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AN ADDENDUM TO A J.M. CORON THEOREM CONCERNING THE CONTROLLABILITY OF THE EULER SYSTEM FOR 2D INCOMPRESSIBLE INVISCID FLUIDS

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RÉSUMÉ. – J.-M. Coron a établi un résultat de contrôlabilité approchée du système d'Euler pour les fluides parfaits incompressibles, dans les espaces L^p pour $p < +\infty$. Lorsque la partie du bord sur laquelle s'applique le contrôle ne rencontre pas toutes les composantes connexes du bord du domaine, on ne peut pas en général obtenir la contrôlabilité L^∞ car la loi de Kelvin impose un certain nombre d'invariants durant le processus. Dans cet article nous prouvons que ces invariants sont les seules objections à la contrôlabilité $W^{1,p}$ pour $p < +\infty$. Sous une hypothèse naturelle supplémentaire sur les profils de vitesse à connecter, on peut assurer un résultat de contrôlabilité approchée $W^{2,p}$. © 2001 Éditions scientifiques et médicales Elsevier SAS

ABSTRACT. – J.-M. Coron established a result of approximate controllability of the 2D Euler system for incompressible inviscid fluids in the L^p spaces for $p < +\infty$. When the controlled part of the boundary does not meet every connected component of the boundary of the domain, one cannot in general extend the result to the L^∞ controllability, because the Kelvin law guarantees some invariants during the process. Here we prove that these invariants are the only objection for the $W^{1,p}$ controllability. Under supplementary natural assumption on the flows we want to connect, we can improve the result to the $W^{2,p}$ approximate controllability. © 2001 Éditions scientifiques et médicales Elsevier SAS

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1. Introduction

1.1. Statement of the results

Consider Ω an open set in $\mathbb{R}^2$, nonempty, bounded, connected, C^∞-regular (precisely whose boundary is composed of a finite number of C^∞ non-intersecting Jordan curves, and which is situated at one side of these curves) and not simply connected. Let Σ be a nonempty open part of its boundary $\partial\Omega$, which does not intersect every connected component of $\partial\Omega$.

The general controllability theorem concerning the 2D Euler system for incompressible inviscid fluids – answering to a problem raised by J.-L. Lions in [7] – was established by J.-M. Coron in [4]. This result proves that this system is "approximately controllable" for (Ω, Σ), with respect to the $L^p(\Omega)$ topology, for all p in $[1, +\infty)$.

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Precisely, for every $T > 0$ and for all y_0 and y_1 in $C^\infty(\overline{\Omega};\mathbb{R}^2)$ such that

$$\operatorname{div} y_0 = \operatorname{div} y_1 = 0 \quad \text{in } \Omega, \tag{1}$$

$$y_0 \cdot \nu = y_1 \cdot \nu = 0 \quad \text{on } \partial\Omega \setminus \Sigma, \tag{2}$$

(where we denote by ν the unit exterior normal vector on the boundary), there exists a sequence $(y^n)_{n\in\mathbb{N}}$ of functions in $C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^2)$, which is composed of solutions of the Euler system for 2D incompressible inviscid fluids, that is:

$$\operatorname{div} y^n(t,x) = 0, \quad \forall (t,x) \in [0,T] \times \Omega, \tag{3}$$

$$\partial_t y^n(t,x) + \big(y^n(t,x)\cdot\nabla\big) y^n(t,x) = \nabla P^n(t,x), \quad \forall (t,x) \in [0,T]\times\Omega, \tag{4}$$

(for some function P^n in $C^\infty(\overline{\Omega}\times[0,T];\mathbb{R})$), which satisfies the condition on the boundary:

$$y^n(t,x)\cdot\nu(x) = 0, \quad \forall t \in [0,T], \forall x \in \partial\Omega\setminus\Sigma, \tag{5}$$

and moreover

$$y^n_{|t=0} = y_0 \quad \text{on } \Omega, \tag{6}$$

$$y^n_{|t=T} \to y_1 \quad \text{with respect to the } L^p(\Omega)\text{norm when } n \to +\infty, \tag{7}$$

for all $1 \leqslant p < +\infty$. One may furthermore require that y^n should coincide with y_1 at points situated at a distance superior to $1/n$ from the components of $\partial\Omega$ which do not intersect Σ.

Remark 1. – If we were not in the case where Σ does not meet every connected component of the boundary (in particular, this remark is valid when Ω is simply connected), but on the contrary in the case where it meets every one of them, the system would be exactly controllable (as shown in [4]). That is, we could substitute to the result (7) the following one:

$$y_{|t=T} = y_1 \quad \text{in } \Omega. \tag{8}$$

But in our precise situation, we cannot even obtain a better convergence result than the one in L^p, for all $p < +\infty$. For example, we have a negative result for the L^∞ approximate controllability problem.

Indeed, let us denote by $\Gamma_0, \dots, \Gamma_k$ the connected components of $\partial\Omega$ which meet Σ, and $\Gamma_{k+1}, \dots, \Gamma_g$ the ones which do not meet Σ. Let also Γ^b be the union of all connected components of $\partial\Omega$ which do not intersect Σ, i.e. $\bigcup_{i=k+1}^g \Gamma_i$.

Now consider $i \in \{k+1,\dots,g\}$. Then if one chooses y_0 and y_1 such that

$$\int_{\Gamma_i} y_0 \cdot \vec{\mathrm{d}x} \neq \int_{\Gamma_i} y_1 \cdot \vec{\mathrm{d}x},$$

the Kelvin law, which ensures

$$\int_{\Gamma_i} y_0 \cdot \vec{\mathrm{d}x} = \int_{\Gamma_i} y^n_{|t=T} \cdot \vec{\mathrm{d}x},$$

for a solution of the Euler system satisfying (5) (because, since $y(x,t)\cdot\nu(x,t)=0\ \forall x\in\Gamma_i$ and $\forall t\in[0,T]$, the loop Γ_i does not change when following the flow of the velocity), makes the L^∞ convergence impossible.

But if we restrict the problem to y_0 and y_1 satisfying

$$\int_{\Gamma_i} y_0\cdot\vec{\mathrm{d}x}=\int_{\Gamma_i} y_1\cdot\vec{\mathrm{d}x},\quad \forall i\in\{k+1,\dots,g\},\tag{9}$$

one can wonder if we can expect a better convergence of the sequence (y^n).

The purpose of this paper is to show that, indeed, if we are in the case described by (9), one can find a sequence (y^n), satisfying (3), (4), (5) and (6), and whose final value $y^n_{|t=T}$ converges to y_1 in the $W^{1,p}(\Omega)$ sense, for all p in $[1,+\infty)$. Moreover, one can require in addition the above coincidence property.

Let us remark that one cannot expect a really better convergence than this one, because the vorticity of y_0, viz. $\operatorname{curl} y_0$, in the process (3)–(4), is transported by the flow of y^n. In particular, $\operatorname{curl} y_{0|\Gamma^b}$ is transported inside each connected component of Γ^b. But $\operatorname{curl} y_{1|\Gamma^b}$ may be very different from any function obtained this way.

To have a precise counter-example, one may choose as a domain the annulus $B(0,2)\backslash\overline{B}(0,1)$, and take $\Sigma:=\partial B(0,2)$ as a control zone. We choose $y_0=0$ and define y_1 the following way: let $\psi_1:\overline{\Omega}\to\mathbb{R}$ be defined by:

$$\begin{aligned}&\Delta\psi_1=1\quad\text{in }\Omega,\\&\psi_1=0\quad\text{on }\partial\Omega.\end{aligned}\tag{10}$$

Now consider the function τ_1 defined by:

$$\begin{aligned}&\Delta\tau_1=0\quad\text{in }B(0,2)\backslash\overline{B}(0,1),\\&\tau_1=1\quad\text{on }\partial B(0,1),\\&\tau_1=0\quad\text{on }\partial B(0,2).\end{aligned}\tag{11}$$

We then set λ such that

$$\lambda\int_\Omega|\nabla\tau_1|^2=-\int_\Omega\tau_1.\tag{12}$$

Then we choose

$$y_1:=\nabla^\perp\psi_1+\lambda\nabla^\perp\tau_1.\tag{13}$$

One easily checks that y_0 and y_1 satisfy (9). (For that, remark that

$$\int_{\partial B(0,1)}\nabla^\perp\psi\cdot\vec{\mathrm{d}x}=\int_{\partial\Omega}\tau_1\partial_\nu\psi_1\,\mathrm{d}x,$$

and then integrate by parts.) But the $W^{1,\infty}$ controllability does not occur, because $\operatorname{curl} y_0=0$ on $\partial B(0,1)$, whereas $\operatorname{curl} y_1=1$ on $\partial B(0,1)$.

In consequence we set up the:

THEOREM 1. – *Let $T > 0$, and let y_0 and y_1 be two functions in $C^\infty(\overline{\Omega};\mathbb{R}^2)$ satisfying* (1), (2) *and* (9). *Then there exists a sequence (y^n) of functions in $C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^2)$ which satisfy* (3), (4), (5), (6), *and moreover*

$$y^n_{|t=T} \to y_1 \quad \text{in norm } W^{1,p}(\Omega), \tag{14}$$

for all p such that $1 \leqslant p < +\infty$. In addition to that, one can choose y^n in order that it satisfies:

$$y^n(T,x) = y_1(x) \quad \text{for all } x \text{ in } \Omega \text{ such that } \operatorname{dist}(x,\Gamma^b) \geqslant \frac{1}{n}. \tag{15}$$

Now one can wonder if the fact that during the process $\operatorname{curl} y_0$ is transported by the flow of the velocity of the fluid along any component of Γ^b is the only objection – in addition to (9) – to the $W^{2,p}$ approximate controllability. This is the aim of our second result. Precisely, we show the following theorem:

THEOREM 2. – *Let $T > 0$, and y_0 and y_1 two functions in $C^\infty(\overline{\Omega})$ satisfying* (1), (2), (9), *and moreover the condition*

"there exists $g-k$ diffeomorphisms $\mathcal{A}_{k+1},\dots,\mathcal{A}_g$, $\mathcal{A}_i:\Gamma_i\to\Gamma_i$,

preserving orientation, such that:

$$\forall i \in \{k+1,\dots,g\}, \quad \operatorname{curl} y_1 = \operatorname{curl} y_0 \circ \mathcal{A}_i \quad \text{on } \Gamma_i\text{''}. \tag{16}$$

Then there exists a sequence (y^n) of functions in $C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^2)$ which satisfy (3), (4), (5), (6), *and moreover*

$$y^n_{|t=T} \to y_1 \quad \text{in norm } W^{2,p}(\Omega), \tag{17}$$

for all p such that $1 \leqslant p < +\infty$. In addition to that, one can choose y^n in order that it satisfies (15).

Again, one cannot expect any better convergence, particularily the $W^{2,\infty}$ one. To have a counter-example, one can for example consider in the same way the annulus $B(0,2)\backslash\overline{B}(0,1)$ as a domain Ω, and all the same consider a control distributed on $\Sigma := \partial B(0,2)$. Then take $y_1 := \nabla^\perp\psi_1$ where ψ_1 is defined by (10), and $y_0 := \nabla^\perp\psi_0$, where ψ_0 is chosen in order that:

$$\begin{aligned}
&\Delta\psi_0 = 2-r, \quad \text{in } B(0,3/2)\backslash\overline{B}(0,1),\\
&\int_\Omega \tau_1\Delta\psi_0\,\mathrm{d}x = -\int_\Omega \tau_1\,\mathrm{d}x,\\
&\psi_0 = 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{18}$$

where $r = |x|$ and τ_1 is defined by (11).

From the construction, (9) and (16) are satisfied. But one cannot expect the $W^{2,\infty}$ (and hence the C^2) approximate controllability, because this would imply for any $\varepsilon > 0$ the existence of an orientation and area-preserving diffeomorphism $\mathcal{A}_\varepsilon$ from a neighbourhood of $\partial B(0,1)$ of Ω into another one, say $B(0,1+\tilde{\varepsilon})\backslash\overline{B}(0,1)$ such that

$$\|\operatorname{curl} y_0 - \operatorname{curl} y_1 \circ \mathcal{A}_\varepsilon\|_{C^1(B(0,1+\tilde{\varepsilon})\backslash\overline{B}(0,1))} \leqslant \varepsilon. \tag{19}$$

Indeed, for any n, there exists $\tilde{\varepsilon} > 0$ such that no point situated in $B(0, 1+\tilde{\varepsilon})\setminus\overline{B}(0,1)$ at the end of the flow of $-y^n$ corresponds to a point coming from Σ. (Of course, in that case, the vorticity of the point is constant when following the flow.)

