Controllability and asymptotic stabilization of the Camassa-Holm equation

Olivier Glass *

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Abstract. We investigate the problems of exact controllability and asymptotic stabilization of the Camassa-Holm equation in the circle, by means of a distributed control. The results are global, and in particular the control prevents the solution from blowing up.

Keywords. Camassa-Holm equation; controllability; asymptotic stabilization.

1 Introduction

1.1 Statement of the problem

In this paper, we are interested in two control problems concerning the Camassa-Holm equation in the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$:

$$u_t - u_{txx} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx} \text{ for } (t,x) \in [0,T] \times \mathbb{T}. \tag{1}$$

The Camassa-Holm equation describes one-dimensional surface waves at a free surface of shallow water under the influence of gravity. The function $u(t,x)$ represents the fluid velocity at time t and position x, and the constant κ is a nonnegative parameter. Equation (1) was first introduced by Fokas and Fuchssteiner [18] as a bi-Hamiltonian model, and was derived as a water wave model by Camassa and Holm [3]. It turns out that this equation was also obtained as a model of propagating waves in cylindrical elastic rods, see Dai [14].

We investigate this equation from the point of view of distributed control. Consider ω a nonempty open set in $\mathbb{T}$. The problems under view concern equation (1) with an additional force term, supported in ω, used as a control:

$$u_t - u_{txx} + 2\kappa u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx} + g(t,x)\mathbf{1}_\omega(x) \text{ for } (t,x) \in [0,T] \times \mathbb{T}. \tag{2}$$

Note that it is essentially equivalent to control the equation on $\mathbb{T} \setminus \omega$ (which can typically be an interval), via boundary controls on $\partial\omega$ (typically both sides of the interval), but without making the boundary conditions explicit.

We consider the two following problems.

- *Controllability problem:* given u_0 and u_1 in some functional space, given $T > 0$, can one find g such that the solution of (2) with initial state u_0 is defined until time T and satisfies $u_{|t=T} = u_1$?
- *Asymptotic stabilizability:* can one define a stationary feedback law $g = g[u(t,\cdot)]$ such that the closed-loop system (2) obtained with this g is globally well defined and asymptotically stable at an equilibrium point?

We prove the following results.

*Université Pierre et Marie Curie–Paris6, UMR 7598 Laboratoire Jacques-Louis Lions, Paris, F-75005, France. E-mail: glass@ann.jussieu.fr

Theorem 1 (Global exact controllability of the Camassa-Holm equation) *Consider $s > 3/2$. Let $u_0, u_1 \in H^s(\mathbb{T})$ and $T > 0$. Then there exists $g \in C^0([0,T]; H^{s-3}(\omega))$ such that equation* (2) *has a unique solution $u \in C^0([0,T]; H^s(\mathbb{T})) \cap C^1([0,T]; H^{s-1}(\mathbb{T}))$ satisfying*

$$u_{|t=0} = u_0 \text{ in } \mathbb{T}, \tag{3}$$

and moreover this solution satisfies that

$$u_{|t=T} = u_1 \text{ in } \mathbb{T}. \tag{4}$$

Theorem 2 (Global asymptotic stabilization by stationary feedback law of (2)**)** *There exists a stationary feedback law $g = g[u]$ ($g : H^2(\mathbb{T}) \to H^{-1}(\omega)$ is made explicit below — see* (45)*) such that one has*

- *for any $u_0 \in H^2(\mathbb{T})$, there exists a solution of* (2) *and* (3) *belonging to $C^0([0,T]; H^2(\mathbb{T})) \cap C^1([0,T]; H^1(\mathbb{T}))$, moreover any such maximal solution is global in time,*
- *for any $u_0 \in H^2(\mathbb{T})$, any maximal solution u satisfies*

$$u(t,x) \underset{t \to +\infty}{\longrightarrow} -\kappa \text{ in } H^2(\mathbb{T}), \tag{5}$$

- *for any $\eta > 0$, there exists $\varepsilon > 0$ such that any maximal solution u of the closed-loop system for some initial state $u_{|t=0} = u_0$ satisfying*

$$\|u_0 + \kappa\|_{H^2(\mathbb{T})} < \varepsilon,$$

satisfies

$$\|u(t,\cdot) + \kappa\|_{H^2(\mathbb{T})} < \eta, \quad \forall t \in \mathbb{R}^+. \tag{6}$$

Remark 1 *It is a known fact that the solutions of the Camassa-Holm equation generally develop singularities in finite time in $H^s(\mathbb{T})$, $s > 3/2$, see in particular [5, 6]. Hence the above control also prevents the blow-up from occurring. In the context of the three-dimensional Euler equation for incompressible inviscid fluids [19], the control has a similar role (although, as a matter of fact, the possibility of a blow-up in the 3D Euler equation is still an open problem).*

Remark 2 *It should be noted that equation* (2) *is obtained in [3] after going to a reference frame moving with speed κ (see [3, p. 1662]). Hence the constant $-\kappa$ in* (5) *is natural since it corresponds to the rest state in the original frame.*

The Camassa-Holm equation has been intensively investigated lately from the point of view of the Cauchy problem, see for instance [2, 4, 5, 6, 7, 15, 16, 17, 22, 23, 24, 27, 33] and references therein. Other important issues such as blow-up mechanism, solitary waves structures, stability properties, geometric approaches, etc. have also been widely analyzed. On the other hand, it seems to the author that the study of this equation from the point of view of control theory was a completely open field.

Finally, concerning the controllability of other equations modelling surface water waves, let us mention [1, 12, 13, 28, 29, 30, 32, 34] for what concerns the Korteweg-de Vries equation, [26] for the linearized Benjamin-Bona-Mahony equation and [25] for the linearized Benjamin-Ono equation.

1.2 Preliminaries

Before establishing Theorems 1 and 2, let us make some transformations on the equation. It is elementary to see that equation (2) can also be written in the following frequently used form:

$$u_t + uu_x = -\partial_x(1 - \partial^2_{xx})^{-1}\Big[u^2 + \frac{1}{2}u_x^2 + 2\kappa u + g\Big]. \tag{7}$$

Another form of the equation is given by the following

$$\left\{ \begin{array}{l} \partial_t y + 2\kappa \partial_x u + \partial_x(uy) = -y\partial_x u + g \\ y = (1 - \partial^2_{xx})u. \end{array} \right. \tag{8}$$

The equivalence of (8) with (2) is straightforward when u has a space regularity $H^s(\mathbb{T})$ with $s \geq 3$; in the case $s > 3/2$, this can be shown by using paradifferential calculus, see Section 2 (and in particular Lemma 2) below.

It should be noted that (8) has some similarities with the three dimensional Euler equation for incompressible inviscid fluids: here in some sense y plays the role of the vorticity, which satisfies a transport equation (with a stretching term), the corresponding velocity field being recovered from the vorticity though an elliptic equation.

1.3 Strategies

Let us briefly discuss the strategies that we use to establish Theorems 1 and 2.

Both the results stated in the previous paragraph rely on the so-called *return method*, introduced by J.-M. Coron in [8] in the context of finite-dimensional control systems. This method consists in finding a particular trajectory of the system, typically starting and ending at 0, such that the linearized equation around this particular solution has good controllability properties.

Here when considering the linearized equation around $\overline{u} \equiv 0$, we get

$$u_t - u_{txx} + 2\kappa u_x = g(t,x)\mathbf{1}_\omega(x) \text{ for } (t,x) \in [0,T] \times \mathbb{T}. \tag{9}$$

This equation is clearly not controllable (nor stabilizable) when $\kappa = 0$, since in that case $u - u_{xx}$ is constant outside ω. For $\kappa \neq 0$, we set y as in (8) and write (9) as follows:

$$y_t = (1 - \partial^2_{xx})^{-1}(-2\kappa y_x + \mathbf{1}_\omega g).$$

It follows in particular that outside ω, y_t is more regular than y, which proves that the equation is not controllable in this case either. Note that a stronger result on the non-controllability of equation (9) was established by Micu [26] (since the Benjamin-Bona-Mahony equation has the same linearized equations around constant states as the Camassa-Holm equation.)

To overcome the bad behaviour of the linearized equation around 0, the general idea is hence to introduce a particular solution of the system, close to which the system has good controllability (or stabilization) properties. In both problems, the corresponding solution $\overline{u}$ has a somewhat trivial form (it is in some sense close to the one used for the Vlasov-Poisson equation, see [20]):

$$\overline{u} = (1 - \partial^2_{xx})^{-1}\overline{y} \;\text{ with }\; \overline{y} \;\text{ satisfying }\; \text{Supp}\, \overline{y}(t,\cdot) \subset \omega, \quad \forall t \in [0,T]. \tag{10}$$

In fact, one can more or less imagine the functions $\overline{u}$ and $\overline{y}$ as constant in time. The function $\overline{u}$ will be roughly the same for both problems, but two different effects are in order in these two results; let us briefly sketch the main ideas.

Controllability problem. As for the incompressible Euler equation (see in particular Coron [9], see also [19]), the plan concerning the controllability problem is to rely on the effect of **transport**. Indeed, if one chooses the function $\overline{y}$ properly, in particular, in order to make sure that the first equation in (8) has characteristics which all reach the control zone, one can hope that the linearized system around $\overline{u}(t,x)$ is controllable. This is due to the fact that inside the control zone, one can in some sense "replace" the value of y with a suitable one (typically 0). Then one can hope to obtain a control for the nonlinear system, in the case of a small initial data and zero final value, via a fixed-point scheme. The construction of the solution of the nonlinear system is inspired by Danchin's work [15].

Two arguments are in order to deduce the general case from this local to zero controllability property (as for the incompressible Euler equation, see in particular [9]).

First, a simple argument of translation will allow to ignore the term $\kappa\partial_x u$, and to work only with

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} + g(t,x)\mathbf{1}_\omega(x) \text{ for } (t,x) \in [0,T] \times \mathbb{T}. \tag{11}$$

When we ignore this term, the equation becomes completely reversible with respect to time, i.e.

$$u(t,x) \text{ is a solution of (11)} \Longrightarrow -u(T-t,x) \text{ is a solution of (11) (with control } g(T-t,x)). \tag{12}$$

One can rely on this reversibility to restrict to the case where $u_1 = 0$ (one can first reach 0 from u_0, then u_1 from 0).

Next, we will see that the smallness of the initial data is not restrictive either, because of the scale invariance of the solutions of the equation. This particular invariance, which affects only the variable t, is given by the following property: for $\lambda > 0$,

$$\begin{aligned} &u(t,x) \text{ is a solution of (11) defined in } [0,T]\times\mathbb{T} \\ &\qquad \Longrightarrow u^\lambda(t,x) := \lambda u(\lambda t,x) \text{ is a solution of (11) defined in } [0,T/\lambda]\times\mathbb{T} \text{ (with control } \lambda^2 g(\lambda t,x)). \end{aligned} \tag{13}$$

Hence using this invariance, we see that we can control larger states in *smaller* times (if needed, one can complete the solution by 0, if the "new" time is too short). This is also an important ingredient in the controllability of the Euler equation, see [9, 19]. In particular, this means that in order to control and prevent blow-up when the initial state is large, one has to act fast and strongly. This will complete the argument.

