e^{\mu W_2^{\varepsilon'}},$$

for suitable $\lambda$ and $\mu$ in $[-4(K+1)\varepsilon, 4(K+1)\varepsilon]$ ($K$ introduced in (A.18)), we will be able to get (A.12). Indeed, we have

$$\begin{cases} \mathfrak{Re}\left[\int_D (\Phi - e^{f_\varepsilon})\, dz\right] \geqslant (1-\varepsilon)\lambda - C\varepsilon', & \text{for } \lambda \geqslant 0, \\ \mathfrak{Re}\left[\int_D (\Phi - e^{f_\varepsilon})\, dz\right] \leqslant (1-\varepsilon)\frac{\lambda}{2} - C\varepsilon', & \text{for } \lambda \leqslant 0, \\ \mathfrak{Im}\left[\int_D (\Phi - e^{f_\varepsilon})\, dz\right] \geqslant (1-\varepsilon)\mu - C\varepsilon' & \text{for } \mu \geqslant 0, \\ \mathfrak{Im}\left[\int_D (\Phi - e^{f_\varepsilon})\, dz\right] \leqslant (1-\varepsilon)\frac{\mu}{2} - C\varepsilon' & \text{for } \mu \leqslant 0. \end{cases} \tag{A.23}$$

From now, we take $\varepsilon < 1/2$ and $\varepsilon' := \frac{\varepsilon}{10C}$. Then we consider the application:

$$\mathscr{H} : \begin{cases} \mathbb{R}^2 \to \mathbb{R}^2, \\ (\lambda, \mu) \mapsto \left[\mathfrak{Re}(\int_D \Phi(z)dz), \mathfrak{Im}(\int_D \Phi(z)dz)\right]. \end{cases}$$

We endow $\mathbb{R}^2$ with the norm $||(x_1, x_2)|| := \max(|x_1|, |x_2|)$. If we restrict the application $\mathscr{H}$ to the sphere (in fact, the square) with center 0 and radius $4(K+1)\varepsilon$, say $\mathscr{S}(0, 4(K+1)\varepsilon)$ (denote by $B(0, 4(K+1)\varepsilon)$ the corresponding ball), then from (A.23), we deduce that 0 is not reached. So we can define

$$\mathscr{H}' : \begin{cases} \mathscr{S}(0, 4(K+1)\varepsilon) \to \mathscr{S}(0, 4(K+1)\varepsilon), \\ (\lambda, \mu) \mapsto 4(K+1)\varepsilon \dfrac{\mathscr{H}(\lambda)}{||\mathscr{H}(\lambda)||}. \end{cases}$$

This application has a non-null degree (for instance, by (A.23), no point is sent to its antipodal point). Hence,

$$\exists \bar{\lambda} \in B(0, 4(K+1)\varepsilon) \quad \text{such that } \mathscr{H}(\bar{\lambda}) = 0.$$

Finally, one finds a solution of the system.

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