But this is clearly impossible for ε small for

$$\big\|(\nabla \operatorname{curl} y_0)(x)\big\| = 1 \quad \text{on } \partial B(0,1),$$

whereas

$$\nabla \operatorname{curl} y_1 \equiv 0 \quad \text{on } \partial B(0,1).$$

1.2. Notations

Let us introduce a few notations. We recall that $g+1$ is the number of connected components of $\partial\Omega$, and $k+1$ the number of connected components of $\partial\Omega$ which meet Σ. For i between $k+1$ and g, we will consider the curves Γ_ε^i obtained by regrouping the points situated at distance ε from Γ^i. These curves are regular and do not intersect themselves nor each other if ε is small enough (say $\varepsilon < \varepsilon_0$), which we will systematically suppose. We will denote by Ω^ε the part of Ω which is composed of all points situated at distance at less ε from Γ_i, for all i between $k+1$ and g; precisely:

$$\Omega^\varepsilon := \left\{ x \in \Omega \mid \operatorname{dist}\left(x, \bigcup_{i=k+1}^{g} \Gamma_i\right) \geqslant \varepsilon \right\}. \tag{20}$$

We will denote by Ω_ε^i the part of Ω consisting in points situated at distance at most ε from Γ_i, viz.

$$\Omega_\varepsilon^i := \big\{x \in \Omega \mid \operatorname{dist}(x, \Gamma_i) \leqslant \varepsilon\big\}.$$

Let us finally denote by $\gamma^\varepsilon := \bigcup_{i=k+1}^{g} \Gamma_i^\varepsilon$.

We also introduce a real $R > 0$ large enough in order that $\overline{\Omega}$ is included in the open ball in $\mathbb{R}^2$ centered in 0 with radius R, which we denote by B_R. The unit outward normal vector on B_R will be denoted by $\tilde{\nu}$.

We will also be given a continuous operator π, which extends functions on $\overline{\Omega}$ of regularity $C^\infty(\overline{\Omega})$ to functions on B_R of regularity $C^\infty(\overline{B_R})$, with compact support in B_R.

For $V \in C^\infty(\overline{B_R} \times \mathbb{R}, \mathbb{R}^2)$ such that $V \cdot \tilde{\nu} = 0$ on $\partial B_R \times \mathbb{R}$ (where we denote by $\tilde{\nu}$ the unit normal exterior vector field on ∂B_R), we define the application $\phi^V : \mathbb{R}^2 \times B_R \to B_R$ as the flow of the vector field V, that is the application satisfying:

$$\frac{\partial \phi}{\partial t_2}(t_1, t_2, x) = V\big(\phi(t_1,t_2,x), t_2\big), \quad \phi(t_1, t_1, x) = x, \ \forall (t_1,t_2,x) \in \mathbb{R}^2 \times B_R. \tag{21}$$

The functions $\tau^i \in C^\infty(\overline{\Omega}; \mathbb{R})$, defined for all $i \in \{1, \dots, g\}$ by:

$$\Delta \tau^i = 0 \quad \text{in } \Omega, \tag{22}$$

$$\tau^i = 1 \quad \text{on } \Gamma_i, \tag{23}$$

$$\tau^i = 0 \quad \text{on } \partial\Omega \setminus \Gamma_i, \tag{24}$$

will also be useful.

Let b be a $C^\infty(\mathbb{R}; [0,1])$ function such that

$$b = 1 \quad \text{on } (-\infty, 1/2], \qquad b = 0 \quad \text{on } [3/4; +\infty), \qquad |b'| \leqslant 5 \quad \text{on } \mathbb{R}. \tag{25}$$

For $x \in \overline{B}_R$ and $d > 0$, we set $b^d(x) := b(\operatorname{dist}(x, \overline{\Omega})/d)$.

Finally, we will use the following notations: $|\cdot|_{W^{2,p}}$ is the sum of all L^p norms of second derivatives of a function; $|\cdot|_\delta$ is the Hölder norm with index $\delta \in (0,1)$.

1.3. The control

Let us remark that in the previous presentation, the control is not explicited, and we study an under-determined system. As a control, one may consider the normal local velocity of the fluid on Σ and the vorticity of the fluid on the points of Σ which enter the domain Ω, that is on the set

$$\big\{x \in \Sigma \mid y(x,t) \cdot \nu(x) < 0\big\}.$$

When given these supplementary boundary conditions, the Cauchy problem associated to the system (3)–(4) has at most one solution.

1.4. Structure of the article

In Section 2, we introduce some tools necessary in the construction.

In Section 3, we give, for a fixed time-dependent velocity field, the construction of a vector field which will be "reachable" for the linearized equation around some W, with initial value y_0 (under some assumptions on W).

In Section 4, we reproduce the construction of [4], in the hope of making the article clearer. Precisely, we show that this velocity field is actually reachable, if W is close enough to a given solution of the Euler system denoted by $\overline{y}$.

In Section 5, we prove Theorem 1, by deducing a non-linear solution from a sequence of solutions of linear problems, and by showing that the obtained solution solves the $W^{1,p}$ controllability problem.

In Section 6, we prove Theorem 2, by using a construction slightly modified with respect to Section 3, and also a proposition which allows to reduce the problem to the case when $\operatorname{curl} y_1$ has "the good shape" (by modifying the function $\overline{y}$).

Section 7 is devoted to the proof of the proposition of Section 6.

2. Some preliminary results

We will use the following lemma, due to J.-L. Lions (see [8, Théorème 5.1]):

LEMMA 1. – *Consider Ω a nonempty bounded regular open set in $\mathbb{R}^n$, with boundary $S_1 \cup S_2$, where S_1 and S_2 are two nonempty disjoint open sets of the boundary $\partial\Omega$. Let $p \in (1,+\infty)$. We consider the mapping from $W^{1-\frac{1}{p},p}(S_2)$ into $W^{1,p}(\Omega)$ defined by*:

$$u \mapsto y(u) \quad \text{such that} \begin{cases} \Delta y(u) = 0 & \text{in } \Omega, \\ y(u) = 0 & \text{on } S_1, \\ y(u) = u & \text{on } S_2. \end{cases} \tag{26}$$

Then $\partial_\nu y(u)_{|S_1}$ describes a dense subspace of $W^{-\frac{1}{p},p}(S_1)$ when u describes $W^{1-\frac{1}{p},p}(S_2)$.

Proof. – We reproduce the proof of [8], which is placed in the "$H^1(S_2)$" framework, instead of the "$W^{1-\frac{1}{p},p}(S_2)$" one as here, in order to make sure that the proof is still valid.

We argue by contradiction; suppose that there exists a certain non-zero distribution ψ in the dual of $W^{-\frac{1}{p},p}(S_1)$, that is in $W^{1-\frac{1}{q},q}(S_1)$ ($q \in (1,+\infty)$ being defined by $1/p + 1/q = 1$), such that for all $u \in W^{1-\frac{1}{p},p}(S_2)$, one has:

$$\langle \psi, \partial_\nu y(u)\rangle_{W^{1-\frac{1}{q},q}(S_1)\times W^{-\frac{1}{p},p}(S_1)} = 0. \tag{27}$$

Then one may define the function $\phi \in W^{1,q}(\Omega)$ by:

$$\begin{aligned} &\Delta\phi = 0 \quad \text{in } \Omega,\\ &\phi = \psi \quad \text{on } S_1,\\ &\phi = 0 \quad \text{on } S_2. \end{aligned} \tag{28}$$

Then by computing $\int_\Omega \nabla\phi\cdot\nabla y(u)$, one obtains that for all $u \in W^{1-\frac{1}{p},p}(S_2)$

$$\langle \partial_\nu\phi, y(u)\rangle_{W^{-\frac{1}{q},q}(\partial\Omega)\times W^{1-\frac{1}{p},p}(\partial\Omega)} = \langle \partial_\nu y(u), \phi\rangle_{W^{-\frac{1}{p},p}(\partial\Omega)\times W^{1-\frac{1}{q},q}(\partial\Omega)}. \tag{29}$$

With (27), one gets that for all $u \in W^{1-\frac{1}{p},p}(S_2)$, one has

$$\langle \partial_\nu\phi; y(u)\rangle_{W^{-\frac{1}{q},q}(\partial\Omega)\times W^{1-\frac{1}{p},p}(\partial\Omega)} = 0,$$

which involves, with (26), $\partial_\nu\phi = 0$ on S_2, which with (28) implies $\phi = 0$ in Ω, and consequently $\psi = 0$, which is contradictory. □

We add here two classical results to which we will refer in the next sections. The first one is a particular extension theorem:

LEMMA 2. – *For every $k \in \mathbb{N}$, for all $p \in (1,+\infty)$, there exists a constant C depending only on Ω, k and p such that: for all f in $C^\infty(\overline{\Omega};\mathbb{R})$, there exists another function $g \in C^\infty(\overline{\Omega};\mathbb{R})$ satisfying the two following properties*:

$$\exists\varepsilon > 0, \quad f \equiv g \quad \text{in } \bigcup_{i=k+1}^{g} \overline{\Omega}_i^{\varepsilon}, \tag{30}$$

$$\|g\|_{W^{k,p}(\Omega)} \leqslant C\|f\|_{W^{k-1/p,p}(\bigcup_{i=k+1}^{g}(\Gamma_i))}. \tag{31}$$

The proof is clear and left to the reader: let us just remark, that, as a constant C, one can take for example the best constant in the trace formula $W^{k,p}(\Omega) \to W^{k-1/p,p}(\partial\Omega)$ plus 1.

The second following lemma is a kind of Poincaré inequality. Recall that we denote by Γ^b the union of the connected components of $\partial\Omega$ which do not intersect Σ.

LEMMA 3. – *Given $k \in \mathbb{N}$, there exists some constant $C > 0$ such that for every $\varepsilon \in (0,\varepsilon_0)$, and every $f \in W^{k+1,p}(\Omega\backslash\overline{\Omega^\varepsilon})$, one has the following relation*:

$$|f|_{W^{k,p}(\Omega\backslash\overline{\Omega^\varepsilon})} \leqslant C\big(\|f\|_{W^{k-\frac{1}{p},p}(\Gamma^b)} + \varepsilon|f|_{W^{k+1,p}(\Omega\backslash\overline{\Omega^\varepsilon})}\big). \tag{32}$$

Proof. – This is a classical Poincaré's lemma. To obtain this, one first reduces to the case when f has a null trace on Γ^b by finding $\overline{f} \in W^{k+1,p}(\Omega)$ such that

$$f - \overline{f} = 0 \quad \text{on } \Gamma^b, \tag{33}$$

$$\|\overline{f}\|_{W^{k,p}(\Omega)} \leqslant C\|f\|_{W^{k-\frac{1}{p},p}(\Gamma^b)}. \tag{34}$$

Then, for f with 0 trace on Γ^b, relation (32) follows from the L^p Poincaré's inequality for a band with width ε, which one can adapt, by means of pasting, to an open set with "width" ε, such as $\Omega \backslash \overline{\Omega^\varepsilon}$. □

3. The construction of a particular accessible function

In this section, our goal is mainly to construct, for a given $W : B_R \times [0,T] \to \mathbb{R}^2$ which satisfies $W \cdot \tilde{\nu} = 0$ on $\partial B_R \times [0,T]$, a solenoidal vector field which is a target for the controlled linearized system around W. Let us precise here that by linearized system around W we mean in the whole paper the following system in z ("in vorticity"):

$$\begin{aligned}
&\partial_t \omega + (W \cdot \nabla)\omega = 0 \quad \text{in } B_R \times [0,T],\\
&\operatorname{div} z = 0 \quad \text{in } \Omega \times [0,T],\\
&\operatorname{curl} z = \omega \quad \text{in } \Omega \times [0,T],\\
&z \cdot \nu = 0 \quad \text{on } \partial\Omega \backslash \Sigma.
\end{aligned}$$

In the next section, we will actually prove that for proper W, the vector field presented here can be achieved as the final value of a solution of the linearized equation around W. Of course, this vector field is intended to be "close" to y_1 and particularily to satisfy a coincidence property such as (15). More precisely, we construct a family of targets y^β indexed by a positive number β, which will satisfy

$$y^\beta = y^1 \quad \text{in } \Omega^\beta.$$

In this whole part, p is a fixed real number, in $(1 + \infty)$. (In fact, we construct a reachable vector field which will give (14) for a fixed p; we will later on prove that one can require (14) "for all p".)