Stabilization problem. Concerning the stabilization problem, we will not quite rely on the transport effect in equation (8), but rather on the **stretching effect** of the equation. It is indeed rather clear from equation (8) that if one could ensure that

$$\partial_x u \geq c > 0, \tag{14}$$

then the solution of the equation would naturally decrease, presumably sufficiently fast (if c is large enough) to prevent any possible blow-up. Of course, due to the periodic structure of $\mathbb{T}$, one clearly cannot expect (14) to be valid on the whole domain $\mathbb{T}$. However, one can choose $\overline{u}$ satisfying (14) outside the control zone ω, so that, should we be able to force the solution of the system to be close to $\overline{u}$ (or to a multiple of $\overline{u}$), the solution would naturally decrease outside ω. Hence, one has to design a feedback law which

- forces in some sense the system to be close to a multiple of $\overline{u}$ (this is done by adding a term of the form $\hat{g}[u(t,\cdot)]\overline{y}$ on the right hand side of the transport equation (8)),
- apply a strong damping on the quantity y inside ω, to compensate the fact that $\overline{u}$ does not satisfy (14) inside ω, and obtain the decrease of the quantity y inside the control zone.

Finally, one can construct a Lyapunov functional that yields the conclusion of Theorem 2, and which reflects the fact that

- either the state of the system is already close to a multiple of $\overline{u}$ and the function decreases due to the joint effects of the stretching outside ω and the damping inside ω,
- or the feedback law is indeed making the system move towards a state close to a multiple of $\overline{u}$.

Concerning the problem of stabilization for the bidimensional Euler equation, which relies on the transport effect of the equation and on the return method, let us cite Coron [11] (see also [21]).

Structure of the paper. Section 2 is devoted to the proof of Theorem 1, while in Section 3 we establish Theorem 2.

2 Controllability

In this section, we prove Theorem 1. At the end of the section, a slightly more general result is discussed (when the Sobolev spaces are replaced with Besov spaces).

To simplify the description of the control zone, we describe the circle $\mathbb{T}$ by the interval $[0,1)$ (with periodic conditions), and we consider ω as the nonempty open interval $[0,a) \cup (1-a,1]$, for some $a \in (0,1/2)$ (we can always assume this, shrinking ω and translating the problem if necessary).

2.1 Reduction

As we explained in Paragraph 1.3, it will be sufficient to prove Theorem 1 in the case where

$$\kappa = 0, \tag{15}$$

$$u_1 = 0, \tag{16}$$

$$\|u_0\|_{H^s(\mathbb{T})} < \epsilon, \tag{17}$$

for a given $\epsilon > 0$.

Let explain how one can reduce the problem to this case: let us assume temporarily that we have proven the following weaker statement.

Proposition 1 *Consider $s > 3/2$ and $T > 0$. There exists $\epsilon > 0$, such that, for all $u_0 \in H^s(\mathbb{T})$ satisfying* (17)*, there exists $g \in C^0([0,T]; H^{s-3}(\omega))$, vanishing in some neighborhood of T, such that equation* (2) *with $\kappa = 0$ has a unique solution $u \in C^0([0,T]; H^s(\mathbb{T})) \cap C^1([0,T]; H^{s-1}(\mathbb{T}))$ satisfying* (3)*, and moreover this solution satisfies* (4) *with $u_1 = 0$.*

Then Theorem 1 is deduced as follows.

Proof of Theorem 1. First, the scaling and reversibility arguments give us Theorem 1 for $\kappa = 0$. Indeed, given $T > 0$, u_0 and u_1 in $H^s(\mathbb{T})$, we introduce $\epsilon > 0$ given by Proposition 1 for time $T/3$. Then we introduce $\lambda > 1$ such that

$$\max(\|u_0\|_{H^s(\mathbb{T})}, \|u_1\|_{H^s(\mathbb{T})}) < \lambda\epsilon.$$

Then by Proposition 1, $\frac{1}{\lambda}u_0$ and $-\frac{1}{\lambda}u_1$ can be driven to 0 in a time $T/3$, via controls g_1 and g_2 respectively; call U_0 and U_1 the corresponding solutions of (2). Now, from (12)-(13), the function

$$u(t,x) := \begin{cases} U_0^\lambda(t,x) \text{ for } t \in [0, T/(3\lambda)], \\ 0 \text{ for } t \in [T/(3\lambda), T - T/(3\lambda)], \\ -U_1^\lambda(T-t,x) \text{ for } t \in [T - T/(3\lambda), T], \end{cases}$$

where the exponents λ refer to the notation in (13), answers to Theorem 1.

Now let us discuss the case where $\kappa \neq 0$. The standard ingredient is that if u is a solution of (2) with $\kappa \neq 0$, then

$$u_\kappa(t,x) := \kappa + u(t, x - \kappa t),$$

is a solution of (2) with $\kappa = 0$. However, by making this transformation we modify the problem, since ω is not invariant by the above change of space variable.

Now given $\omega \subset \mathbb{T}$, for which $[1-a,1] \cup [0,a] \subset \omega$, $T > 0$, u_0 and u_1, we solve the problem of controllability with

$$\tilde{T} := \min(T, a/(2|\kappa|)),$$

$$\tilde{u}_0 := \kappa + u_0,$$

$$\tilde{u}_1(t,x) := \kappa + u_1(x - \kappa\tilde{T}),$$

$$\tilde{\omega} := (1 - a/2, 1] \cup [0, a/2).$$

One can check that returning to the variables $(t, x+\kappa t)$, we get a solution of the original problem, with a control located in ω. Of course the time of controllability may be shorter, but this is not a problem (bring u_0 to 0 and wait some time if necessary). This ends the proof that Proposition 1 implies Theorem 1. $\square$

For the rest of Section 2, we will consider that (15), (16) and (17) are satisfied (with ϵ to be chosen later), and hence we aim at proving Proposition 1. Now the construction of solutions to Proposition 1 is done in three consecutive steps:

- In a first time, we construct a solution that starts from u_0 and reaches at time $T/3$ a state which is in some sense close to $\overline{u}$ (which is yet to be defined).
- Then we construct a solution (close to $\overline{u}$), which starts at time $T/3$ from the previous state, and which satisfies $(1-\partial_{xx})u(2T/3,\cdot)=0$ in $\mathbb{T}\setminus\omega$.
- In a last (easy) step, we bring the latter state to 0.

Moreover, We expose these three steps in separate paragraphs.

2.2 First step

Let us first describe $\overline{u}$ more precisely. Consider

$$\mu := 4/T. \tag{18}$$

Then $\overline{u}(x)$ is any function satisfying:

$$\begin{cases} \overline{u} \in C^\infty(\mathbb{T}), \\ \overline{u} = \mu e^x \text{ in } [a/3, 1-a/3], \\ \overline{u} \geq \mu \text{ in } \mathbb{T}. \end{cases} \tag{19}$$

Note that (10) is clearly satisfied, that is the "vorticity"

$$\overline{y} := (1-\partial^2_{xx})\overline{u}, \tag{20}$$

is supported in ω.

Now during the first time interval $[0,T/3]$, we construct a solution of (11), starting from u_0 at $t=0$ and reaching a state close to $\overline{u}$ at $t=T/3$ in the sense that we have

$$\|u(T/3,\cdot)-\overline{u}\|_{H^s(\mathbb{T})} \leq \eta_1, \tag{21}$$

for some η_1 which we are going to choose later, and moreover

$$(1-\partial^2_{xx})\big[u(T/3,\cdot)-\overline{u}\big] = 0 \text{ in } [0,a/2]\cup[1-a/2,1]. \tag{22}$$

This is done through a Schauder fixed point scheme. First, consider $\psi \in C^\infty([0,T/3];\mathbb{R})$ satisfying

$$\begin{cases} \psi \geq 0,\ \psi' \geq 0, \\ \psi \equiv 0 \text{ in } [0,T/9] \text{ and } \psi \equiv 1 \text{ in } [2T/9, T/3]. \end{cases} \tag{23}$$

and $M \in C^\infty(\mathbb{T};\mathbb{R})$ satisfying

$$\begin{cases} 0 \leq M \leq 1, \\ M \equiv 1 \text{ in } [0,a/2]\cup[1-a/2,1] \text{ and } M \equiv 0 \text{ in } [a, 1-a]. \end{cases} \tag{24}$$

We consider the set

$$X_1 := \Big\{ u \in C^0([0,T/3]; H^s(\mathbb{T})) \cap C^1([0,T/3]; H^{s-1}(\mathbb{T})) \ \Big/ \ \|u - \psi(t)\overline{u}\|_{L^\infty([0,T/3];H^s(\mathbb{T}))} + \|u - \psi(t)\overline{u}\|_{\mathcal{L}ip([0,T/3];H^{s-1}(\mathbb{T}))} \leq \eta_1 \Big\},$$

with $\eta_1 \in (0,1)$ small enough to be defined. We embed X_1 in the space $L^\infty([0,T/3]; H^{s'}(\mathbb{T}))$ for some $\frac{3}{2} < s' < s$. Then it is a classical matter to observe that X_1 is a (nonempty) convex compact subset of $L^\infty([0,T/3]; H^{s'}(\mathbb{T}))$.

Now, we first define the operator S_1 on X_1 as follows:

$$S_1 : X_1 \ni u \longmapsto \text{ the solution } y \in C^0([0,T]; H^{s-2}(\mathbb{T})) \text{ of the transport equation:}$$
$$\left\{ \begin{array}{l} y_{|t=0} = y_0 := (1 - \partial^2_{xx})(u_0) \text{ in } \mathbb{T}, \\ \partial_t y + u \partial_x y = -2y \partial_x u \text{ in } [0,T/3] \times \mathbb{T}. \end{array} \right. \tag{25}$$

That the solution of the above equation (25) is well-defined follows by regularization: for regular initial data (u, y_0), these solutions are well-defined through characteristics (since $H^s(\mathbb{T}) \hookrightarrow \mathcal{L}ip(\mathbb{T})$ for $s > 3/2$); next the passage to the limit is for instance ensured by Lemma 1 below. The uniqueness of this solution also comes for instance from Lemma 1.

Then the operator $\mathcal{T}$ is given by

$$\mathcal{T} : X_1 \ni u \mapsto (1 - \partial^2_{xx})^{-1} \big[\big(1 - \psi(t) M(x)\big) S_1(u) \big] + \psi(t) \overline{u}(x).$$

We have to prove that, provided that $\|u_0\|_{H^s(\mathbb{T})}$ is small enough, $\mathcal{T}$ maps X_1 to X_1 and that it is continuous; then we have left to prove that a fixed point is a solution to (11) (for some g) satisfying (21).