The construction makes use of a potential solution of the Euler equation, that we call $\overline{y}$. It is not explicited here, but it will be in Section 4.

The first step consists in constructing a solution "without control". Precisely, we construct a function $y^w \in C^\infty(\overline{\Omega} \times [0,T]; \mathbb{R}^2)$ as a fixed point of the following process.

First, we define the functionnal space in which this fixed point is to be found. For this, we introduce functions r and $\overline{q}$ the following way: we introduce the following function from $\mathbb{R}^{+*}$ into $\mathbb{R} \cup \{+\infty\}$ defined by:

$$\begin{aligned}
&\xi(s) = s + s \log\frac{1}{s} \quad \text{for } s \in (0,1),\\
&\xi(s) = s \quad \text{for } s \geqslant 1,
\end{aligned} \tag{35}$$

and the function $\overline{q}$ from $C^0(\overline{\Omega} \times [0,T])$ into $\mathbb{R} \cup \{+\infty\}$:

$$\overline{q}(y) := \sup\left\{ \left|y(\cdot,t)\right|_0 + \sup\left\{ \frac{|y(x_2,t) - y(x_1,t)|}{\xi(|x_2 - x_1|)}, x_1, x_2 \in \overline{\Omega}, x_1 \neq x_2 \right\}, t \in [0,T] \right\}. \tag{36}$$

Now, to any y in

$$S := \left\{ y \in C^0\big(\overline{\Omega} \times [0,T]\big),\ \overline{q}(y) < +\infty,\ y \cdot \nu = b\left(\frac{2t}{T}\right) y_0 \cdot \nu \right\}, \tag{37}$$

one associates $P(y)$ by:

$$\begin{aligned}
&\operatorname{div} P(y) = 0 \quad \text{in } \overline{\Omega} \times [0,T],\\
&\operatorname{curl} P(y) = \omega^* \quad \text{in } \overline{\Omega} \times [0,T],\\
&P(y) \cdot \nu = b\Big(\frac{2t}{T}\Big) y_0 \cdot \nu \quad \text{on } \partial\Omega \times [0,T],\\
&\int_\Omega \big[\partial_t P(y) + (P(y) \cdot \nabla) P(y)\big] \cdot \nabla^\perp \tau_i = 0 \quad \text{on } [0,T], \forall i \in \{1,\ldots,g\},\\
&\int_\Omega P(y)(\cdot,0) \cdot \nabla^\perp \tau_i = \int_\Omega y_0 \cdot \nabla^\perp \tau_i \quad \forall i \in \{1,\ldots,g\},
\end{aligned} \tag{38}$$

where ω^* is a function in $C^\infty(\overline{B_R}; \mathbb{R})$ defined by:

$$\begin{aligned}
&\omega^*(0,\cdot) = \operatorname{curl}(\pi y_0) \quad \text{in } B_R,\\
&\partial_t \omega^* + \big(\pi(y) \cdot \nabla\big) \omega^* = 0 \quad \text{in } B_R \times [0,T].
\end{aligned} \tag{39}$$

One can find a unique fixed point of P (this follows from the classical method of [6] and [11], except that here we impose non-homogenous boundary conditions; we refer to these articles for more precisions) which gives us a regular solution of the Euler system. Let us denote this fixed point by y^w. (Note that by "without control" we do not mean that the control described in Section 1.3 is 0, but that we do not make the decisive control here.)

Let us now introduce the functions ψ^w in $C^\infty(\overline{\Omega} \times [0,T])$, ψ_0 and ψ_1 in $C^\infty(\overline{\Omega})$ the following way:

$$\begin{aligned}
&\Delta \psi^w = \operatorname{curl} y^w \quad \text{in } \Omega \times [0,T],\\
&\psi^w = 0 \quad \text{on } \partial\Omega \times [0,T],
\end{aligned} \tag{40}$$

$$\begin{aligned}
&\Delta \psi_0 = \operatorname{curl} y_0 \quad \text{in } \Omega,\\
&\psi_0 = 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
&\Delta \psi_1 = \operatorname{curl} y_1 \quad \text{in } \Omega,\\
&\psi_1 = 0 \quad \text{on } \partial\Omega.
\end{aligned} \tag{42}$$

We obtain this way the following unique decomposition of y_0 and y_1:

$$y_i = \nabla^\perp \psi_i + \nabla \theta_i + \sum_{j=1}^{j=g} l_i^j \nabla^\perp \tau_j, \tag{43}$$

for $i \in \{0, 1\}$, where θ_i is a function defined up to a constant by:

$$\begin{aligned} \Delta \theta_i &= 0 \quad \text{in } \Omega, \\ \partial_\nu \theta_i &= y_i \cdot \nu \quad \text{on } \partial\Omega. \end{aligned} \tag{44}$$

We consider $\beta > 0$ a fixed number. For this β, according to Lemma 1 (for which we choose $S_1 = \Gamma^b$ and $S_2 = \partial\Omega \backslash \Gamma^b$), there exists a function u defined in $W^{1-\frac{1}{p}}(\bigcup_{i=0}^k \Gamma_i)$ such that

$$\left\| \partial_\nu y(u) + \partial_\nu \psi_1 - \partial_\nu \psi^w(T) \right\|_{W^{-\frac{1}{p},p}(\Gamma^b)} < \beta/2.$$

Regularizing u if needed, one can require that u should satisfy

$$u \in C^\infty\left(\bigcup_{i=0}^k \Gamma_i\right), \tag{45}$$

$$\left\| \partial_\nu y(u) + \partial_\nu \psi_1 - \partial_\nu \psi^w(T) \right\|_{W^{-\frac{1}{p},p}(\Gamma^b)} < \beta. \tag{46}$$

For this u, one applies Lemma 2 to $y(u)$. For β chosen small enough (in terms of Ω and p), one obtains a function $H \in C^\infty(\overline{\Omega})$ such that

$$H = y(u) \quad \text{in } \overline{\Omega} \backslash \Omega^{r(\beta)}, \tag{47}$$

for some $0 < r(\beta) < \varepsilon_0$, in such a way that relation (46) occurs when we replace $y(u)$ by H and such that

$$|H|_{W^{2,p}(\Omega \backslash \overline{\Omega^\beta})} \leqslant 2. \tag{48}$$

Besides, we may choose $r(\beta)$ in order that it satisfies $r(\beta) < \beta/2$.

Note that by construction, one has

$$\Delta H = 0 \quad \text{in } \Omega \backslash \overline{\Omega^{r(\beta)}}. \tag{49}$$

Now for our considered W, we introduce the vector field $\tilde{W}$ defined in $C^\infty(\overline{B_R \times [0, T]}, \mathbb{R}^2)$ by

$$\tilde{W} := \overline{y} + \pi(W - \overline{y}). \tag{50}$$

Note that this implies in particular that

$$\tilde{W} \cdot \tilde{\nu} = 0 \quad \text{on } \partial B_R \times [0, T].$$

For this fixed $\tilde{W}$, we define ω^* as the function in $C^\infty([0,T]\times\overline{B_R},\mathbb{R})$ satisfying:

$$\begin{aligned}&\omega^*(\cdot,0)=\operatorname{curl}(\pi y_0)\quad \text{in } B_R,\\ &\partial_t\omega^*+(\tilde{W}\cdot\nabla)\omega^*=0\quad \text{in } B_R\times[0,T].\end{aligned}\tag{51}$$

We deduce from it the function $\psi^*\in C^\infty([0,T]\times\overline{\Omega},\mathbb{R})$ as follows:

$$\begin{aligned}&\Delta\psi^*=\omega^*\quad \text{in } \Omega\times[0,T],\\ &\psi^*=0\quad \text{on } \partial\Omega\times[0,T].\end{aligned}\tag{52}$$

Here, as in [4], the point is to "glue" $\psi^*(T)$ and ψ_1 in order to obtain a $\tilde{\psi}$ in such a way that its second derivatives are not "too big".

We define the following family of functions indexed by $\alpha\in(0,\varepsilon_0)$:

$$\begin{aligned}&\rho_\alpha=0\quad \text{in } \Omega^\alpha,\\ &\rho_\alpha=1\quad \text{in } \Omega\backslash\Omega^{\alpha/2},\\ &\|\rho_\alpha\|_{C^0}=1,\\ &\|\nabla\rho_\alpha\|_{C^0}<K/\alpha,\\ &\|\nabla^2\rho_\alpha\|_{C^0}<K/\alpha^2,\end{aligned}\tag{53}$$

for some constant K independant of α in $(0,\varepsilon_0)$.

We then define $\tilde{\psi}$ on Ω by:

$$\tilde{\psi}=(1-\rho_\beta)\psi_1+\rho_\beta\big(\psi^*(T)+H\big).\tag{54}$$

But we still have to modify once again this $\tilde{\psi}$. For $i\in\{k+1,\dots,g\}$ and for $\alpha>0$ small, we introduce the function η^i_α from $\overline{\Omega}$ into $\mathbb{R}$, C^∞-regular, such that

$$\begin{aligned}&0\leqslant\eta^i_\alpha\leqslant 1,\\ &\big|\nabla\eta^i_\alpha\big|\leqslant\frac{C}{\alpha}\\ &\big|\nabla\nabla\eta^i_\alpha\big|\leqslant\frac{C}{\alpha^2}\\ &\operatorname{Supp}\eta^i_\alpha\subset\Omega\backslash\Omega^\alpha,\\ &\operatorname{Supp}\big(1-\eta^i_\alpha\big)\subset\Omega^{\alpha/2}.\end{aligned}\tag{55}$$

One may consider then the following function for $i\in\{k+1,\dots,g\}$:

$$\tilde{\psi}_i=\mu_i\eta^i_\beta(1-\tau_i),\tag{56}$$

where we have ruled $\mu_i \in \mathbb{R}$ in order that

$$\int_{\Omega} (\Delta\tilde{\psi} - \operatorname{curl} y_1) \cdot \tau_i = \mu_i \int_{\Omega} |\nabla\tau_i|^2. \tag{57}$$

(Note that this expression is different from the one of [4]; it is equivalent only because we have the supplementary assumption (9).)

The "$\nabla^{\perp}\psi$" part of the searched accessible function (in a decomposition such as (43)) is then given by:

$$\hat{\psi} := \tilde{\psi} + \sum_{i=k+1}^{g} \tilde{\psi}_i. \tag{58}$$

Finally, one defines the searched solenoidal vector field by:

$$y^{\beta} = \nabla^{\perp}\hat{\psi} + \nabla\theta_1 + \sum_{j=1}^{j=g} l_1^j \nabla^{\perp}\tau_j. \tag{59}$$

4. The reachability of the presented velocity field

In this section, we recall the construction due to J.-M. Coron. The general idea is that the linearized equation around 0 is not controllable, but nevertheless one can hope to control the one around a particular solution $\overline{y}$ of the Euler system which begins and ends at 0.

First, we describe this $\overline{y}$. Then, we describe a solution of the linearized Euler system around W (which is regular for W close enough to $\overline{y}$). Finally, we show that for W close enough to $\overline{y}$, this solution actually reaches the vector field given by (59).

4.1. The function $\overline{y}$

The function $\overline{y}$ is chosen as a potential solution of the Euler system (i.e $\overline{y} = \nabla\theta$) with local support in time. In fact we choose (for the moment) two types of "$\nabla\theta$".