$\mathcal{T}$ maps X_1 into itself. We use the following estimates for the solutions of linear transport equations:

Lemma 1 *Let $\sigma > -1/2$. Consider v a vector field in $L^1(0,T; (H^{\frac{3}{2}} \cap \mathcal{L}ip)(\mathbb{T}))$ if $\sigma \leq 3/2$, in $L^1(0,T; H^\sigma(\mathbb{T}))$ if $\sigma > 3/2$, and $a \in L^\infty(0,T; H^\sigma(\mathbb{T})) \cap C([0,T]; \mathcal{S}'(\mathbb{T}))$ the solution of the transport equation*

$$\partial_t a + v \partial_x a = f, \quad a_{|t=0} = a_0,$$

where $a_0 \in H^\sigma(\mathbb{T})$ and $f \in L^1(0,T; H^\sigma(\mathbb{T}))$. Then $a \in C^0([0,T]; H^\sigma(\mathbb{T}))$ and the following estimate holds for $t \in [0,T]$ and some constant $C > 0$:

$$\|a(t)\|_{H^\sigma} \leq e^{CV(t)} \Big(\|a_0\|_{H^\sigma} + \int_0^t e^{-CV(\tau)} \|f(\tau)\|_{H^\sigma} d\tau \Big), \tag{26}$$

where

$$V(t) := \left\{ \begin{array}{l} \int_0^t \|\partial_x v(\tau, \cdot)\|_{(H^{1/2} \cap L^\infty)(\mathbb{T})} \, d\tau \text{ if } \sigma \leq 3/2, \\ \int_0^t \|\partial_x v(\tau, \cdot)\|_{H^{\sigma-1}(\mathbb{T})} \, d\tau \text{ if } \sigma > 3/2. \end{array} \right. \tag{27}$$

This result is rather classical in this form (and for positive indices); one can find a much more general statement in Danchin [15, Proposition A.1] (see Lemma 3 below).

Also, we will need the following product lemma.

Lemma 2 *Consider $s, t \in \mathbb{R}$ such that $s \leq 1/2$, $t > 1/2$ and $s + t > 0$. Then the map*

$$\left\{ \begin{array}{c} H^s(\mathbb{T}) \times H^t(\mathbb{T}) \longrightarrow H^s(\mathbb{T}) \\ (u, v) \longmapsto uv, \end{array} \right. \quad \text{if } s \leq 1/2 \text{ and } t > 1/2,$$

is well-defined and continuous.

This result is also classical and can be proven by means of paradifferential calculus. See for instance [31]. Recall also that $H^s(\mathbb{T})$ is an algebra for $s > 1/2$.

Remark that the condition $s > 3/2$ implies that the products $y\partial_x u$ and $u\partial_x y$ (for $u \in H^s(\mathbb{T})$ and hence $y \in H^{s-2}(\mathbb{T})$) are well-defined in $H^{s-2}(\mathbb{T})$ and $H^{s-3}(\mathbb{T})$, respectively. (Distinguish the cases $s \leq 5/2$ and $s > 5/2$.)

Now using Lemmas 1 and 2 on (25) yields that for $u \in X_1$, we have for all $t \in [0, T/3]$,

$$\|S_1(u)(t)\|_{H^{s-2}} \leq e^{CV(t)}\Big(\|y_0\|_{H^{s-2}} + \int_0^t e^{-CV(\tau)}\|S_1(u)\partial_x u\|_{H^{s-2}} d\tau\Big) \tag{28}$$

$$\leq e^{CV(t)}\Big(\|y_0\|_{H^{s-2}} + \int_0^t e^{-CV(\tau)}\|S_1(u)\|_{H^{s-2}}\|\partial_x u\|_{H^{s-1}} d\tau\Big), \tag{29}$$

where we have denoted as in Lemma 2

$$V(t) := \int_0^t \|\partial_x u(\tau,\cdot)\|_{(H^{1/2}\cap L^\infty)(\mathbb{T})} d\tau \text{ is } s \leq 7/2 \text{ and } V(t) := \int_0^t \|\partial_x u(\tau,\cdot)\|_{H^{s-3}(\mathbb{T})} d\tau \text{ if } s > 7/2. \tag{30}$$

Using Gronwall's lemma and the definition of X_1 immediately yields the following estimate on $S_1(u)$:

$$\|S_1(u)\|_{L^\infty(0,T/3;H^{s-2}(\mathbb{T}))} \leq C\|y_0\|_{H^{s-2}(\mathbb{T})}, \tag{31}$$

where the constant depends on $\overline{u}$ and ψ but is independent of $\eta_1 \in (0,1)$.

Now using the continuity of $(1-\partial^2_{xx})^{-1}$ from $H^{s-2}(\mathbb{T})$ to $H^s(\mathbb{T})$, and the one of the multiplication by $(1-\psi(t)M(x))$ in $L^\infty(0,T/3;H^{s-2}(\mathbb{T}))\cap \mathcal{L}ip(0,T/3;H^{s-3}(\mathbb{T}))$, we see that for any $\eta_1 > 0$, there exists $\epsilon > 0$ such that under the assumption $\|u_0\|_{H^s(\mathbb{T})} < \epsilon$, $\mathcal{T}$ maps X_1 into itself.

The operator $\mathcal{T}$ is continuous. Consider $u \in X_1$ and a sequence $u_n \in X_1$ converging to u for the $L^\infty(0,T/3;H^{s'}(\mathbb{T}))$ norm; introduce together with these $y := S_1(u)$ and $y_n := S_1(u_n)$. Then using Lemma 2 it is not difficult to see that the equation

$$\begin{cases} y_{n|t=0} = y_0 := (1-\partial^2_{xx})(u_0) \text{ in } \mathbb{T}, \\ \partial_t y_n + u_n\partial_x y_n = -2y_n\partial_x u_n \text{ in } [0,T/3]\times\mathbb{T}. \end{cases}$$

passes to the limit. This ensures that $\mathcal{T}$ is continuous.

Conclusion. It follows from Schauder's fixed point theorem that $\mathcal{T}$ admits a fixed point in X_1, which clearly satisfies that

$$(1-\partial^2_{xx})u = (1-\psi M)S_1(u) + \psi\overline{y}$$

and consequently, keeping the notation $y := S_1(u)$,

$$y = (1-\partial^2_{xx})u - \psi(t)\overline{y}(x) + \psi(t)M(x)y \text{ in } [0,T/3]\times\mathbb{T}. \tag{32}$$

On another side, we have from (25) that

$$\partial_t y + u\partial_x y = -2y\partial_x u \text{ in } [0,T/3]\times\mathbb{T}. \tag{33}$$

Plugging (32) into (33) and recalling (10) and (23) yields that this fixed point satisfies (11) for some g supported in ω. The continuity in time is a direct consequence of Lemma 1. Note that (22) is satisfied.

Uniqueness. This is done exactly as [15, Proposition 2.1]. To see that uniqueness holds at this level or regularity, it is easier to consider the equation in "integrated form" (7). Then considering two solutions u and u', and their difference $w := u' - u$, we see that the following holds

$$\partial_t w + u\partial_x w = -w\partial_x u' - \partial_x(1-\partial^2_{xx})^{-1}\big(w(u+u') + \frac{1}{2}\partial_x w\partial_x(u+u') + 2\kappa w\big).$$

Now using Lemmas 1 and 2, and taking into account that the operator $\partial_x(1-\partial^2_{xx})^{-1}$ continuously maps $H^{s-2}(\mathbb{T})$ into $H^{s-1}(\mathbb{T})$, we reach the following estimate in $H^{s-1}(\mathbb{T})$:

$$\|w(t,\cdot)\|_{H^{s-1}(\mathbb{T})} \leq \exp\big\{CV(t)\big\} \\ \times \Big(\|w(0,\cdot)\|_{H^{s-1}(\mathbb{T})} + \int_0^t C\exp\{-CV(\tau)\}\big(\|w\|_{H^{s-1}}\|\partial_x u'\|_{H^{s-1}}(\tau) + \|w\|_{H^{s-1}}(\tau)\|u+u'\|_{H^{s-1}}(\tau)\big)d\tau\Big), \tag{34}$$

where again $V(t)$ is given by (30).

Using the uniform estimate in $H^s(\mathbb{T})$ for u and u' and Gronwall's lemma, we clearly reach the uniqueness.

2.3 Second step

Here we construct a solution u during $[T/3,2T/3]$, which starts from the above

$$\tilde{u}_0 := u(T/3,\cdot)$$

and reaches at time $2T/3$ a state satisfying

$$(1-\partial^2_{xx})u(2T/3,\cdot) = 0 \text{ in } \mathbb{T}\setminus\omega. \tag{35}$$

The idea is to use as an intermediate step a problem defined on the whole real line. When considering the real line as the domain, it is easy to construct a solution which makes the “vorticity” leave the domain $[0,1]$; from this we can deduce a solution of our problem on $\mathbb{T}$, modulo errors which are located inside ω. The solution is obtained again as a Schauder's fixed-point.

First we introduce $\Lambda \in C^\infty(\mathbb{T};\mathbb{R})$ such that

$$\left\{ \begin{array}{l} \Lambda \geq 0, \\ \Lambda \equiv 0 \text{ in } [0,a/3]\cup[1-a/3,1] \text{ and } \Lambda\equiv 1 \text{ in } [a/2,1-a/2], \end{array} \right.$$

Define $\hat{\Lambda}$ as the operator which maps functions on $\mathbb{T}$ to functions on $\mathbb{R}$ in the following way:

$$\hat{\Lambda}u := \left\{ \begin{array}{lll} \Lambda(x)u & \text{in} & [a/3,1-a/3], \\ 0 & \text{in} & \mathbb{R}\setminus[a/3,1-a/3]. \end{array} \right.$$

In the same way, given a function u defined on $[0,1]$ or on $\mathbb{R}$, we can define $\check{\Lambda}(u)$ as a function on $\mathbb{T}$ by considering $\Lambda(x)u_{|[0,1]}$, and extending it into a 1-periodic function.

We introduce

$$X_2 := \Big\{ u \in C^0([T/3,2T/3];H^s(\mathbb{T}))\cap C^1([T/3,2T/3];H^{s-1}(\mathbb{T})) \ \Big/ \\ \|u-\overline{u}\|_{L^\infty([T/3,2T/3];H^s(\mathbb{T}))} + \|u-\overline{u}\|_{\mathcal{L}ip([T/3,2T/3];H^{s-1}(\mathbb{T}))} \leq \eta_2 \Big\},$$

with $\eta_2>0$ to be determined. Again X_2 is a nonempty compact convex subset of $L^\infty([T/3,2T/3];H^{s'}(\mathbb{T}))$ for $s'\in(3/2,s)$. The operator $\mathcal{T}_2$ defined on X_2 is constructed as follows. First introduce the operator Π:

$$\Pi : X_2 \ni u \longmapsto \mu + \hat{\Lambda}(u-\mu) \in u \in C^0([T/3,2T/3];H^s(\mathbb{R}))\cap C^1([T/3,2T/3];H^{s-1}(\mathbb{R})).$$

Introduce

$$\tilde{y}_0 := \hat{\Lambda}\big[(1-\partial^2_{xx})(\tilde{u}_0-\overline{u})\big], \tag{36}$$

and

$S_2 : u \longmapsto$ the solution y on $\mathbb{R}$ of the transport equation

$$\left\{ \begin{array}{l} y_{|t=T/3} = \tilde{y}_0 \text{ in } \mathbb{R}, \\ \partial_t y + \Pi[u]\partial_x y = -2y\partial_x\Pi(u) \text{ in } [T/3,2T/3]\times\mathbb{R}. \end{array} \right. \tag{37}$$

$$\mathcal{T}_2 : X_2 \ni u \mapsto (1-\partial^2_{xx})^{-1}\Big[\check{\Lambda} y + \overline{y}\Big]. \tag{38}$$

That $\mathcal{T}_2$ admits a (unique) fixed point is done as in the previous paragraph: one needs to have that

$$\|u_{|t=T/3} - \overline{u}\|_{H^s(\mathbb{T})} \text{ is small enough,}$$

hence one uses (31) (or equivalently chooses η_1 small enough) and restricts to smaller y_0 if necessary. This can be done for any $\eta_2 > 0$. We omit the details.