The first type of θ is given by the following lemma. To state it, we introduce a function $a \in C^{\infty}([0,1],[0,1])$, different from 0, and with support in $(0,1)$. Then one has:

LEMMA 4 ([4], Proposition 2.1). – *For any i in $\{1,\dots,k\}$, there exist θ^i in $C_0^{\infty}(B_R;\mathbb{R})$ and ω_0^i in $C^{\infty}(\overline{B}_R;\mathbb{R})$ such that*

$$\Delta\theta^i = 0 \quad \text{in } \overline{\Omega}, \tag{60}$$

$$\partial_{\nu}\theta^i = 0 \quad \text{on } \partial\Omega\backslash\big[(\Gamma_0 \cup \Gamma_i) \cap \Sigma\big], \tag{61}$$

$$\operatorname{Supp}\omega_0^i \subset B_R\backslash\overline{\Omega}, \tag{62}$$

and if we define the function $\overline{\omega}^i : \overline{B_R} \times [0,1] \to \mathbb{R}$ by:

$$\begin{aligned} &\overline{\omega}^i(\cdot,0) = \omega_0^i \quad \text{in } \overline{B_R}, \\ &\partial_t\overline{\omega}^i(x,t) + \big[(a(t)\nabla\theta^i(x)) \cdot \nabla\big]\overline{\omega}^i(x,t) = 0 \quad \text{in } \overline{B_R} \times [0,1], \end{aligned} \tag{63}$$

then one has

$$\operatorname{Supp} \overline{\omega}^i(\cdot, 1) \subset B_R \backslash \overline{\Omega}, \tag{64}$$

$$\int_{\Gamma_i \times [0,1]} a(t) \partial_\nu \theta^i(x) \overline{\omega}^i(x,t) \,\mathrm{d}x \,\mathrm{d}t = 1. \tag{65}$$

The second type of θ is given by the following lemma:

LEMMA 5 ([4], Proposition 2.2). – *For any x in $\overline{\Omega} \backslash \bigcup_{i=k+1}^{g} \Gamma_i$, there exists θ in $C_0^\infty(B_R; \mathbb{R})$ such that*

$$\Delta\theta = 0 \quad \text{in } \overline{\Omega}, \tag{66}$$

$$\partial_\nu \theta = 0 \quad \text{on } \partial\Omega \backslash \Sigma, \tag{67}$$

$$\phi^{a\nabla\theta}(0, 1, x) \notin \overline{\Omega}. \tag{68}$$

Now "the" function $\overline{y}$ is constructed the following way. We consider $\varepsilon > 0$. For this ε, by Lemma 5 and using the compactness of $\overline{\Omega}$, there exists $l \in \mathbb{N}$ such that $l > k$, there exist $l - k$ functions $\theta^{k+1}, \ldots, \theta^l$ such that for all x in $\overline{\Omega}$ such that $\operatorname{dist}(x, \Gamma^b) > \varepsilon$, one has

$$\operatorname{dist}\big(\phi^{a\nabla\theta^i}(0, 1, x), \overline{\Omega}\big) \geqslant 2d, \tag{69}$$

for a certain $i \in \{k+1, \ldots, l\}$, the real number $d > 0$ being fixed, satisfying for $i \in \{1, \ldots, k\}$:

$$\overline{\omega}^i(x, 1) = \omega_0^i(x) = 0 \quad \text{if } \operatorname{dist}(x, \overline{\Omega}) \leqslant 2d.$$

Now for a $T > 0$ and $\eta > 0$ (which will be small) one defines:

$$t_{i/2} := T - \eta\left(l + 1 - \frac{i}{2}\right) \quad \text{for } i \text{ in } \big\{0, \ldots, 2(l+1)\big\}. \tag{70}$$

Then one defines $\overline{y} \in C^\infty(\overline{B_R} \times [0, T])$ (slightly differently from [4]) by the following formulas:

$$\overline{y} := \nabla\overline{\theta}, \tag{71}$$

where $\overline{\theta}$ is defined by:

$$\begin{aligned}
&\overline{\theta}(x,t) = 0 \quad \text{for } t \text{ in } [0, t_0],\\
&\overline{\theta}(x,t) = \frac{2}{\eta} a\left(\frac{2(t - t_{i-1})}{\eta}\right)\theta^i(x), \quad \forall i \in \{1, \ldots, l\}, \forall t \in [t_{i-1}, t_{i-1/2}],\\
&\overline{\theta}(x,t) = -\frac{2}{\eta} a\left(\frac{2(t_i - t)}{\eta}\right)\theta^i(x), \quad \forall i \in \{1, \ldots, l\}, \forall t \in [t_{i-1/2}, t_i],\\
&\overline{\theta}(x,t) = 0 \quad \text{for } t \text{ in } [t_l, T].
\end{aligned} \tag{72}$$

Reducing η if necessary, we can demand that $t_0 > 1/2$ and $\eta < \beta$ in Section 3. We underline here that $\overline{y}$ depends on two constants ε and η.

Note that $\overline{y}$ is actually a solution of the Euler system (3)–(4) in $\Omega \times [0, T]$, where the pressure is given by:

$$\overline{p} = \partial_t \overline{\theta} + \frac{1}{2}|\nabla \overline{\theta}|^2 \quad \text{in } \Omega \times [0, T],$$

and moreover satisfies (5).

4.2. The construction of a solution of the linearized system around W

Here we describe how the solution of the linearized control problem around W is constructed. We limit the study to W satisfying:

$$W \cdot \nu = \overline{y} \cdot \nu + b\left(\frac{2t}{T}\right) y_0 \cdot \nu + b\left(\frac{T-t}{\eta}\right) y_1 \cdot \nu. \tag{73}$$

For such a W, one defines $\tilde{W}$ by (50). To this $\tilde{W}$, one associates $\hat{\psi}$ by Section 3. Then one defines the function $\hat{\omega}$ by:

$$\begin{aligned} &\partial_t \hat{\omega} + (\tilde{W} \cdot \nabla)\hat{\omega} = 0 \quad \text{in } \overline{B_R} \times [0, T], \\ &\hat{\omega}(\cdot, T) = \pi(\Delta \hat{\psi}) \quad \text{in } \overline{B_R}. \end{aligned} \tag{74}$$

We consider the functions ω^i for i in $\{k+1, \ldots, l\}$ respectively defined on $\overline{B_R} \times [t_{i-1/2}, t_{i+1/2}]$, given by induction by the formulas:

$$\omega^{k+1}(\cdot, t_{k+1/2}) = \omega^*(\cdot, t_{k+1/2}) \quad \text{on } \overline{B_R}, \tag{75}$$

$$\partial_t \omega^i + (\tilde{W} \cdot \nabla)\omega^i = 0 \quad \text{in } \overline{B_R} \times [t_{i-1/2}, t_{i+1/2}], \tag{76}$$

$$\omega^i(x, t_{i-1/2}) = b^d(x)\omega^{i-1}(x, t_{i-1/2}) + \big(1 - b^d(x)\big)\hat{\omega}(x, t_{i-1/2}), \quad \forall i \in \{k+2, \ldots, l\}. \tag{77}$$

(ω^* is defined by (51).)

Now we consider the functions ω^i for $i \in \{1, \ldots, k\}$ defined respectively in $\overline{B_R} \times [t_{i-1}, t_i]$ by the following formulas:

$$\omega^i(\cdot, t_{i-1}) = \omega^*(\cdot, t_{i-1}) + \mu_i \omega_0^i \tag{78}$$

(μ_i is a real number to be precised) and

$$\omega^i(\cdot, t_{i-1/2}) = \omega^*(\cdot, t_{i-1/2}). \tag{79}$$

Between times t_{i-1} and $t_{i-1/2}$ and between $t_{i-1/2}$ and t_i, one requires that ω^i should satisfy:

$$\partial_t \omega^i + (\tilde{W} \cdot \nabla)\omega^i = 0 \quad \text{in } \overline{B_R} \times [t_{i-1}, t_{i-1/2}) \text{ and in } \overline{B_R} \times [t_{i-1/2}, t_i). \tag{80}$$

The μ_i are defined for $i \in \{1, \ldots, k\}$ by the following equation:

$$\int\limits_{\Gamma_i \times [t_{i-1}, t_{i-1/2}]} (W \cdot \nu)\omega^i = \int\limits_{\Omega} (y_0 - y_1) \cdot \nabla^{\perp}\tau_i + \big(\omega_0 - \hat{\omega}(\cdot, T)\big)\tau_i$$

$$
\begin{aligned}
&- \int\limits_{[0,t_0]\times\Gamma_i} (W\cdot\nu)\omega^* - \sum_{j=k+1}^{l} \int\limits_{[t_{j-1/2},t_{j+1/2}]\times\Gamma_i} (W\cdot\nu)\omega^j \\
&- \int\limits_{\Gamma_i\times[t_k,t_{k+1/2}]} (y\cdot\nu)\omega^* - \sum_{j=1}^{k} \int\limits_{\Gamma_i\times[t_{j-1/2},t_j]} (y\cdot\nu)\omega^*. \qquad (81)
\end{aligned}
$$

(We will show that this equation actually has a solution in the next section.)

Then one finally defines:

$$
\begin{aligned}
&\omega(x,t) = \omega^*(x,t) \quad \text{in } \overline{B_R}\times([0,t_0]\cup[t_k,t_{k+1/2}]),\\
&\omega(x,t) = \omega^i(x,t) \quad \text{in } \overline{B_R}\times[t_{i-1},t_i], \forall i\in\{1,\dots,k\},\\
&\omega(x,t) = \omega^i(x,t) \quad \text{in } \overline{B_R}\times[t_{i-1/2},t_{i+1/2}], \forall i\in\{k+1,\dots,l\},\\
&\partial_t\omega + (\tilde{W}\cdot\nabla)\omega = 0 \quad \text{in } \overline{B_R}\times[t_{l+1/2},T].
\end{aligned} \qquad (82)
$$

The searched solution of the linear system is $z = F(W)$ defined in $\overline{\Omega}\times[0,T]$ in the following way:

$$\operatorname{div} z = 0 \quad \text{in } \overline{\Omega}\times[0,T], \qquad (83)$$

$$\operatorname{curl} z = \omega \quad \text{in } \overline{\Omega}\times[0,T], \qquad (84)$$

$$z\cdot\nu = y\cdot\nu \quad \text{on } \partial\Omega\times[0,T], \qquad (85)$$

$$\int\limits_{\Omega} (\partial_t z + (z\cdot\nabla)z)\cdot\nabla^{\perp}\tau_i = 0, \quad \forall i\in\{1,\dots,g\}, \qquad (86)$$

$$\int\limits_{\Omega} z(\cdot,0)\cdot\nabla^{\perp}\tau_i = \int\limits_{\Omega} y_0\cdot\nabla^{\perp}\tau_i. \qquad (87)$$

4.3. Why the previous solution of the linear system is correctly defined

In this section, we show that F is correctly defined if W satisfies some assumptions. One defines (recall $\overline{y}$ depends on η):

$$
\begin{aligned}
S_{M,\eta} := \Big\{ & W\in C^0\big(\overline{\Omega}\times[0,T]\big),\ \overline{q}(W) < +\infty, |W-\overline{y}^{\eta}|_0 < M,\\
& W\cdot\nu = \overline{y}^{\eta}\cdot\nu + b\left(\frac{2t}{T}\right)y_0\cdot\nu + b\left(\frac{T-t}{\eta}\right)y_1\cdot\nu \text{ on } \partial\Omega\times[0,T]\Big\},
\end{aligned} \qquad (88)
$$

where $\overline{q}$ is defined by (35) and (36).

For M large enough, $S_{M,\eta}\neq\emptyset$ for all η. We fix such a M, and show that ω (and hence F) is correctly defined for $\eta(M)$ small enough and $W\in S_{M,\eta}$.

The problem for the correct definition of ω in $C^{\infty}(\overline{\Omega}\times[0,T])$ is the definition of the μ_i and the continuity at times $t_{i/2}$ for $i\in\{0,\dots,2k\}$. (Indeed, for other times ω is given by the composition of a regular function by a regular flow.)

We deduce from Gronwall's lemma the following formula:

$$\big|\phi^{\tilde{W}}(x,s,t) - \phi^{\overline{y}}(x,s,t)\big| \leqslant \eta\big[e^{C|t-s|/\eta} - 1\big]\|\tilde{W}-\overline{y}\|_{C^0(\overline{B_R}\times[0,T])}. \qquad (89)$$

Consequently, for fixed M, for η small enough, one has:

$$\omega^i(\cdot, t_{i-1/2}) = \omega^*(\cdot, t_{i-1/2}) \quad \text{in } \overline{\Omega}, \text{ for } i \in \{1, \dots, k\}. \tag{90}$$

Hence, with the construction of ω^i for $i \in \{1, \dots, l\}$, one gets

$$\omega \in C^\infty\big(\overline{\Omega} \times [0, T]\big). \tag{91}$$

Let us now specify why the μ_i are well defined. For η small enough, one has for any $W \in S_{M,\eta}$,

$$\left| \int_{\Gamma_i \times [t_{i-1}, t_{i-1/2}]} \tilde{\omega}^i (W \cdot \nu) - 1 \right| \quad \text{is small,} \tag{92}$$

as a consequence of (89), where $\tilde{\omega}^i$ is defined for $i \in \{1, \dots, k\}$, on $B_R \times [t_{i-1}, t_i]$ by:

$$\begin{aligned} &\tilde{\omega}^i(\cdot, t_{i-1}) = \omega_0^i \quad \text{in } \overline{B_R}, \\ &\partial_t \tilde{\omega}^i + (\tilde{W} \cdot \nabla)\tilde{\omega}^i = 0 \quad \text{in } B_R \times [t_{i-1}, t_{i-1/2}), \\ &\tilde{\omega}^i = 0 \quad \text{in } B_R \times [t_{i-1/2}, t_i) \end{aligned} \tag{93}$$

(in such a way that $\mu_i \tilde{\omega}^i = \omega^i - \omega^*$ in $B_R \times [t_{i-1}, t_i)$). Hence the linear system (81) has one and only one solution $(\mu_1, \dots, \mu_k)$.

4.4. The final value of the solution of the linear system

Let us now explain why the function $F(W)$ actually reaches the desired vector field, at least if ε and η are chosen small enough in the definition of $\overline{y}$ (this ε is chosen as a function of β). For this, we use the following decomposition of $z := F(W)(T)$:

$$z = \nabla^\perp \psi_z + \nabla \theta_z + \sum_{j=1}^{j=g} \lambda_z^j \nabla^\perp \tau_j, \tag{94}$$

where ψ_z equals 0 on $\partial\Omega$, and where θ_z is a harmonic function on Ω. We want to precise the different terms in this decomposition and then to compare them with those of y_1 in (43). (Let us remark that in this decomposition, all of the three terms are L^2-orthogonal one to another.)

First, for the "gradient" part, one has:

$$\nabla \theta_z = \nabla \theta_1, \tag{95}$$

as a consequence of the normal velocity imposed on the boundary by (85).

Now we are interested in the "$\nabla^\perp \tau_i$" terms. Thanks to (86), we can affirm that for all i in $\{1, \dots, g\}$, one has:

$$\sum_{j=1}^{g} \left(\int_\Omega \nabla \tau_i \cdot \nabla \tau_j \right) \frac{\mathrm{d}\lambda^j}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \omega \tau_i + \int_{\Gamma_i} (W \cdot \nu)\omega = 0. \tag{96}$$

With (81) and (82), this leads to the fact that for all i in $\{1,\dots,k\}$ one has:

$$\sum_{j=1}^{g}\left(\int_{\Omega}\nabla\tau_i\cdot\nabla\tau_j\right)\left(\lambda_z^j-l_1^j\right)=0. \tag{97}$$

For $i\in\{k+1,\dots,g\}$, equation (96) becomes:

$$\sum_{j=1}^{g}\left(\int_{\Omega}\nabla\tau_i\cdot\nabla\tau_j\right)\lambda^j+\int_{\Omega}\omega\tau_i\quad\text{is constant for } t\in[0,T]. \tag{98}$$

But on the other hand, by the further assumption (9), and by (43) one deduces that

$$\int_{\partial\Omega}\tau_i\left(\nabla^{\perp}\psi_1+\sum_{j=1}^{g}l_i^1\nabla^{\perp}\tau_j\right)\cdot\vec{\mathrm{d}x}=\int_{\partial\Omega}\tau_i\left(\nabla^{\perp}\psi_0+\sum_{j=1}^{g}l_i^0\nabla^{\perp}\tau_j\right)\cdot\vec{\mathrm{d}x}, \tag{99}$$

which leads to

$$\sum_{j=1}^{g}\left(\int_{\Omega}\nabla\tau_i\cdot\nabla\tau_j\right)\left(l_1^j-l_0^j\right)=\int_{\Omega}\tau_i(\operatorname{curl}y_1-\operatorname{curl}y_0)+\int_{\partial\Omega}\partial_\nu(\psi_1-\partial_\nu\psi_0).$$

With (98), we deduce that for all $i\in\{k+1,\dots,g\}$, one has:

$$\sum_{j=1}^{j=g}\left(\int_{\Omega}\nabla\tau_i\cdot\nabla\tau_j\right)\left(\lambda_j^z(T)-l_j^1\right)\mathrm{d}x=\int_{\Gamma_i}(\partial_\nu\hat{\psi}-\partial_\nu\psi_1)\,\mathrm{d}x. \tag{100}$$

Now (56) and (58) imply

$$\int_{\Gamma_i}\partial_\nu\hat{\psi}=\int_{\Gamma_i}\partial_\nu\tilde{\psi}-\mu_i\partial_\nu\tau_i.$$

Hence, using (57), one gets:

$$\int_{\Gamma_i}\partial_\nu\hat{\psi}=\int_{\Gamma_i}\partial_\nu\psi_1. \tag{101}$$

Transferring this into (100), one gets:

$$\sum_{j=1}^{j=g}\left(\int_{\Omega}\nabla\tau_i\cdot\nabla\tau_j\right)\left(\lambda_j^z(T)-l_j^1\right)=0 \tag{102}$$

for i in $\{k+1,\dots,g\}$. We deduce together with (97) that

$$\lambda_z^j=l_1^j\quad\text{for all } j\in\{1,\dots,g\}. \tag{103}$$

Now we show that the "$\nabla^\perp \psi_z$" part equals in fact $\nabla^\perp \hat{\psi}$, or equivalently (both ψ_z and $\hat{\psi}$ satisfy the 0 Dirichlet boundary condition) that

$$\operatorname{curl} z = \Delta\hat{\psi} \quad \text{in } \Omega. \tag{104}$$

To obtain this result, we need to have chosen ε small enough, precisely here we take $\varepsilon := r(\beta)$.

Then we distinguish points at distance superior or inferior of ε to Γ^b.

For points at distance at least ε from Γ^b, there is a number i for which one has $\operatorname{dist}(\phi^{\tilde{W}}(x,0,t_i), \overline{\Omega}) \geqslant d$ (if one chooses η small enough). At time t_i, formula (77) gives "the good value" to the vorticity of the point (thanks to formula (74)). Then this value does not change any more, even if the point is sent again out of $\overline{\Omega}$, thanks to (77). For points at distance at most ε, (104) is a consequence of the form of $\Delta\hat{\psi}$ (see in particular (53) to (58)): the vorticity allocated to these points was "the good one" from the beginning.

5. Proof of Theorem 1

5.1. Preliminary step

The main step of the proof will be to establish the following proposition:

PROPOSITION 1. – *Let $T > 0$ and y_0, y_1 be two functions of $C^\infty(\overline{\Omega};\mathbb{R}^2)$ satisfying* (1), (2) *and* (9). *Then for every p in $[1,+\infty)$, there exists a sequence (y^n) of functions in $C^\infty([0,T]\times\overline{\Omega};\mathbb{R}^2)$ which satisfy* (3), (4), (5) *and* (6), *and moreover*

$$y^n_{|t=T} \to y_1 \quad \text{in norm } W^{1,q}(\Omega), \tag{105}$$

for all $q < p$. One can moreover make (y^n) satisfy (15).

Theorem 1 follows easily from Proposition 1 (using a diagonal extraction argument).

5.2. Passing to a non-linear solution

The goal of this section is to make sure that one can obtain, given a $\beta > 0$, a fixed point of the operator F, and to ensure that this solves the problem of approximate controllability for the $W^{1,q}$ topology, for any $1 < q < p$ (for proper ε and η computed in function of β).

One may observe that during the construction of Section 3, if one defines:

$$\begin{aligned} &\Delta\psi = \omega \quad \text{in } \Omega\times[0,T],\\ &\psi_{|\partial\Omega} = 0, \end{aligned} \tag{106}$$

then one obtains

$$\|\Delta\psi\|_0 \leqslant C,$$

for any W, because ω is "made" from bounded functions (as the vorticity is transported by the flow of the velocity). Using techniques due to Wolibner in [11] and following Kato (see [6]), one may deduce from this that for all W obtained as the image by F of a certain function, we get

$$\left|\operatorname{curl} F(W)\right|_\delta \leqslant C, \tag{107}$$

for a certain $\delta > 0$.

That involves that $\|F(W)\|$ is C^{δ}-bounded in time, and $C^{1+\delta}$-bounded in space.

On another side, one gets that for proper η, $S_{M,\eta}$ is sent into itself (consequence of (89)). The previous boundedness implies then that $F(S_{M,\eta})$ is a compact subset of $S_{M,\eta}$. Hence, we get, using the Leray–Schauder theorem, the existence of a fixed point to F, which is moreover in the class C^{∞}. For further precisions, we refer to [4, Part 4].

Proceeding this way, we obtain a solution of the Euler system y^{β}, such that $y^{\beta}_{|t=T}$ is the vector field described by (59), ψ being the function $\hat{\psi}^{\beta}$ of the previous section computed for $W = y^{\beta}$.

5.3. The $W^{1,q}$ convergence

We have left to prove that this implies

$$\left\| y^{\beta} - y^{1} \right\|_{W^{1,q}(\Omega)} \to 0 \quad \text{when } \beta \to 0, \tag{108}$$

for any $q < p$ and moreover (15) for $1/n = \beta$.

First, we prove that, β being fixed, one has

$$\left\| \psi^{*}(T) - \psi^{w}(T) \right\|_{2,p} \leqslant C\eta. \tag{109}$$

Let us first remark that it is a consequence of the construction that

$$\psi^{w}(t_0) = \psi^{*}(t_0).$$

(We consider non-linear solutions here.) So (109) is a consequence of the form of $\overline{y}$ (note particularly that $\psi^{\nabla\overline{\theta}}(\cdot, t_0, T) = \mathrm{Id}$) and of (89).

We now show that $\hat{\psi}^{\beta}$ converges to ψ_1 for the $W^{2,q}$ topology (for $q < p$). As these functions have a zero trace on the boundary, it suffices to show the convergence of their Laplacian in the L^q sense, as a consequence of the classical elliptic estimate (in fact we will bound it in the L^p norm).

Now,

$$\Delta\left(\hat{\psi}^{\beta} - \psi_1\right) = \Delta\left[\rho_{\beta}\left(\psi^{*}(T) + H - \psi_1\right)\right] + \sum_{i=k+1}^{g} \mu_i \Delta\tilde{\psi}_i. \tag{110}$$

Of course, this is null on Ω^{β}, and only what happens on $\Omega \backslash \Omega^{\beta}$ does interest us.

There are three terms to estimate in the development of the first term in the previous expression. We are going to bound them in L^p. These three terms are the following:

- $\rho_{\beta}\Delta(\psi^{*}(T) + H - \psi_1) = \rho_{\beta}(\Delta H + \omega^{*}(T) - \operatorname{curl} y_1)$ which is bounded in L^p when $\beta \to 0$ (because of (48)).
- $2\nabla\rho_{\beta}\nabla(\psi^{*}(T) + H - \psi_1)$. The first factor has a norm L^{∞} which is a term of order $1/\beta$. The other factor has a norm L^p of order β, as a consequence of Lemma 3, (46) and (48).
- $\Delta\rho_{\beta}(\psi^{*}(T) + H - \psi_1)$. The first factor has a L^{∞} norm of order $1/\beta^2$. The other factor has an L^p norm of order β^2, as a consequence all the same of Lemma 3, (46) and (48).

Now, let us deal with the second term in (110). Precisely, let us show that:

$$\|\tilde{\psi}_i\|_{W^{2,p}(\Omega)} \quad \text{is bounded as } \beta \to 0. \tag{111}$$

The same way as for $\hat{\psi}$, one may reduce the problem, for each $i \in \{k+1,\dots,g\}$, to the estimating of $\|\Delta\tilde{\psi}_i\|_{L^p(\Omega)}$. In the same way, there is three terms to deal with, in the domain Ω^i_β. These three terms are the following:

- $\eta^i_\beta \Delta(1-\tau_i) = 0$.
- $-2\nabla\eta^i_\beta \nabla\tau_i$. The second factor is bounded in the L^∞ norm, and the first factor has its L^∞ norm of order $1/\beta$.
- $\Delta\eta^i_\beta(1-\tau_i)$. The first factor has an L^∞ norm of order $1/\beta^2$. The other factor has an L^∞ of order β, since $1-\tau_i$ has a zero trace on Γ_i.