Such a fixed point starts from the state $u(T/3)$ as seen from (22), (36), (37) and (38). Moreover, it satisfies equation (11) for some g, because, **in** $[\mathbf{a}, \mathbf{1}-\mathbf{a}]$, one has

$$\Pi(u) = u, \ \ y\partial_x \Pi(u) = y\partial_x u \text{ and } (1-\partial^2_{xx})u = y.$$

Hence injecting the last identity in (37) yields that u fulfills the equation for some g which satisfies $g = 0$ in $[a, 1-a]$. The regularity of the corresponding function g in $\mathbb{T}$ can be check from the other terms in (37).

Let us briefly explain why this fixed point (that we still call u) satisfies (35). By construction, using (19), (38), the Sobolev embedding of $H^s(\mathbb{T})$ into $L^\infty(\mathbb{T})$ and the definition of X_2, $\Pi[u]$ satisfies

$$\Pi[u] \geq \mu - c\eta_2 \text{ on } \mathbb{R},$$

for some $c > 0$. We choose $\eta_2 > 0$ such that $\mu - c\eta_2 \geq 3/T$ (recall (18)). It follows then from (36) and (37) that

$$\text{Supp } (y(2T/3, \cdot)) \subset \mathbb{R} \setminus [0, 1]. \tag{39}$$

Then (35) follows easily.

2.4 Final step

This step is obvious, since, because of (39),

$$[2T/3, T] \ni t \longmapsto \psi(T-t) u(2T/3, \cdot), \tag{40}$$

is a solution of (2) for some g, which drives the state of the system to 0. Note in particular that indeed the solution and the control are zero in a neighborhood of T.

The conclusion of Proposition 1 follows, except for what concerns the regularity in time. Indeed, as we glue three different parts of solutions, we obtain for the solution the regularity $C^0([0,T]; H^s(\mathbb{T}))$, but merely the regularity $\mathcal{L}ip([0,T]; H^{s-1}(\mathbb{T}))$, instead of $C^1([0,T]; H^{s-1}(\mathbb{T}))$ as claimed. To get this regularity, we observe that, due to the fact that all the three parts of the solution satisfy

$$\partial_t y + u\partial_x y = -2y\partial_x u \text{ outside } \omega, \tag{41}$$

the "reconnections" of y at times $T/3$ and $2T/3$ yield a function in $C^1([0,T]; H^{s-3}(\mathbb{T} \setminus \omega))$ outside ω. Hence it is sufficient to consider this function and to extend it in ω into a function of $C^0([0,T]; H^{s-2}(\mathbb{T})) \cap C^1([0,T]; H^{s-3}(\mathbb{T}))$, in such a way that one keeps the old values of y during $[0, T/9]$ and $[8T/9, T]$ (in these intervals, the "old" y has the correct regularity). The "new" y satisfies (41) with

$$y = (1-\partial^2_{xx})u \text{ outside } \omega, \tag{42}$$

and hence fulfills all the requirements.

This concludes the proof of Proposition 1, and hence of Theorem 1.

2.5 Remark on the result in Besov spaces

In fact it is not more difficult (due do the results already proved in [15] and [31]) and a little more general to establish the above result in the context of Besov spaces. We sketch here how the above results can be extended in this context.

Let us first briefly recall the definition of Besov spaces. Introduce $(\chi, \varphi) \in C_0^\infty(\mathbb{R};\mathbb{R})^2$ such that

$$\mathrm{Supp}\,(\chi) \subset \{\xi \in \mathbb{R} \;/\; |\xi| \leq 4/3\}, \;\; \mathrm{Supp}\,(\varphi) \subset \{\xi \in \mathbb{R} \;/\; 3/4 \leq |\xi| \leq 8/3\}$$
$$\text{and } \chi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1 \text{ on } \mathbb{R}.$$

The Littlewood-Paley decomposition is introduced as follows: for u a distribution on $\mathbb{T}$ decomposed in Fourier series

$$u = \sum_{\alpha\in\mathbb{Z}} \hat{u}_\alpha e^{2i\pi\alpha},$$

we consider the localization operators:

$$\Delta_q u = 0 \text{ for } q < -1, \;\; \Delta_{-1} u = \sum_{\alpha\in\mathbb{Z}} \chi(\alpha)\hat{u}_\alpha e^{2i\pi\alpha} = \hat{u}_0,$$
$$\text{and } \; \Delta_q u = \sum_{\alpha\in\mathbb{Z}} \varphi(2^{-q}\alpha)\hat{u}_\alpha e^{2i\pi\alpha} \;\text{ for } q \geq 0.$$

The Littlewood-Paley decomposition of a distribution u is

$$u = \sum_{q\geq -1} \Delta_q u.$$

Now for $(s,p,r) \in \mathbb{R} \times [1,+\infty]^2$, the Besov space $B^s_{p,r}(\mathbb{T})$ is defined as the Banach space of distributions on $\mathbb{T}$ satisfying

$$\|u\|_{B^s_{p,r}(\mathbb{T})} := \Big(\sum_{q\geq -1} \big[2^{sq}\|\Delta_q u\|_{L^p(\mathbb{T})}\big]^r \Big)^{1/r} < +\infty.$$

The Sobolev spaces correspond to $H^s(\mathbb{T}) = B^s_{2,2}(\mathbb{T})$.

It turns out that the whole construction before can be extended in the context of Besov spaces; it suffices to explain why the above fixed-point strategies can still apply. This is a consequence of the fact that Lemmas 1 and 2 can be generalized as follows (we refer to [15] and [31] respectively).

Lemma 3 *Consider $(p,r) \in [1,+\infty]^2$. Let $\sigma > -1/p$. Consider v a vector field in $L^1(0,T;(B^{1+\frac{1}{p}}_{p,r} \cap \mathcal{L}ip)(\mathbb{T}))$ if $\sigma \leq 1+\frac{1}{p}$, in $L^1(0,T;B^\sigma_{p,r}(\mathbb{T}))$ if $\sigma > 1+\frac{1}{p}$, and $a \in L^\infty(0,T;B^\sigma_{p,r}(\mathbb{T})) \cap C([0,T];\mathcal{S}'(\mathbb{T}))$ the solution of the following transport equation*

$$\partial_t a + v\partial_x a = f, \quad a_{|t=0} = a_0,$$

where $a_0 \in B^\sigma_{p,r}(\mathbb{T})$ and $f \in L^1([0,T];B^\sigma_{p,r}(\mathbb{T}))$. Then $a \in L^\infty([0,T];B^\sigma_{p,r}(\mathbb{T})) \cap C^0([0,T];\mathcal{S}')$ and the following estimate holds for $t \in [0,T]$ and some constant $C > 0$:

$$\|a(t)\|_{B^\sigma_{p,r}} \leq e^{CV(t)}\Big(\|a_0\|_{B^\sigma_{p,r}} + \int_0^t e^{-CV(\tau)}\|f(\tau)\|_{B^\sigma_{p,r}} d\tau\Big),$$

where

$$V(t) := \begin{cases} \int_0^t \|\partial_x v(\tau,\cdot)\|_{(B^{1/p}_{p,\infty}\cap L^\infty)(\mathbb{T})}\, d\tau \;\; \text{if } \sigma \leq 1+\frac{1}{p}, \\ \int_0^t \|\partial_x v(\tau,\cdot)\|_{B^{\sigma-1}_{p,r}(\mathbb{T})} d\tau \;\; \text{if } \sigma > 1+\frac{1}{p}. \end{cases}$$

Moreover $a \in C^0([0,T];B^\sigma_{p,r}(\mathbb{T}))$ when $r < +\infty$.

Lemma 4 *For $(s,t,p,r) \in \mathbb{R}^2 \times [1,+\infty]^2$, the following holds:*

$$B^s_{p,r}(\mathbb{T}) \text{ is an algebra when } s > 1/p,$$

and the map

$$\begin{cases} B^s_{p,r}(\mathbb{T}) \times B^t_{p,r}(\mathbb{T}) \longrightarrow B^s_{p,r}(\mathbb{T}) \\ \quad (u,v) \longmapsto uv, \end{cases} \quad \text{if } s+t>0, s \leq 1/p \text{ and } t > 1/p,$$

is well-defined and continuous.

Finally, the Besov spaces satisfy the following classical embedding results (see [31]).

Lemma 5 *The following properties hold:*
i. For $(s,p,r) \in \mathbb{R} \times [1,+\infty]^2$ such that $s > 1+1/p$, $B^s_{p,r}(\mathbb{T}) \hookrightarrow \mathcal{L}ip(\mathbb{T})$.
ii. For $(s_1,s_2,p,r) \in \mathbb{R}^2 \times [1,+\infty]^2$ with $s_1 < s_2$ the injection $B^{s_2}_{p,r}(\mathbb{T}) \hookrightarrow B^{s_1}_{p,r}(\mathbb{T})$ is compact.
iii. The operators ∂_x and $(1-\partial^2_{xx})^{-1}_{\mathbb{T}}$ map $B^s_{p,r}(\mathbb{T})$ to $B^{s-1}_{p,r}(\mathbb{T})$ and $B^{s+2}_{p,r}(\mathbb{T})$ respectively.

Once this is taken into account, Theorem 1 can be extended as follows (compare with [15, Theorem 2.3] for the Cauchy problem):

Theorem 3 *Consider $(p,r) \in [1,+\infty]^2$. Let $s > \max(3/2, 1+1/p)$. Let $u_0, u_1 \in B^s_{p,r}(\mathbb{T})$ and $T > 0$. Then there exists $g \in C^0([0,T]; B^{s-3}_{p,r}(\omega))$ if $r < +\infty$ (respectively in $L^\infty([0,T]; B^{s-3}_{p,r}(\omega))$ if $r = +\infty$) such that equation* (2) *has a unique solution $u \in C^0([0,T]; B^s_{p,r}(\mathbb{T})) \cap C^1([0,T]; B^{s-1}_{p,r}(\mathbb{T}))$ if $r < +\infty$ (in $L^\infty([0,T]; B^s_{p,r}(\mathbb{T})) \cap \mathcal{L}ip([0,T]; B^{s-1}_{p,r}(\mathbb{T}))$ if $r = +\infty$) satisfying* (3)*, and moreover this solution satisfies* (4)*.*

The main point is the product estimates used in (29) and in (34). One has to be able to define the products of the type $B^{s-1}_{p,r} \times B^{s-2}_{p,r}$ in $B^{s-2}_{p,r}$ —hence $s > 3/2$—, and to apply Lemma 3 —hence $s > 1+1/p$.

3 Stabilization

Here we establish Theorem 2. We begin by giving the explicit form of our feedback law.

3.1 Design of the feedback law

As in the previous section, we introduce $a > 0$ such that $[0,a) \cup (1-a,1) \subset \omega$, when considering $[0,1) \subset \mathbb{T} = \mathbb{R}/\mathbb{Z}$, translating ω if necessary. We introduce $\overline{u}$ and $\overline{y}$ approximately as in (19):

$$\begin{cases} \overline{u} \in C^\infty(\mathbb{T}), \\ \overline{u} = \mu e^x \text{ in } [a/3, 1-a/3], \\ \displaystyle\int_{\mathbb{T}} \overline{u} = 0, \end{cases} \tag{43}$$

and

$$\overline{y} := (1-\partial^2_{xx})\overline{u}.$$

Here $\mu > 0$ is chosen in order to allow that

$$\|\overline{y}\|_{L^2(\mathbb{T})} = 1. \tag{44}$$

The last condition in (43) is not really necessary (and in fact we could have used the same $\overline{u}$ as in Section 2), but it will reduce the computations in the sequel, because it involves that both $\overline{u}$ and $\overline{y}$ are orthogonal to constant states in $L^2(\mathbb{T})$.

We introduce the following functions $\mathcal{M}$ and $\mathcal{N}$ in $C^\infty(\mathbb{T})$ such that

$$\begin{cases} 0 \leq \mathcal{M} \leq 1 \text{ in } \mathbb{T}, \\ \mathcal{M} \equiv 0 \text{ in } [0,a/3] \cup [1-a/3,1] \text{ and } \mathcal{M} \equiv 1 \text{ in } [a/2, 1-a/2], \end{cases}$$

$$\begin{cases} 0 \leq \mathcal{N} \leq 1 \text{ in } \mathbb{T}, \\ \mathcal{N} \equiv 1 \text{ in } [0,a/2] \cup [1-a/2,1] \text{ and } \mathcal{N} \equiv 0 \text{ in } [a, 1-a]. \end{cases}$$

We introduce in $L^2(\mathbb{T})$ the orthogonal projectors π and $\pi^{\perp}$ on $\mathrm{Span}\{\overline{y}\}$ and $(\mathrm{Span}\{\overline{y}\})^{\perp}$ respectively. By (44), these are given by:

$$\pi(y) :=< \overline{y}, y >_{L^2} \overline{y} \ \text{ and } \ \pi^{\perp}(y) := y - \pi(y).$$

The feedback law $g = g[u]$ is given by the following formula: denote as previously $y := (1 - \partial^2_{xx})u$; then

$$\begin{aligned} g[u] \ &:= \ (1-\mathcal{M}(x))u(x)y_x(x) + 2(1-\mathcal{M}(x))u_x(x)y(x) + 2\kappa(1-\mathcal{M}(x))u_x(x) \\ &+ \ \kappa(1-\mathcal{M}(x))y_x(x) + \kappa < y, \overline{y}_x >_{L^2} \overline{y}(x) - \kappa < y, \overline{y} >_{L^2} \overline{y}_x(x) \\ &- \ \beta\|y+\kappa\|_{L^2}\Big[< \overline{y}, y >_{L^2} - K\|\pi^{\perp}(y) + \kappa\|_{L^2}\Big]\overline{y}(x) \\ &- \ \gamma\|y+\kappa\|_{L^2}(\pi^{\perp}(y)+\kappa)(x)\mathcal{N}(x), \end{aligned} \tag{45}$$

where the positive constants K, β and γ are to be determined later. One can easily check (using in particular Lemma 2) that $g : u \mapsto g[u]$ is continuous from $H^2(\mathbb{T})$ to $H^{-1}(\mathbb{T})$, and that all the terms in the above expression are supported in ω.

Remark 3 *We recall that for $\kappa = 0$, the system is naturally endowed with the scale invariance described in Paragraph 1.2. In that case, the feedback law* (45) *shares the same scale invariance (for $\lambda > 0$) as the other terms of the equation.*

Taking (45) into account, the "closed loop system", that is, equation (8) in which g is given by (45), is precisely the following system:

$$\begin{cases} \partial_t y + 2\kappa\mathcal{M}u_x + \mathcal{M}uy_x + 2\mathcal{M}yu_x = \kappa(1-\mathcal{M})y_x + \kappa < y, \overline{y}_x >_{L^2} \overline{y} - \kappa < y, \overline{y} >_{L^2} \overline{y}_x \\ \qquad\qquad -\beta\|y+\kappa\|_{L^2}\Big[< \overline{y}, y >_{L^2} - K\|\pi^{\perp}(y)+\kappa\|_{L^2}\Big]\overline{y} - \gamma\|y+\kappa\|_{L^2}(\pi^{\perp}(y)+\kappa)\mathcal{N}, \\ y = (1-\partial^2_{xx})u. \end{cases} \tag{46}$$

In fact it will be simpler to work with a slightly transformed version of equation (46). This is done by changing of unknown:

$$\check{u}(t,x) := u(t,x) + \kappa, \quad \check{y}(t,x) := y(t,x) + \kappa. \tag{47}$$

The system now reduces to

$$\begin{cases} \partial_t \check{y} - \kappa\check{y}_x + \mathcal{M}\check{u}\check{y}_x + 2\mathcal{M}\check{y}\check{u}_x = \kappa < \check{y}, \overline{y}_x >_{L^2} \overline{y} - \kappa < \check{y}, \overline{y} >_{L^2} \overline{y}_x \\ \qquad\qquad -\beta\|\check{y}\|_{L^2}\Big[< \overline{y}, \check{y} >_{L^2} - K\|\pi^{\perp}\check{y}\|_{L^2}\Big]\overline{y}(x) - \gamma\|\check{y}\|_{L^2}\mathcal{N}\pi^{\perp}\check{y}, \\ \check{y} = (1-\partial^2_{xx})\check{u}. \end{cases} \tag{48}$$

Note indeed that the convention $\int_{\mathbb{T}} \overline{u} = 0$ involves that

$$< \overline{y}, \kappa >= 0 \ \text{ and } \ \pi^{\perp}(\kappa) = \kappa.$$

Hence it is clear that proving Theorem 2 is reduced to proving the following proposition.

Proposition 2 *The system* (48) *is globally* 0*-asymptotically stable.*

In order to reduce the notations, we will from now on use u and y to refer in fact to $\check{u}$ and $\check{y}$.

The main idea yielding the result is the following: given the above parameter $K > 0$, which we will choose large enough later, we distinguish two situations for the state u:

- *Favorable situation:* we will refer to the situation as favourable when

$$< \overline{y}, y >_{L^2} \geq \frac{K}{2}\|\pi^{\perp}y\|_{L^2}. \tag{49}$$

  In this situation, the part of u determined by $\pi(y)$ dominates in some sense the part $\pi^{\perp}(y)$. Hence outside ω, u is somewhat close to $\mu < \overline{y}, y >_{L^2} e^x$, and one can rely on the stretching effect of the equation to get some decrease on y. To get a decrease inside ω, one has to extend u in a proper way and to rely on an additional damping term (the last one in (45)).

- *Unfavourable situation:* we consider that the situation is unfavourable when on the contrary
$$< \overline{y}, y >_{L^2} < \frac{K}{2} \|\pi^\perp y\|_{L^2}. \tag{50}$$
In that case, one would like to move towards the previous situation. This motivates the β term in (45). The form given to this term is intended to make the "additional amount" of $\overline{y}$ naturally disappear afterwards, when the favourable situation has been reached.

Finally, the other terms in (45) are intended to decouple the evolution of $\pi(y)$ (which is central in particular in the unfavourable situation) from the rest of the equation.

It will be convenient to introduce the following convex cones in $H^2(\mathbb{T})$ depending on $\chi \in \mathbb{R}^+$:
$$\Omega_\chi = \big\{u \in H^2(\mathbb{T});\ < \overline{y}, y >_{L^2} > \chi \|\pi^\perp y\|_{L^2}\big\},$$
$$\overline{\Omega}_\chi = \big\{u \in H^2(\mathbb{T});\ < \overline{y}, y >_{L^2} \geq \chi \|\pi^\perp y\|_{L^2}\big\}.$$
The behaviour of the solutions which we described above will be established by introducing the following functional of the state $u(t) \in H^2(\mathbb{T})$:
$$\mathcal{L}(u) := \Big(K\|\pi^\perp y\|_{L^2} - < \overline{y}, y >_{L^2} \Big)_+ + K\|\pi^\perp y\|_{L^2} + \frac{1}{2}| <\overline{y}, y>_{L^2} |, \tag{51}$$
and proving that it is a continuous Lyapunov functional. This is done in Paragraph 3.4.

Remark 4 *Of course, the corresponding Lyapunov functional for the original system* (46) *is given by*
$$\tilde{\mathcal{L}}(u) := \Big(K\|\pi^\perp(y) + \kappa\|_{L^2} - < \overline{y}, y >_{L^2} \Big)_+ + K\|\pi^\perp(y) + \kappa\|_{L^2} + \frac{1}{2}| <\overline{y}, y>_{L^2} |.$$

3.2 Local existence for the closed-loop system

The local-in-time existence of a solution of (48) is given by the following proposition.

Proposition 3 *There exists $c_\star > 0$ such that, for any u_0 in $H^2(\mathbb{T})$, there exists $T = T(u_0) \geq c_\star / \max(1, \|u_0\|_{H^2})$, and a function u in $C^0([0,T], H^2(\mathbb{T})) \cap C^1([0,T], H^1(\mathbb{T}))$, which satisfies equation* (48) *with initial value $u_{|t=0} = u_0$.*

Proof of Proposition 3.

Rewriting the system. We begin by rewriting the system: the main point is to separate in the equation the part of y in $\mathrm{Span}\{\overline{y}\}$ and the part in $(\mathrm{Span}\{\overline{y}\})^\perp$.

Note that in the functional framework of Proposition 3, all the terms in (48) are (at least) in $C^0([0,T]; H^{-1}(\mathbb{T}))$. We take the $H^{-1}(\mathbb{T}) \times H^1(\mathbb{T})$ duality product of (48) with $\overline{y}$. Noting that
$$\mathrm{Supp}(\overline{y}) \cap \mathrm{Supp}(\mathcal{M}) = \emptyset \text{ and } \mathcal{N} \equiv 1 \text{ on } \mathrm{Supp}(\overline{y}), \tag{52}$$
that
$$< \overline{y}_x, \overline{y} >_{L^2} = 0, \tag{53}$$
and that of course $\overline{y} \perp \pi^\perp y$, we easily arrive at
$$\frac{d}{dt} < \overline{y}, y >_{L^2} = -\beta \|y\|_{L^2} \big[< \overline{y}, y >_{L^2} - K\|\pi^\perp y\|_{L^2}\big]. \tag{54}$$

It follows immediately that

$$\frac{\partial}{\partial t}\pi(y) = -\beta\|y\|_{L^2}\big[< \overline{y}, y >_{L^2} - K\|\pi^\perp y\|_{L^2}\big]\overline{y}.$$

Withdrawing this from (48) one gets, using again (52),

$$\partial_t \pi^\perp(y) - \kappa y_x + \mathcal{M}(x)u\big[\pi^\perp(y)\big]_x + 2\mathcal{M}(x)\pi^\perp(y)u_x = \\ - \gamma\|y\|_{L^2}\mathcal{N}(x)\pi^\perp(y) + \kappa < \overline{y}_x, y >_{L^2} \overline{y} - \kappa < y, \overline{y} >_{L^2} \overline{y}_x.$$

It follows that

$$\partial_t \pi^\perp(y) + (\mathcal{M}(x)u - \kappa)\big[\pi^\perp(y)\big]_x = -\big[2\mathcal{M}(x)u_x + \gamma\|y\|_{L^2}\mathcal{N}(x)\big]\pi^\perp(y) + \kappa < \overline{y}_x, y >_{L^2} \overline{y}. \tag{55}$$

From now we will work either with (48) or with the system (54)-(55).