In order to prove (111), it is sufficient hence to prove that $\mu_i = \mathrm{O}(\beta)$. But considering (57), one deduces:

$$\mu_i \int_\Omega |\nabla\tau_i|^2 = \int_{\Gamma_i} \partial_\nu\big(\psi^*(T) + H - \psi_1\big), \tag{112}$$

hence

$$|\mu_i| \leqslant C\big\|\partial_\nu\big(\psi^*(T) + H - \psi_1\big)\big\|_{W^{-\frac{1}{p},p}(\Gamma_i)}. \tag{113}$$

We deduce that:

$$|\mu_i| \leqslant C\big[\big\|\partial_\nu\big(\psi^w(T) + H - \psi_1\big)\big\|_{W^{-\frac{1}{p},p}(\Gamma_i)} + \big\|\partial_\nu\big(\psi^w(T) - \psi^*(T)\big)\big\|_{W^{-\frac{1}{p},p}(\Gamma_i)}\big], \tag{114}$$

which implies

$$|\mu_i| \leqslant C\big(\beta + \big\|\psi^w(T) - \psi^*(T)\big\|_{W^{1,p}(\Omega)}\big).$$

But thanks to (109) and with $\eta < \beta$, one can deduce:

$$|\mu_i| \leqslant C\beta,$$

that is, the researched estimate.

As the support of $\Delta(\hat{\psi}^\beta - \psi_1)$ "tends" to 0, one deduces:

$$\Delta\big(\hat{\psi}^\beta - \psi_1\big) \to 0 \quad \text{for the } L^q \text{ topology}, \forall q < p. \tag{115}$$

Now with (95), (103) and (115), we obtain as claimed that $y^\beta \to y_1$ as $\beta \to 0$ in the $W^{1,q}$ sense for all $q < p$.

6. Proof of Theorem 2

The proof of Theorem 2 is approximately the same as the one of Theorem 1, but in that case, we have to "prepare" the solution before the beginning of the active control at time t_0 and to modify the construction of Section 3 a little. If one has prepared the solution well, one can obtain better estimates in the study of Section 5.3.

6.1. Supplementary propositions

In this section, we use the direct orientation on the plane $\mathbb{C}$. This is useful only to ensure that a diffeomorphism of a Jordan curve which conserves orientation is homotopic to the identity of the Jordan curve.

To "prepare" the solution properly, we need the following proposition:

PROPOSITION 2. – *Consider $g-k$ diffeomorphisms $\psi_{k+1},\dots,\psi_g$, respectively from Γ_i into itself, which each conserve orientation. Then for all $\varepsilon>0$, there exists a function $\check{\theta}^\varepsilon$ defined in $C^\infty(\overline{\Omega}\times[0,1];\mathbb{R})$ satisfying the following properties*:

$$\operatorname{Supp}\check{\theta}^\varepsilon \subset \overline{\Omega}\times(0,1), \tag{116}$$

$$\Delta\check{\theta}^\varepsilon = 0 \quad \textit{in } \Omega\times[0,1], \tag{117}$$

$$\partial_\nu\check{\theta}^\varepsilon = 0 \quad \textit{on } (\partial\Omega\backslash\Sigma)\times[0,1], \tag{118}$$

$$\left\|\phi^{\nabla\check{\theta}^\varepsilon}(0,1,\cdot)-\psi_i\right\|_{C^1(\Gamma_i)}<\varepsilon, \quad \forall i\in\{k+1,\dots,g\}. \tag{119}$$

Proving this proposition is the goal of Section 7.

Simultaneously with Proposition 2, we will prove the following result, which is an improvement of Proposition 1, but limited to the 2-dimensional case and to the assumption "Γ_1 connected".

LEMMA 6. – *Consider Ω a nonempty bounded regular open set in $\mathbb{R}^2$, with boundary $\Gamma_1\cup\Gamma_2$, with Γ_1 connected and disjoint from Γ_2. Let $k\in\mathbb{N}$. We consider the mapping y from $C^\infty(\Gamma_2)$ into $C^\infty(\overline{\Omega})$ defined by* (26). *Then $\partial_\nu y(u)_{|\Gamma_1}$ describes a dense subspace of $C^k(\Gamma_1)$ when u describes the space $C^\infty(\Gamma_2)$* (*for any integer k*).

6.2. A new construction for the reachable velocity field

Our goal is principally to reduce to the case when

$$\left\|\omega^*(T)-\operatorname{curl}y_1\right\|_{C^1(\Gamma^b)} \quad \text{is small.} \tag{120}$$

This is possible as a consequence of Proposition 2. Indeed, by our assumption (16), one can write $\operatorname{curl}y_1$ on each Γ_i for $i\in\{k+1,\dots,g\}$ as the composition of $\operatorname{curl}y_0$ by a certain direct diffeomorphism $\mathcal{A}_i$ of Γ_i. As during the movement of the perfect fluid, the vorticity is transported by the flow of the velocity, one can obtain all the same $\operatorname{curl}y_0$ on each Γ_i for $i\in\{k+1,\dots,g\}$ as the composition of $\operatorname{curl}y^w(T)$ by a certain direct diffeomorphism $\mathcal{B}_i$ of Γ_i.

Now for $\varepsilon^r>0$, we consider a function $\overline{\overline{\theta}}$ given by Proposition 2 for the diffeomorphisms $\mathcal{B}_i\circ\mathcal{A}_i:\Gamma_i\to\Gamma_i$.

We then define $\overline{\overline{y}}:\overline{\Omega}\times[0,T]\to\mathbb{R}^2$ by:

$$\overline{\overline{y}}(x,t):=\overline{y}(x,t)+\nabla\overline{\overline{\theta}}\left(x,\frac{t-t_0-\eta}{\eta}\right), \tag{121}$$

where $\overline{\overline{\theta}}$ is extended to $\overline{\Omega}\times\mathbb{R}$ by 0 at the exterior of $\overline{\Omega}\times[0,1]$.

Let us now describe the construction of a new target for the velocity field at time T. Indeed, we need here to modify the construction of Section 3 at two points: we consider a reference solution y^w which takes $\overline{\overline{y}}$ into account, and we use a stronger approximating function $y(u)$ and then H.

Here, y_2^w is defined as the fixed point of the following operator P_2:

$$
\begin{aligned}
&\omega^*(0,\cdot) = \operatorname{curl}(\pi y_0) \quad \text{in } B_R,\\
&\partial_t \omega^* + \big(\pi(W)\cdot\nabla\big)\omega^* = 0 \quad \text{in } B_R \times [0,T],\\
&\operatorname{div} P_2\big(W\big) = 0 \quad \text{in } \Omega \times [0,T],\\
&\operatorname{curl} P_2(W) = \omega^* \quad \text{in } \Omega \times [0,T],\\
&P_2(W).\nu = b\left(\frac{2t}{T}\right) y_0\cdot\nu + \overline{\overline{y}}\cdot\nu \quad \text{on } \partial\Omega \times [0,T],\\
&\int_\Omega \big[\partial_t P_2(W) + \big(P_2(W)\cdot\nabla\big)P_2(W)\big]\cdot\nabla^\perp\tau_i = 0 \quad \text{on } [0,T], \forall i \in \{1,\dots,g\},\\
&\int_\Omega P_2(W)(\cdot,0)\cdot\nabla^\perp\tau_i = \int_\Omega y_0\cdot\nabla^\perp\tau_i, \quad \forall i \in \{1,\dots,g\}.
\end{aligned}
\tag{122}
$$

We still consider $\beta > 0$, and now, by Lemma 6, there exists for each $i \in \{k+1,\dots,g\}$ a function $u_i \in C^\infty(\partial\Omega\backslash\Gamma_i;\mathbb{R})$ such that

$$\big\|\partial_\nu y(u_i) + \partial_\nu\psi_1 - \partial_\nu\psi^w(T)\big\|_{C^2(\Gamma_i)} < \beta^2. \tag{123}$$

And then, as in Section 3, we define $H \in C^\infty(\overline{\Omega\backslash\Omega^\beta})$ and $r(\beta)$ in order to satisfy:

$$\Delta H = 0 \quad \text{in } \Omega\backslash\overline{\Omega^{2r(\beta)}}, \tag{124}$$

$$|H|_{W^{3,p}(\Omega\backslash\overline{\Omega^\beta})} \leqslant 2, \tag{125}$$

and such that H coincides with $y(u_i)$ in a boundary of Γ_i, in such a way that

$$\big\|\partial_\nu H + \partial_\nu\psi_1 - \partial_\nu\psi^w(T)\big\|_{C^2(\Gamma^b)} < \beta^2. \tag{126}$$

In the same way as for H, one considers $r'(\beta)$ such that for all $x \in \Omega^{r'(\beta)}$, one has:

$$\big|\operatorname{curl} y_0\big(\phi^{\overline{\overline{y}}}(t_0,0,x)\big) - \operatorname{curl} y_1(x)\big| \leqslant \big\|\operatorname{curl} y_0\big(\phi^{\overline{\overline{y}}}(t_0,0,\cdot)\big) - \operatorname{curl} y_1\big\|_{C^1(\Gamma^b)} + \beta. \tag{127}$$

Now we constuct a function $G \in C^\infty(\overline{\Omega\backslash\Omega^\beta})$ such that

$$
\begin{aligned}
&G(x) = \psi^* - \psi_1 \quad \text{in } \overline{\Omega^{r'(\beta)}},\\
&\|G\|_{C^3(\overline{\Omega\backslash\Omega^\beta})} = \|G\|_{C^3(\overline{\Omega^{r'(\beta)}})}.
\end{aligned}
\tag{128}
$$

Then we replace formula (54) of Section 3 by

$$\tilde\psi = \psi_1 + \rho_\beta(G+H). \tag{129}$$

The rest of the construction, that is (55) to (59), is kept as in Section 3.

Then the arguments for reachability of Section 4 and the one for passing to a non-linear solution in Section 5.2 are still valid, in such a way that we can consider a fixed point of the obtained process (with a new "$\hat{\psi}$" and a new "$\overline{y}$"), whose final value is given again by the corresponding formula (59). Indeed, for $\varepsilon < \max(r(\beta), r'(\beta))$ fixed for the choice of $\overline{y}$, the same proof as before can be done for the reachability of the velocity field. (By the way, one may impose to η and ε to be inferior to β.)

For what concerns the fixed point of the process, the point is that here we want to find it in a different functional space, viz.

$$S'_{M,\eta} := \Big\{ y \in C^1\big(\overline{\Omega} \times [0,T]\big),\ \overline{q}(y) < +\infty,\ \big|y - \overline{y}^{\eta}\big|_{C^0([0,T],C^1(\overline{\Omega};\mathbb{R}^2))} < M, \\ y \cdot \nu = \overline{y}^{\eta} \cdot \nu + b\left(\frac{2t}{T}\right) y_0 \cdot \nu + b\left(\frac{T-t}{\eta}\right) y_1 \cdot \nu \text{ on } \partial\Omega \times [0,T] \Big\}. \tag{130}$$

One gets all the same some compactness of the operator in this new space. One has to verify that for proper η, $S'_{M,\eta}$ is sent by F into itself. This point follows from the fact that one can obtain the following Gronwall-type estimate for s and t in $[0,T]$:

$$\big\|\phi^{\tilde{W}}(t,s,\cdot) - \phi^{\overline{\overline{y}}}(t,s,\cdot)\big\|_{C^1(\overline{B_R})} \leqslant \eta \mathrm{e}^{|t-s| \|\overline{\overline{y}}\|_{C^0([0,T],C^2(\overline{B_R}))}} \big\|\tilde{W} - \overline{\overline{y}}\big\|_{C^0([0,T],C^1(\overline{B_R}))}. \tag{131}$$

The problem is now reduced to check that with this new preparation, one gets the $W^{2,q}$ convergence for $q < p$.

6.3. The $W^{2,q}$ convergence

First, one easily deduces from the previous construction that, at the end of the process, one has in addition to above, the result:

$$\big\|\operatorname{curl} y^{\beta} - \operatorname{curl} y^1\big\|_{C^1(\Gamma^b)} < C\big(\varepsilon^r + \eta\big), \tag{132}$$

where C is a constant depending on the domain, y_0 and y_1. This is indeed a consequence of (131).