The operator $\mathcal{T}$. Now as in Section 2 we prove Proposition 3 by means of a fixed-point scheme of Schauder's type. To $u \in C^0([0,T]; H^2(\mathbb{T}))$, we associate $y := (1-\partial^2_{xx})u \in C^0([0,T]; L^2(\mathbb{T}))$, and then $< \overline{y}, y >_{L^2} \in C^0([0,T];\mathbb{R})$ and $y^\perp := \pi^\perp(y) \in C^0([0,T]; L^2(\mathbb{T}))$. Then the operator $\mathcal{T}$ maps a couple (u,Υ) in $C^0([0,T]; H^2(\mathbb{T})) \times C^0([0,T];\mathbb{R}^+)$ to $(\tilde{u}, \tilde{\Upsilon})$ in the following way:

$$\tilde{u} := (1-\partial^2_{xx})^{-1}(\tilde{y}^\perp + \alpha(t)\overline{y}) \tag{56}$$

and

$$\tilde{\Upsilon}(t) := \|\tilde{y}^\perp(t,\cdot)\|^2_{L^2(\mathbb{T})}, \tag{57}$$

where α and $\tilde{y}^\perp$ are the solutions of

$$\begin{cases} \dfrac{d}{dt}\alpha := -\beta\|y\|_{L^2}(\alpha - K\|y^\perp\|_{L^2}), \\ \alpha(0) := < (1-\partial^2_{xx})u_0, \overline{y} >_{L^2}, \end{cases} \tag{58}$$

and

$$\begin{cases} \partial_t \tilde{y}^\perp + (\mathcal{M}u - \kappa)\tilde{y}^\perp_x = Q(u,\Upsilon,\tilde{y}^\perp) + L(y), \\ \tilde{y}^\perp_{|t=0} = y_0^\perp := \pi^\perp(1-\partial^2_{xx})u_0, \end{cases} \tag{59}$$

where Q is the "quadratic" map

$$H^2(\mathbb{T}) \times \mathbb{R} \times L^2(\mathbb{T}) \ni (w,\nu,z) \mapsto -\big[2\mathcal{M}\partial_x w + \gamma\sqrt{|\nu| + < (1-\partial^2_{xx})w, \overline{y} >^2}\ \mathcal{N}\big]z,$$

and L is the linear map

$$L^2(\mathbb{T}) \ni y \mapsto \kappa < \overline{y}_x, y > \overline{y}.$$

It is clear that L and Q are continuous from $L^2(\mathbb{T})$ and $H^2(\mathbb{T}) \times \mathbb{R} \times L^2(\mathbb{T})$ respectively to $L^2(\mathbb{T})$; moreover Q also satisfies

$$\|Q(w,\nu,z)\|_{L^2} \lesssim \max(\|w\|_{H^2}, |\nu|^{1/2})\|z\|_{L^2}, \tag{60}$$

as follows easily from Sobolev inclusions.

Estimates. Now to prove that the operator $\mathcal{T} : (u,\Upsilon) \mapsto (\tilde{u},\tilde{\Upsilon})$ admits a fixed point, we need some estimates to determine a relevant domain for $\mathcal{T}$. Set

$$v(t) := \int_0^t \max\big[\|y(\tau)\|_{L^2}, \Upsilon(\tau)^{1/2}\big]\, d\tau.$$

First, from (58), one can see that for some $C > 0$,

$$|\alpha(t)| \le |< y_0, \overline{y} >_{L^2}| + C \int_0^t \|y(\tau)\|_{L^2}^2 \, d\tau. \tag{61}$$

Next, one clearly has for some $C > 0$,

$$\int_0^t \|\partial_x[\mathcal{M}u(\tau)]\|_{H^1} d\tau \le Cv(t).$$

Using Lemma 1 and (59)-(60), we reach the following estimate on $\tilde{y}^\perp$, for some $C > 0$:

$$\|\tilde{y}^\perp(t)\|_{L^2} \le e^{Cv(t)} \Big(\|y_0^\perp\|_{L^2} + C \int_0^t e^{-Cv(\tau)} \big[\max\big\{ \|y(\tau)\|_{L^2}, \Upsilon(\tau)^{1/2} \big\} \|\tilde{y}^\perp(\tau)\|_{L^2} + \|y(\tau)\|_{L^2} \big] d\tau \Big).$$

(As a matter of fact, the Υ term is not really necessary due to the sign of the second term in Q, but this has no importance.) We use the following form of Gronwall's lemma: for non-negative functions,

$$y(t) \le y(0) + \int_0^t f(\tau) d\tau + \int_0^t g(\tau) y(\tau) d\tau \text{ on } [0,T]$$
$$\Longrightarrow y(t) \le \exp\big[\int_0^t g(\tau) d\tau \big] \Big(y(0) + \int_0^t f(\tau) \exp\big[- \int_0^\tau g(s) ds \big] d\tau \Big) \text{ on } [0,T].$$

Applying this on $\exp[-Cv(t)] \|\tilde{y}^\perp(t)\|_{H^2}$ yields that

$$\|\tilde{y}^\perp(t)\|_{L^2} \le e^{2Cv(t)} \Big(\|y_0^\perp\|_{L^2} + C \int_0^t e^{-2Cv(\tau)} \|y(\tau)\|_{L^2} d\tau \Big).$$

Finally with (56) and (61) this leads to the following estimate on $\tilde{u}$: for some (new) constant $C > 0$, one has

$$\max\big[\|\tilde{u}(t)\|_{H^2}, \tilde{\Upsilon}(t)^{1/2} \big] \le e^{Cv(t)} \Big(2\|u_0\|_{H^2} + C \int_0^t e^{-Cv(\tau)} \|u(\tau)\|_{H^2} d\tau \Big) + \frac{C}{2} \int_0^t \|u(\tau)\|_{H^2}^2 d\tau. \tag{62}$$

Set

$$N := \max(2\|u_0\|_{H^2}, 1) \text{ and } \rho(t) := \frac{2N}{1 - 2CNt} \text{ for } 0 \le t < 1/(2CN). \tag{63}$$

It is then elementary to check that for any $T > 0$ such that

$$2CTN < 1, \tag{64}$$

the set

$$Y := \Big\{ (u, \Upsilon) \in L^\infty(0,T; H^2(\mathbb{T})) \times C^0([0,T]; \mathbb{R}^+) \ / \ \forall t \in [0,T], \ \max\big[\|u(t)\|_{H^2}, \Upsilon^{1/2}(t) \big] \le \rho(t) \Big\},$$

is stable by the process (59) (just inject (63) into (62)). We suppose from now on that (64) is satisfied.

Using this and (56)-(58)-(59), we see that for some $C_* > 0$, we can reach an estimate

$$\|\partial_t \tilde{u}(t)\|_{H^1(\mathbb{T})} \le C_*(\rho(t) + \rho^2(t)), \tag{65}$$

for u in Y.

We will need another estimate, provided by the following elementary lemma.

Lemma 6 *Consider $v \in L^1(0,T; \mathcal{L}ip(\mathbb{T}))$, $f \in L^\infty(0,T; L^2(\mathbb{T}))$ and $a \in C^0([0,T]; L^2(\mathbb{T}))$ the solution of the transport equation*

$$\partial_t a + v \partial_x a = f, \quad a_{|t=0} = a_0.$$

Then the following holds for all $s, t \in [0,T]$:

$$\|a(t)\|_{L^2}^2 - \|a(s)\|_{L^2}^2 = \int_s^t \int_{\mathbb{T}} v_x a^2 + 2fa.$$

Of course, Lemma 6 is clear for regular a, and is easily established by regularizing (a_0, f).

Now we infer from Lemma 6 and (59) that the following estimate holds for $s, t > 0$:

$$\left| \|\tilde{y}^\perp(t)\|_{L^2}^2 - \|\tilde{y}^\perp(s)\|_{L^2}^2 \right| \leq C|t-s| \left(\|u\|_{L_t^\infty \mathcal{L}ip_x} \|\tilde{y}^\perp\|_{L_t^\infty L_x^2}^2 + \left[\|y\|_{L_t^\infty L_x^2} \|\tilde{y}^\perp\|_{L_t^\infty L_x^2} + \|y\|_{L_t^\infty L_x^2} \right] \|\tilde{y}^\perp\|_{L_t^\infty L_x^2} \right).$$

It follows from what precedes that for any $(u, \Upsilon) \in Y$, we have also $\mathcal{T}(u, \Upsilon) \in Y$, and hence for a suitable constant $R_1 > 0$, the inequality holds for all $(u, \Upsilon) \in Y$ and $(s, t) \in [0, T]^2$:

$$\left| \|\tilde{y}^\perp(t)\|_{L^2}^2 - \|\tilde{y}^\perp(s)\|_{L^2}^2 \right| \leq R_1 |t-s|. \tag{66}$$

Also from (58), one can see that the following inequality is obtained for some $R_2 > 0$, for any $(u, \Upsilon) \in Y$:

$$\Big| < \tilde{y}, \overline{y} > (t) - < \tilde{y}, \overline{y} > (s) \Big| \leq R_2 |t-s|. \tag{67}$$

Fixed point and conclusion. Let us fix

$$\begin{aligned} X := \Big\{ (u, \Upsilon) \in C^0(0, T; H^2(\mathbb{T})) \times C^0([0,T]; \mathbb{R}) \ / \ \forall t \in [0, T],\ \|u(t)\|_{H^2} \leq \rho(t) \text{ and} \\ \text{i.} \quad \|\partial_t \tilde{u}(t)\|_{H^1(\mathbb{T})} \leq C_*(\rho(t) + \rho^2(t)), \\ \text{ii.} \quad \Upsilon \geq 0 \text{ and } \big|\Upsilon\big|_{\mathcal{L}ip(0,T)} \leq R_1, \\ \text{iii.} \quad \big| < \tilde{y}(\cdot), \overline{y} >_{L^2} > \big|_{\mathcal{L}ip(0,T)} \leq R_2 \Big\}, \end{aligned}$$

where $\big| \cdot \big|_{\mathcal{L}ip(0,T)}$ is the semi-norm on $\mathcal{L}ip(0, T)$:

$$\big|h\big|_{\mathcal{L}ip(0,T)} := \sup_{0 \leq s < t \leq T} \frac{|h(t) - h(s)|}{|t-s|}.$$

Clearly X is a nonempty closed convex subset of $C^0([0, T]; H^2(\mathbb{T})) \times C^0([0, T])$. We have already proved that $\mathcal{T}(X) \subset X$, since we obtained conditions i. to iii. by relying only on the estimate on $\|u(t)\|_{H^2}$. Hence, by Schauder's fixed-point theorem, it remains to prove that $\mathcal{T}$ is continuous and that $\mathcal{T}(X)$ is relatively compact in $C^0(0, T; H^2(\mathbb{T})) \times C^0([0, T])$, in order to get that $\mathcal{T}$ has a fixed point.

The continuity of $\mathcal{T}$ in the $C^0([0, T]; H^2(\mathbb{T})) \times C^0([0, T])$ topology is rather straightforward: let $(u_n, \Upsilon_n)_{n \in \mathbb{N}} \in X^{\mathbb{N}}$ converging to $(u, \Upsilon) \in X$. Then one introduces the corresponding flows associated to $(\mathcal{M} u_n - \kappa)$:

$$\partial_t \Phi^n(t, s, x) = [\mathcal{M} u_n - \kappa](t, \Phi^n(t, s, x)), \quad \Phi^n(t, t, x) = x. \tag{68}$$

Due to the uniform Lipschitz bound on $(\mathcal{M} u_n - \kappa)$ and Gronwall's lemma, these flows converge uniformly to the one associated to $(\mathcal{M} u - \kappa)$. We use the same notations $y_n := (1 - \partial_{xx}^2) u_n$ and $y := (1 - \partial_{xx}^2) u$ as previously, and so on for $\tilde{y}_n$, etc. We have:

$$L(y_n) \to L(y) \in C^0([0, T]; L^2(\mathbb{T})),$$

and

$$\lambda_n := -\big[2\mathcal{M} \partial_x u_n + \gamma \sqrt{\Upsilon_n + < y_n, \overline{y} >^2} \mathcal{N}\big] \to \lambda := -\big[2\mathcal{M} \partial_x u + \gamma \sqrt{\Upsilon + < y, \overline{y} >^2} \mathcal{N}\big] \text{ in } C^0([0, T] \times \mathbb{T}). \tag{69}$$

Now (59) yields that

$$\begin{aligned} \tilde{y}_n^\perp(t, x) = \exp\Big\{ \int_0^t \lambda_n(\tau, \Phi^n(\tau, t, x))\, d\tau \Big\} \\ \times \Big(y_0^\perp(\Phi^n(0, t, x)) + \kappa \overline{y} \int_0^t \exp\Big\{ \int_\tau^t \lambda_n(s, \Phi^n(s, t, x)) ds \Big\} < y_n(\tau, \cdot), \overline{y}_x > \, d\tau \Big), \end{aligned} \tag{70}$$

and the same is valid for $\tilde{y}^{\perp}$. Then the convergence of $\tilde{y}_n^{\perp}$ to $\tilde{y}^{\perp}$ in $L^\infty(0,T;L^2(\mathbb{T}))$ follows (and hence so does the one of $\tilde{\Upsilon}_n$ to $\tilde{\Upsilon}$). Since the right-hand side of (58) converges in $L^\infty(0,T)$, the convergence of α_n to α follows as well, which yields the continuity of $\mathcal{T}$.

Hence it remains to prove the relative compactness of $\mathcal{T}(X)$ in $C^0([0,T];H^2(\mathbb{T})) \times C^0([0,T])$. Consider a sequence $(u_n,\Upsilon_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, and we take the same notations as previously, that is, $\tilde{u}_n$ is the first part of $\mathcal{T}(u_n,\Upsilon_n)$, etc. To each u_n, we associate the flow $\Phi^n = \Phi^n(t,s,x)$ of $(\mathcal{M}u_n - \kappa)$ by (68). The uniform $L^\infty(0,T;H^2(\mathbb{T}))$ bound on u_n yields a uniform Lipschitz bound on Φ^n, hence up to a subsequence we can assume that

$$\Phi^n \to \Phi \in C^0([0,T]\times[0,T]\times\mathbb{T}).$$

On another side, using the $L^\infty(0,T;H^2(\mathbb{T}))$ and $\mathcal{L}ip(0,T;H^1(\mathbb{T}))$ bounds on u_n, we see that up to a subsequence,

$$u_n \to u \in C^0([0,T];H^{s'}(\mathbb{T})) \text{ for } \frac{3}{2} < s' < s.$$

In particular, we have the convergence of this subsequence in $C^0([0,T];\mathcal{L}ip(\mathbb{T}))$. Using the condition iii. in the definition of X, we see that, by extracting a subsequence again, we may also assume that

$$< y_n, \overline{y} > \to < y, \overline{y} > \text{ in } C^0([0,T]).$$

Finally, as we have a uniform Lipschitz estimate on Υ_n, we also have, up to a subsequence,

$$\Upsilon_n \to \Upsilon \text{ in } C^0([0,T]).$$

Of course, because of the last two convergences, one has

$$N_n(t) := \sqrt{< y_n, \overline{y} >^2 + \Upsilon_n} \to N(t) := \sqrt{< y, \overline{y} >^2 + \Upsilon} \text{ in } C^0([0,T]).$$

Recall that which $\tilde{y}_n^{\perp}$ can be written as in (70), where λ_n is defined in (69). Gathering the above convergences, we deduce that we have the convergence for the corresponding subsequence of $\tilde{y}_n^{\perp}$, and hence the one of $\tilde{\Upsilon}_n$. It is straightforward to get a converging subsequence in $C^0([0,T])$ for α_n since we clearly have a uniform Lipschitz estimate from (58); hence we get the convergence of the corresponding subsequence of $\tilde{u}^n$ in $C^0([0,T];H^2(\mathbb{T}))$. This completes the proof of the relative compactness of $\mathcal{T}(X)$.

Consequently, we get a fixed point of $\mathcal{T}$ in X. Let us briefly show that it gives a solution to the problem. Given such a fixed point (u,Υ), we take the $H^{-1}\times H^1$ product of (59) with $\overline{y}$. Using that $\mathrm{Supp}(\mathcal{M}) \cap \mathrm{Supp}(\overline{y}) = \emptyset$, that $\mathcal{N} = 1$ on $\mathrm{Supp}(\overline{y})$, and "integrating by parts", we deduce that for some function of the time $C(t)$,

$$\frac{d}{dt} < \tilde{y}^{\perp}, \overline{y} >_{L^2} = C(t) < \tilde{y}^{\perp}, \overline{y} >_{L^2} .$$

With the initial condition for $\tilde{y}^{\perp}$, this involves that $< \tilde{y}^{\perp}, \overline{y} > = 0$ for all times, and hence, with (56), that

$$\tilde{y}^{\perp} = \pi^{\perp}(y).$$

With (57) it follows that

$$\sqrt{|\Upsilon| + < y, \overline{y} >^2} = \|y\|_{L^2(\mathbb{T})}.$$

Hence we recover (55) from (59), and (54) from (56) and (58). Finally, the $C^1([0,T];H^1(\mathbb{T}))$ regularity of u follows from (59). This ends the proof of Proposition 3. □

Remark 5 *The uniqueness of the solutions in Proposition 3 is an open problem. As a matter of fact, this is not very important since the asymptotic stability property applies to all solutions. Remark that this is due to the feedback law: at this level of regularity, one can elementarily show the uniqueness of the free trajectories* *of* (2) *(see e.g. [15]). Note that even in the context of finite dimensional control systems, one may have to consider feedback laws which are not necessarily locally Lipschitz and for which uniqueness may be lost. See e.g. [10] for a discussion of this in the context of the Euler equation for bidimensional inviscid incompressible fluids.*

Remark 6 *Proposition 3 can be proven in any Sobolev space $H^s(\mathbb{T})$ for $s \geq 2$. It is essentially a matter of noticing that the above Q is continuous from $H^s(\mathbb{T}) \times \mathbb{R} \times H^{s-2}(\mathbb{T})$ to $H^{s-2}(\mathbb{T})$, with the corresponding estimate* (60)*. It is easy to check that for $s \geq 3$ uniqueness holds, by performing an estimate for the difference of two solutions of* (48) *in $H^{s-1}(\mathbb{T})$. (But a priori, properties* (5) *and* (6) *only hold for the $H^2(\mathbb{T})$ norm.)*

3.3 Energy estimate

Now the rest of the paper is devoted to the proof of Proposition 2.

In what follows, we will refer to the favourable and unfavourable situations such as described in Paragraph 3.1. The proof that $\mathcal{L}$ (defined in (51)) is a Lyapunov functional will rely on the following lemma.

Lemma 7 *There are some positive numbers γ_0, K_0, c_1 and C_1 independent of the choice of β such that the following holds whenever $\gamma \geq \gamma_0$, $K \geq K_0$. Consider u a solution of* (46) *defined on $[0, T]$, with the regularity of Proposition 3. Then for almost any $t \in (0, T)$, one has the following estimates, according to the situation:*

- *Favorable situation (under condition* (49)*):*

$$\frac{d}{dt}\|\pi^{\perp}(y)\|_{L^2} \leq -c_1\|\pi^{\perp}(y)\|_{L^2} < \overline{y}, y >_{L^2}, \tag{71}$$

- *Unfavourable situation:*

$$\frac{d}{dt}\|\pi^{\perp}(y)\|_{L^2} \leq C_1\|\pi^{\perp}(y)\|_{L^2}\|y\|_{L^2}. \tag{72}$$

Proof of Lemma 7. The above derivatives are defined almost everywhere thanks to (66), which thanks to Lemma 6 is valid for any solution of the closed-loop system having the regularity described in Proposition 3.

Now establishing (71)-(72) is essentially an energy estimate. Let us denote $y^{\perp} := \pi^{\perp}(y)$. We multiply (55) by $y^{\perp}$ and integrate on $\mathbb{T}$ (as a matter of fact, we use Lemma 6). One gets

$$\frac{d}{dt}\int_{\mathbb{T}} \frac{{y^{\perp}}^2}{2} - \int_{\mathbb{T}} \partial_x(\mathcal{M}(x)u - \kappa)\frac{{y^{\perp}}^2}{2} + 2\int_{\mathbb{T}} \mathcal{M}(x)u_x {y^{\perp}}^2 = -\gamma\|y\|_{L^2}\int_{\mathbb{T}} {y^{\perp}}^2\mathcal{N}(x)dx. \tag{73}$$

After simplification we get for almost every time

$$\frac{d}{dt}\int_{\mathbb{T}} \frac{{y^{\perp}}^2}{2} = -\frac{3}{2}\int_{\mathbb{T}} \mathcal{M}u_x {y^{\perp}}^2 + \frac{1}{2}\int_{\mathbb{T}} \mathcal{M}_x u {y^{\perp}}^2 - \gamma\|y\|_{L^2}\int_{\mathbb{T}} {y^{\perp}}^2\mathcal{N}(x)dx. \tag{74}$$

Now we discuss according to the case under view.