Now, we reconsider Section 5.3 in the light of the supplementary information (132).

Of course, the estimates that allowed the $W^{1,q}$ convergence ($q < p$) in Section 5.3 are still valid, and we just deal with $\nabla(\operatorname{curl} y^{\beta})$. As previously, we are concerned only with what happens on $\Omega \backslash \Omega^{\beta}$.

For this we consider first the "$\nabla^2 \hat{\psi}$" part. There are four terms to study (precisely, for which we want to prove the L^p boundedness on Ω^{β}): $\nabla^3 \rho_{\beta}(G + H - \psi_1)$, $\nabla^2 \rho_{\beta} \nabla(G + H - \psi_1)$, $\nabla \rho_{\beta} \nabla^2(G + H - \psi_1)$ and $\rho_{\beta} \nabla^3(G + H - \psi_1)$.

To that effect, let us first study the traces of $G + H$ and of its derivatives on Γ^b:

- 0-th order trace: $G + H = 0$ on Γ_b.
- 1st order normal trace: $\|\partial_{\nu}(G + H)\|_{W^{\frac{1}{p},p}} = \|\partial_{\nu}(\psi^* - \psi^1 + H)\|_{W^{\frac{1}{p},p}} < \beta^2$.
- 2nd order "normal" trace: by (132) one can bound the second derivatives on the boundary in the direction of the normal in terms of second derivatives in the tangent direction (which are null) plus first derivatives corresponding to curvature terms (and controlled with the help of the preceeding point) plus a term of order "$\varepsilon^r + \eta$" (and hence of order β).

Now we can go back to our "four terms" that we want to bound in L^p.

- $\rho_\beta \nabla^3 (G + H - \psi_1)$. = The "$H$" part is actually bounded because of (125). Of course $\nabla^3 \psi_1$ does not change. Finally the term $\nabla^3 G$ is also bounded in L^p as a consequence of (127), (128) and (131).
- $\nabla \rho_\beta \nabla^2 (G + H - \psi_1)$. By the previous study of traces, (124), (125) and (128), one gets (with a proper Poincaré's inequality) that $\|\nabla^2 (G + H - \psi_1)\|_{L^p(\Omega^\beta)}$ is of order β. But $\|\nabla \rho_\beta\|_{L^\infty}$ is of order $1/\beta$.
- $\nabla^2 \rho_\beta \nabla (G + H - \psi_1)$. The same way as previously, one can get by a proper Poincaré's inequality that $\|G + H - \psi_1\|_{L^p(\Omega^\beta)}$ is of order β^2.
- $\nabla^3 \rho_\beta (G + H - \psi_1)$. The same way, by the previous study of the traces and by Poincaré's inequality, one gets that the second part of this product is of order β^3.

Now we study again the second term in (58). Here, we have to prove that $\mu_i = \mathrm{O}(\beta^2)$ to get the result. It follows the same way as in Section 5.3 from (126).

Then, one can conclude as previously.

7. Proof of Proposition 2

The proof of Proposition 2 relies on the proofs of the two following propositions, that we are going to establish before coming back to the principal proof.

PROPOSITION 3. – *Let J be a C^∞-regular Jordan curve of the plane. Let ψ be a C^∞-diffeomorphism of J, which preserves orientation. For all $\varepsilon > 0$ (small), there exists a time-dependent tangent vector field $v : J \times [0,1] \to TJ$ of class C^∞ satisfying the constraints*:

$$\operatorname{Supp} v \subset J \times (0,1), \tag{133}$$

$$\int_J v(x,t) \cdot \vec{\mathrm{d}x} = 0, \quad \forall t \in [0,1], \tag{134}$$

and such that

$$\left\| \phi^v(0,1,\cdot) - \psi \right\|_{C^1(J)} \leqslant \varepsilon. \tag{135}$$

The second proposition is the following:

PROPOSITION 4. – *Let Ω be a regular, bounded, non empty connected open subset of $\mathbb{C}$. Consider Σ an non empty (regular) open part of its boundary. Let $k \in \mathbb{N}$ and $f \in C^k(\partial\Omega \setminus \Sigma; \mathbb{C})$. For all $\varepsilon > 0$, there exists a holomorphic function $\phi \in H(\Omega) \cap C^\infty(\overline{\Omega}; \mathbb{C})$ such that*

$$\|f - \phi\|_{C^k(\partial\Omega\setminus\Sigma;\mathbb{C})} < \varepsilon. \tag{136}$$

7.1. Proof of Proposition 3

Let us explain the general strategy. First, we consider a vector field whose flow between time 0 and time 1 gives ψ on J. We reproduce this vector field everywhere on J except on a small connected subset of J (which moves with the flow), and on which we impose a vector field in order to obtain (134). At the end of this stage, the obtained $\phi^v(1,0,\cdot)$ is close to ψ in the C^0 norm, but its derivatives are certainly very different on the small subset. To obtain the C^1 approximation, we then "dilute" the irregularity of $\phi^v(1,0,\cdot)$ on the whole J, during a second stage.

From the fact that ψ preserves orientation, we deduce that it is in the same connected component as Id.

Let us be given a homotopy $\Phi:[0,1]\to \mathrm{Diff}_\infty(J)$, differentiable in time, for which $\Phi(0)=\mathrm{Id}_J$ and $\Phi(1)=\psi$. If needed, one can add to this the condition that $(\partial_t\Phi)$ has a compact support in time in $(0,1)$. (We will note also Φ_t for $\Phi(t)$.)

The homotopy Φ can be seen as the flow of the time-dependent tangent vector field:

$$\tilde{v}(x,t)=(\partial_t\Phi)\big(\Phi_t^{-1}(x),t\big). \tag{137}$$

The problem is that in general, $\tilde{v}(\cdot,t)$ is not of null circulation on J.

We consider a positive number ε. We introduce a connected closed subset in J, say $\mathcal{I}_\varepsilon$, of length at most ε, and such that its image by the flow of $\tilde{v}$, say $\mathcal{I}_\varepsilon(t):=\phi^{\tilde{v}}(0,t,\mathcal{I}_\varepsilon)$ is of length at most ε. Then $\mathcal{I}_\varepsilon(t)$ has a minimal length; let us denote it by ε_0. (It suffices, by Gronwall's lemma, to choose $\mathcal{I}_\varepsilon$ sufficiently small.)

Now we consider a modification on $\tilde{v}$: we set

$$\hat{v}(x,t)=\tilde{v}(x,t)\quad \forall t,\forall x\in J\backslash\mathcal{I}_\varepsilon(t), \tag{138}$$

and $\hat{v}$ on $\{(x,t)/x\in J\backslash\mathcal{I}_\varepsilon(t)\}$ is ruled in order that

$$\int_J \hat{v}(\cdot,t)\cdot \vec{\mathrm{d}x}=0,\quad \forall t\in[0,1], \tag{139}$$

and in order that $\hat{v}$ is regular in space and in time (and still has a compact support in time).

We consider the flow of $\hat{v}$. For any x in $J\backslash\mathcal{I}_\varepsilon$, one has $\phi^{\hat{v}}(0,t,x)\in J\backslash\mathcal{I}_\varepsilon(t)$ and finally one has

$$\phi^{\hat{v}}(0,1,x)=\psi(x),\quad \forall x\in J\backslash\mathcal{I}_\varepsilon. \tag{140}$$

The problem is that we do not measure well the regularity of the flow for points originally situated in $\mathcal{I}_\varepsilon$ (even if we know that $\mathcal{I}_\varepsilon$ is sent into $\mathcal{I}_\varepsilon(1)$). So to ensure that the researched diffeomorphism $\phi^v(0,1,\cdot)$ is not "too irregular" on $\mathcal{I}_\varepsilon$, we use the following lemma:

LEMMA 7. – *Let J be a C^∞ Jordan curve of the plane. There exist two constants $K(J)$ and $\varepsilon_0(J)$ depending only on J, such that if $\mathcal{D}$ is a C^∞-regular diffeomorphism $J\to J$, which conserves orientation and which moreover satisfies*

$$\forall x\in J\backslash I_\varepsilon,\quad \mathcal{D}(x)=x, \tag{141}$$

where I_ε is a connected subset in J of length at most ε, with $\varepsilon<\varepsilon_0(J)$, then there exists a C^∞-time-dependent tangent vector field $\check{v}:J\times[0,1]\to TJ$ such that

$$\int_J \check{v}(x,t)\cdot \vec{\mathrm{d}x}=0,\quad \forall t\in[0,1], \tag{142}$$

$$\mathrm{Supp}\,\check{v}\subset J\times(0,1), \tag{143}$$

and such that if we use the notation:

$$T:=\phi^{\check{v}}(0,1,\cdot), \tag{144}$$

then one has:

$$\|T-\mathrm{Id}\|_{C^1(J\backslash I_\varepsilon)}\leqslant K(J)\varepsilon, \tag{145}$$

and moreover

$$(T \circ \mathcal{D})_{|I_\varepsilon} = \mathrm{Id}_{I_\varepsilon}. \tag{146}$$

This lemma will allow us to smooth the transform of J given by the flow of $\hat{v}$. The proof of this lemma is delayed till the end of the proof of Proposition 2.

We apply Lemma 7 with $\mathcal{D} := \phi^{\hat{v}}(0, 1, \cdot) \circ \psi^{-1}$ and $I_\varepsilon := \mathcal{I}_\varepsilon(1)$.

Hence, one gets an operator T – which can be represented as the flow between time 0 and time 1 of a null-circulation (in space for each time) tangent vector field of J – such that

$$\|T - \mathrm{Id}\|_{C^1(J \setminus I_\varepsilon)} \leqslant K(J)\varepsilon,$$

which implies

$$\left\| T \circ \phi^{\hat{v}}(0, 1, \cdot) - \phi^{\hat{v}}(0, 1, \cdot) \right\|_{C^1(J \setminus \mathcal{I}_\varepsilon)} \leqslant K(J, \psi)\varepsilon. \tag{147}$$

(For $x \in J \setminus \mathcal{I}_\varepsilon$, on has $\phi^{\hat{v}}(0, 1, \cdot) = \psi(x)$.)

By (146), one has on the interval $\mathcal{I}_\varepsilon$

$$T \circ \phi^{\hat{v}}(0, 1, \cdot) = \psi. \tag{148}$$

Consequently, by (147) and (148),

$$\left\| T \circ \phi^{\hat{v}}(0, 1, \cdot) - \psi \right\|_{C^1(J)} \leqslant K(J, \psi)\varepsilon.$$

But $T \circ \phi^{\hat{v}}(0, 1, \cdot)$ consists of the flow of $\hat{v}$ during $[0, 1]$ followed by the flow of $\check{v}$ corresponding to T during $[1, 2]$:

$$\begin{aligned} v(x, t) &= 2\hat{v}(x, 2t), \quad \forall (x, t) \in J \times [0, \tfrac{1}{2}], \\ v(x, t) &= 2\check{v}(x, 2t - 1), \quad \forall (x, t) \in J \times [\tfrac{1}{2}, 1]. \end{aligned} \tag{149}$$

Hence taking ε small enough, one gets (135).

Proof of Lemma 7. – Let us first introduce the time-dependent vector field $\check{v}$, and then we will show that it satisfies the required properties.

We introduce a C^∞ function $m : [0, 1] \to \mathbb{R}$ such that:

$$\operatorname{Supp} m \subset (0, 1), \tag{150}$$

$$0 \leqslant m(t) \leqslant 2, \quad \forall t \in [0, 1], \tag{151}$$

$$\int_{[0,1]} m = 1. \tag{152}$$

Let M be the primitive of m such that $M(0) = 0$. Let us also introduce the interval $\tilde{I}_\varepsilon$ obtained by extending I_ε of length $\varepsilon/2$ on each side. For ε small, one obviously has $\tilde{I}_\varepsilon \Subset J$.

We introduce a parametrisation of J, say $j : \frac{\mathbb{R}}{L\mathbb{Z}} \to J$ (where L is the total length of J) which is compatible with the arc length, that is if we denote by s the arc length on J starting from $j(0)$, $s : J \to \mathbb{R}$, the one has $j \circ S \circ s = \mathrm{Id}_J$, with S the canonical surjection $\mathbb{R} \to \frac{\mathbb{R}}{L\mathbb{Z}}$.