- *Unfavourable case:* in that case, using the fact that $(1 - \partial_{xx}^2)^{-1} : L^2(\mathbb{T}) \to H^2(\mathbb{T}) \hookrightarrow W^{1,\infty}(\mathbb{T})$, we see that the two first terms in the right hand side of (74) can be estimated together by

$$\left|\int_{\mathbb{T}} \mathcal{M}(x)u\partial_x\left(\frac{{y^{\perp}}^2}{2}\right)\right| \lesssim \|y\|_{L^2}\|y^{\perp}\|_{L^2}^2. \tag{75}$$

The remaining term in (73) is non-positive, and this yields

$$\frac{d}{dt}\|y^{\perp}\|_{L^2}^2 \leq C_1\|y^{\perp}\|_{L^2}^2\|y\|_{L^2}. \tag{76}$$

If $\pi^{\perp}(y(t))$ does not vanish, this gives (72). If it does, it follows also from (73) and Gronwall's lemma that one has $y^{\perp}(\tau) = 0$ for $\tau \geq t$, which gives again (72).

- *Favorable case:* the analysis in this situation relies on the following lemma:

Lemma 8 *In the favourable situation (that is if (49) is satisfied), if K is chosen large enough, we have*

$$\partial_x u \geq \frac{\mu}{2} < \overline{y}, y > \quad on\ \mathbb{T}\setminus\omega. \tag{77}$$

Proof of Lemma 8.
Recall that u and y are connected via

$$\begin{aligned} u &= (1-\partial_{xx}^2)^{-1}y \\ &= (1-\partial_{xx}^2)^{-1}(\pi(y)+\pi^{\perp}(y)) \ \text{ in } \mathbb{T} \\ &= \mu < \overline{y}, y >_{L^2} e^x + (1-\partial_{xx}^2)^{-1}\pi^{\perp}(y) \ \text{ in } \mathbb{T}\setminus\omega. \end{aligned}$$

Now, should K be large enough, we see, using elliptic regularity and Sobolev injections, that the relation $< \overline{y}, y > \geq K\|\pi^{\perp}(y)\|_{L^2}$ implies

$$\|(1-\partial_{xx}^2)^{-1}\pi^{\perp}(y)\|_{W^{1,\infty}(\mathbb{T})} \leq C_2\|\pi^{\perp}(y)\|_{L^2(\mathbb{T})} \tag{78}$$

$$\leq \frac{C_2}{K} < \overline{y}, y >_{L^2}, \tag{79}$$

which yields the result simply by choosing

$$K > \frac{2C_2}{\mu}. \tag{80}$$

□

From now we assume that (80) is satisfied. Using Lemma 8, we see that the only term in (74) whose sign is not easily determined is second one of the right-hand side. But it is easy to see that this term can be absorbed for instance by taking γ such that

$$\gamma \geq 20\|\mathcal{M}_x\|_{\infty}. \tag{81}$$

It follows that if γ is large enough, one has

$$\frac{d}{dt}\int_{\mathbb{T}}\frac{{y^{\perp}}^2}{2} \leq -\frac{3}{2}\int_{\mathbb{T}}\mathcal{M}(x)u_x{y^{\perp}}^2 - \frac{4\gamma}{5}\|y\|_{L^2}\int_{\mathbb{T}}{y^{\perp}}^2\mathcal{N}(x)dx. \tag{82}$$

Using again Lemma 8 and the fact that

$$\Big\{\mathcal{M}=1\Big\}\cup\Big\{\mathcal{N}=1\Big\} = \mathbb{T},$$

we easily arrive at the conclusion, by noting that

$$| < y, \overline{y} >_{L^2} | = < y, \overline{y} >_{L^2} \leq \|y\|_{L^2}.$$

This ends the proof of Lemma 7. □

3.4 Lyapunov functional

The goal of this section is to prove that the functional $\mathcal{L}$ introduced in Paragraph 3.1 is a Lyapunov function associated to the closed loop system, precisely, that it satisfies

$$\mathcal{L}(0) = 0, \tag{83}$$

$$\text{for some } c > 0,\ \mathcal{L}(u) \geq c\|u\|_{H^2} \text{ for all } u \in H^2(\mathbb{T}), \tag{84}$$

$$\mathcal{L}(u) \text{ decreases along a trajectory of (46) unless } u(t) = 0. \tag{85}$$

The existence of solutions in the large and the asymptotic stability of (46) are direct consequences of the existence of such an $\mathcal{L}$ and of Proposition 3.

As previously it will be crucial to distinguish in the state function y, the part that is proportional to $\overline{y}$ and the one that is orthogonal to it. We will use the following notation:

$$v_\perp(t) := \|\pi^\perp(y)\|_{L^2} \text{ and } v_\parallel(t) :=< \overline{y}, y(t) > . \tag{86}$$

Since the two first properties (83)-(84) are obvious regardless of the constant K (although conspicuously the constant c in (84) depends on it), it remains to prove the following proposition.

Proposition 4 *For a suitable choice of β, γ and K, the functional $\mathcal{L}(u)$ decreases along a trajectory of* (46) *unless $u(t)$ vanishes.*

We begin with a lemma.

Lemma 9 *The favourable situation is stable by the evolution of the closed-loop system.*

Proof of Lemma 9. Consider a solution u of the closed-loop system such as described in Proposition 3 and at time t at which $< \overline{y}, y(t) >_{L^2} \geq \frac{K}{2}\|\pi^\perp y(t)\|_{L^2}$. It is elementary to see that, according to Lemma 7 and (54),

$$\frac{d}{dt}(v_\parallel - \frac{K}{2}v_\perp) \geq -\beta(v_\parallel - \frac{K}{2}v_\perp)\|y\|_{L^2} + g, \tag{87}$$

for some non-negative function g. □

Proof of Proposition 4. Let us establish the decrease of the Lyapunov functional $\mathcal{L}$. Suppose that we have a solution u such as described in Proposition 3, defined on $[0, T]$. We discuss according to the situation:

- *First situation:* if $u(t) \in \Omega_K$, that is $v_\parallel > Kv_\perp$.

  In that case, due to the continuity of the solution with respect to time, we have in a neighborhood of t:

  $$\mathcal{L}(u) = Kv_\perp + \frac{v_\parallel}{2}.$$

  Now using Lemma 7, we infer that

  $$\frac{d\mathcal{L}}{dt} \leq -c_1 K v_\perp v_\parallel - \frac{\beta}{2}(v_\parallel - Kv_\perp)\|y\|_{L^2}. \tag{88}$$

  Notice that both terms on the right-hand side are non-positive. Moreover, by distinguishing the two cases $u(t) \in \Omega_{2K}$ and $u(t) \notin \Omega_{2K}$ it is easy to see that (90) implies for some $c > 0$:

  $$c_1 K v_\perp v_\parallel + \frac{\beta}{2}(v_\parallel - Kv_\perp)\|y\|_{L^2} \geq c\mathcal{L}^2(t).$$

- *Second situation:* if $u(t) \in \overline{\Omega}_{K/2} \setminus \Omega_K$ that is, $\frac{K}{2} v_\perp \leq v_\parallel \leq K v_\perp$.

  In that case, due to Lemma 9, we are in the favourable situation in $[t, T]$, and $\mathcal{L}$ reads:

$$\mathcal{L}(u) := 2Kv_\perp - \frac{1}{2} v_\parallel. \tag{89}$$

  Using again the notations of (86), we still have (87), and we deduce

$$\frac{d}{dt}\mathcal{L}(u(t)) \leq -2Kc_1 v_\perp(t) v_\parallel(t) + \frac{\beta}{2}(v_\parallel - K v_\perp)\|y\|_{L^2} \tag{90}$$

  Again the two terms on the right hand side are nonpositive, and one can easily bound from above the first one with a term of the type $-c\mathcal{L}^2(t)$ (note that $v_\parallel$ and $v_\perp$ are of comparable size in this case).

- *Third situation:* $0 \leq v_\parallel < \frac{K}{2} v_\perp$.

  In that case, $\mathcal{L}$ can still be written as in (89). However, as we are no longer in the favourable situation, the estimate becomes

$$\frac{d}{dt}\mathcal{L}(u(t)) \leq 2KC_1 v_\perp(t)\|y\|_{L^2} + \frac{\beta}{2}(v_\parallel - K v_\perp)\|y\|_{L^2}.$$

  On another side, we see that

$$v_\parallel - K v_\perp \leq -\frac{K}{2} v_\perp,$$

  and it follows that

$$\frac{d}{dt}\mathcal{L}(u(t)) \leq K(2C_1 - \frac{\beta}{4}) v_\perp(t)\|y\|_{L^2}.$$

  Hence if we choose β such that

$$\beta > 16 C_1, \tag{91}$$

  we can affirm that

$$\frac{d}{dt}\mathcal{L}(u(t)) \leq -\frac{\beta K}{8} v_\perp(t)\|y\|_{L^2}. \tag{92}$$

  (Recall from Lemma 7 that the constant C_1 does not depend on the choice of β.)

- *Fourth situation:* $v_\parallel < 0$.

  In that case, $\mathcal{L}$ becomes

$$\mathcal{L}(u) := 2Kv_\perp - \frac{3}{2} v_\parallel.$$

  It is then easy to see that under condition (91), one can still obtain (92). (As a matter of fact, the decrease is even better in that case.)

□

3.5 Global existence, asymptotic stabilization

Let us briefly conclude. The decrease of $\mathcal{L}$ ensures that a local solution u of the closed-loop system defined on the (maximal) time interval $[0, T^*)$ has a bounded $H^2(\mathbb{T})$ norm on $[0, T^*)$ (say by M). It follows easily that, should we suppose $T^* < +\infty$, we would get a contradiction by applying Proposition 3 with initial data $u(T^* - c_\star / \max(2, 2M), \cdot)$.

The stability property (6) is a trivial consequence of the monotonicity of $\mathcal{L}$: to get (6), it is for instance sufficient to start from a state sufficiently small in $H^2(\mathbb{T})$ norm to satisfy $\mathcal{L}(u_0) \leq \varepsilon$ to make sure that for all time, one has

$$K\|\pi^\perp(y)\| + \frac{1}{2}|< \overline{y}, y >| \leq \varepsilon.$$

Finally, let us discuss the asymptotic behaviour (5). Consider u a global in time solution of the closed loop system, in $C^0(\mathbb{R}^+; H^2(\mathbb{T})) \cap C^1(\mathbb{R}^+; H^1(\mathbb{T}))$. According to the situation, we may use (88), (90) or (92); it follows that in all the cases, and for some $c > 0$ independent of u,

$$\frac{d}{dt}\mathcal{L}(u(t)) \lesssim -c\mathcal{L}(u(t))^2.$$

It follows immediately that

$$\mathcal{L}(u(t)) \longrightarrow 0 \text{ as } t \to +\infty,$$

which establishes (5).

Remark 7 *One can see that*

$$\mathcal{L}(t) \leq \frac{1}{ct},$$

independently of the initial state. One can check that c can be chosen independent of $K \geq 1$; since K is chosen arbitrarily large, wee see that we can find a feedback law which damps as fast as we want the part $\pi^\perp y$ of the state (and consequently its "part " $y_{|\omega}$).

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