We introduce the following time-dependent transform of J:

$$\begin{aligned}&\varphi : J \times [0,1] \to J,\\ &\varphi(x,t) := j \circ S\{s(x) + M(t)[s(\mathcal{D}(x)) - s(x)]\}.\end{aligned} \tag{153}$$

Let us remark that from (141) and (153) one deduces $\varphi(x,t) = x$ for $J \backslash I_\varepsilon$. The transform is thus internal in $\tilde{I}_\varepsilon$ and by the way, one has $|s \circ \mathcal{D} - s| < \varepsilon$ on J.

From the fact that $\mathcal{D}$ is a direct diffeomorphism, one deduces together with (153) that the transform φ is an homotopy of (direct) diffeomorphisms. At each time, we note $\varphi_t := \varphi(\cdot,t)$.

Then one chooses $\check{v}$ in $C^\infty(J \times [0,1]; \mathbb{R}^2)$ in the set of all the tangent vector fields satisfying:

$$\check{v}(x,t) = (\partial_t \varphi)\big(\varphi_t^{-1}(x), t\big) \quad \text{in } \tilde{I}_\varepsilon \times [0,1], \tag{154}$$

$$\int_J \check{v}(\cdot,t) \cdot \vec{\mathrm{d}x} = 0, \quad \forall t \in [0,1], \tag{155}$$

$$\|\check{v}\|_{C^1(J \backslash \tilde{I}_\varepsilon)} \leqslant 10 \frac{1+L}{L^2} \left| \int_{\tilde{I}_\varepsilon} v \cdot \vec{\mathrm{d}x} \right|, \tag{156}$$

for ε small enough with respect to L (say $\varepsilon < L/10$).

Let us remark that such a $\check{v}$ satisfies $\check{v}(x,t) = 0$ for all x in $\tilde{I}_\varepsilon \backslash \underline{I_\varepsilon}$. Then to obtain (154)–(156), the work consists in finding a regular function with support in $J \backslash \overline{\tilde{I}_\varepsilon}$ with prescribed integral on $J \backslash \tilde{I}_\varepsilon$ (which is an interval of length at least $L - 2\varepsilon$), precisely:

$$-\int_{\tilde{I}_\varepsilon} m(t)(s \circ \mathcal{D} - s)\,\mathrm{d}x.$$

This can clearly be done regularly in time and such that (156) holds (taking for example, C^∞-approximations of piecewise affine functions).

Let us prove that the $\check{v}$ constructed this way is convenient. The point (146) is a trivial consequence of the form of φ, and of the choice of m.

Let us verify the point (145). By (151), (153) and (154), one deduces that $\|\check{v}\|_{C^0(I_\varepsilon)} \leqslant 2\varepsilon$, and consequently with (156) that

$$\|\check{v}\|_{C^1(J \backslash \tilde{I}_\varepsilon)} \leqslant C(J)\varepsilon. \tag{157}$$

So one has for any $x \in J \backslash \tilde{I}_\varepsilon$

$$\big|T(x) - x\big| \leqslant C(J)\varepsilon. \tag{158}$$

So we have left to study $\|\partial_x T - 1\|_{C^0}$ in $J \backslash \tilde{I}_\varepsilon$. But (145) can be easily obtained for ε small by a classical Gronwall's inequality and (157). □

7.2. Proof of Proposition 4

First, it is easy to see that one can suppose f of class C^∞.

We shall cut the proof in two parts. During a first step, we prove that one can approximate f on $\partial\Omega \backslash \Sigma$ by a holomorphic function defined in a neighbourhood of $\partial\Omega \backslash \Sigma$. In a second step we give a holomorphic function approximating f on $\partial\Omega \backslash \Sigma$ and defined globally on Ω.

Part I: The local problem

Step 1: Let us first consider the case when $\partial\Omega\backslash\Sigma = S^1$ is the unit circle.

As the function f is C^∞, the Fourier series

$$P_N^f(\theta) := \sum_{n=-N}^{n=+N} c_n(f)\mathrm{e}^{\mathrm{i}n\theta}, \tag{159}$$

converges to f in the C^k sense on S^1.

Hence, we choose N so that

$$\|f - P_N^f\|_{C^k(S^1)} < \varepsilon. \tag{160}$$

We now consider the rational function on $\mathbb{C}$:

$$Q_N^f(z) := \sum_{n=-N}^{n=+N} c_n(f)z^n. \tag{161}$$

Then the function $z \mapsto Q_N^f(z)$, holomorphic on a neighbourhood of S^1 (in fact, in the whole $\mathbb{C}^*$) is such that

$$\|\psi - f\|_{C^k(S^1)} < \varepsilon. \tag{162}$$

Step 2: We consider the case when $\partial\Omega\backslash\Sigma$ is a real-analytic Jordan curve.

We use a conformal mapping M from the interior of J into the unit disc. Then by real-analyticity of J, this mapping can be enhanced slightly across J, as a consequence of the Schwarz reflexion principle. For this, we refer for example to [10, p. 41, Proposition 3.1].

So if we can solve the problem in a neighbourhood of the unit circle (what was done in Step 1), we can solve it in a neighbourhood of any real-analytic Jordan curve.

Let us remark here that the local property holds *a fortiori* when, instead of a Jordan curve, one considers only an interval in a Jordan curve.

Part II: The global problem

Step 3: Let us consider the case when $\partial\Omega\backslash\Sigma$ is the union of g real-analytic Jordan curves and interval of real-analytic Jordan curves. Let us call $J_1, \ldots, J_g$ these curves. Let us fix $f \in C^\infty(\partial\Omega\backslash\Sigma)$.

There are g neighbourhoods $\mathcal{O}_1, \ldots, \mathcal{O}_g$ (we can suppose these neighbourhoods do not intersect each other, by reducing them if necessary) of respectively $J_1, \ldots, J_g$, and one can find g holomorphic functions $\psi_1, \ldots, \psi_g$ defined respectively on $\mathcal{O}_i$ such that one has:

$$\|f - \psi_i\|_{C^k(J_i;\mathbb{C})} < \varepsilon/2, \quad \forall i \in \{1, \ldots, g\}. \tag{163}$$

Then the problem reduces to extracting from the ψ_i a global holomorphic function ψ.

As $\Sigma \neq \emptyset$, one gets that one of the connected components of $\mathbb{C}\backslash\overline{\Omega}$ in the topological space $\mathbb{C}\backslash(\partial\Omega\backslash\Sigma)$ contains Ω.

Hence, let us consider g points $x_1, \ldots, x_g$ in $\mathbb{C}\backslash\overline{\Omega}$, such that any connected component of $\mathbb{C}\backslash(\partial\Omega\backslash\Sigma)$ contains at least one point x_i.

Then one obtains the global holomorphic function ϕ by Runge's theorem: it gives us a sequence of rational functions with poles in $\{x_1, \ldots, x_g\}$ (and hence, holomorphic on Ω), and which converge to ψ_i uniformly on any compact of $\mathcal{O}_i$. But for holomorphic functions, the uniform convergence on compacts determines the C^k convergence on compacts. Consequently, one can find the solution by getting a element of the sequence sufficiently far.

Step 4: We consider the general case. The general case is a consequence of the Step 3, because any (bounded regular) domain Ω is conformaly equivalent to a domain whose boundary is composed with analytic Jordan curves.

This point is rather classical (see, e.g., [1, p. 244]): it suffices to compose conformal mappings obtained by the Riemann's theorem for either exterior (in the Riemann sphere) or interior domain of the Jordan curves composing the boundary (and its iterated transformations), computed one after another. The important fact that we would like to underline is that during this process each conformal mapping is C^∞ up to the boundary by the Kellogg–Warschawski theorem (see, e.g., [10, Theorem 3.6, p. 46]), because the Jordan curves are all C^∞. The resulting conformal mapping to a domain bounded by real-analytic curves is hence also C^∞ up to the boundary.

Hence, it is sufficient to have the Step 3 solved to solve the general case.

Remark 2. – The local result is a very particular case of the result of R. Nirenberg and R.O. Wells (see [9]), which gives approximating holomorphic functions around (instead of a Jordan curve in $\mathbb{C}$) C^∞ totally real submanifolds of n-dimensional complex manifolds.

7.3. Back to the proof of Proposition 2

In this whole part, we identify $\mathbb{R}^2$ to $\mathbb{C}$ and hence, points in Ω will sometimes be considered as complex numbers.

Let us fix $\varepsilon > 0$. For this ε, and for the ψ_i, one can find by Proposition 3, $g-k$ time-dependent tangent vector fields v_i defined respectively on $\Gamma_i \times [0,1]$ for i in $\{k+1,\dots,g\}$ and such that (133), (134) and (135) hold on Γ_i. For $i \in \{1,\dots,k\}$, we fix $v_i := 0$ on Γ_i, and v_1 is fixed to 0 on $\Gamma_0 \backslash \Sigma$ and "free" on $\Sigma \cap \Gamma_0$ (that is we use $\Sigma \cap \Gamma_0$ as the only control region).

The main work is now to extend these v_i inside Ω in a form $v = \nabla\theta$ in order that (117) and (118) occur.

In a first step, we show that we can limit ourselves to the case when $v_i(x,t)$ is of the form $\sum_j \lambda_j(t) w_j(x)$. For that, we fix $\varepsilon_2 > 0$, to be ruled later (in function of ε).

For this ε_2, we consider $\kappa \in \mathbb{N}$, $\kappa \geqslant 3$, such that for t_1, t_2 in $[0,1]$:

$$|t_1 - t_2| < \frac{2}{\kappa} \Rightarrow \left\| v_i(t_1,\cdot) - v_i(t_2,\cdot) \right\|_{C^2(\Gamma_i)} < \varepsilon_2, \quad \forall i \in \{k+1,\dots,g\}, \tag{164}$$

$$\operatorname{Supp} v_i \subset \left(\frac{1}{\kappa}, \frac{\kappa-1}{\kappa}\right), \quad \forall i \in \{k+1,\dots,g\}. \tag{165}$$

Then we consider a partition of unity adapted to $[0, 3/2\kappa] \cup [1/\kappa, 5/2\kappa] \cup \cdots \cup [(\kappa-2)/\kappa, (2\kappa-1)/2\kappa] \cup [(\kappa-1)/\kappa, 1]$; that is, we consider κ functions $\rho_1,\dots,\rho_\kappa$ in $C^\infty([0,1];\mathbb{R})$ such that:

$$\begin{aligned}
&\operatorname{Supp} \rho_j \subset [0,1] \cap \left[\frac{j-1}{\kappa}, \frac{j+1/2}{\kappa}\right),\\
&0 \leqslant \rho_j \leqslant 1,\\
&\sum_{j=1}^{\kappa} \rho_j = 1 \quad \text{on } [0,1].
\end{aligned} \tag{166}$$

Now we consider the function:

$$w_i(t,x) = \sum_{j=1}^{\kappa} \rho_j(t) v_i\left(\frac{j}{\kappa}, x\right) \quad \text{on } [0,1] \times \Gamma_i. \tag{167}$$

Then the difference between w_i and

$$v_i(t,x) = \sum_{j=1}^{\kappa} \rho_j(t) v_i(t,x), \tag{168}$$

in the $C^2(\Gamma_i)$ norm is majored by ε_2 for all t (note that for a given t, there are at most two non null terms in the previous sums).

Now we consider, for each $j \in \{1,\dots,\kappa\}$, a holomorphic C^2 approximation of the $\overline{v_i}(j/\kappa,\cdot)$ (the complex conjugate of v_i) on the Γ_i for $i \in \{1,\dots,g\}$ and of 0 on $\Gamma_0 \backslash \Sigma$, with error at most ε_2. This is given by Proposition 4. (Note that one can obviously consider that Σ is regular, reducing it if necessary). We obtain κ holomorphic functions defined on Ω, viz. $H_1,\dots,H_\kappa$, such that

$$\begin{aligned} &\left\| H_j - \overline{v_i}\left(\frac{j}{\kappa},\cdot\right)\right\|_{C^2(\Gamma_i)} < \varepsilon_2, \quad \forall i \in \{1,\dots,g\}, \\ &\|H_j\|_{C^2(\Gamma_0\backslash\Sigma)} < \varepsilon_2. \end{aligned} \tag{169}$$

We add as a condition that, if for a given j, the $v_i(j/\kappa,\cdot)$ are all null for all i, then one chooses as function H_j the function 0.

Remark that by (134) and by the choice of v_i for $i \in \{1,\dots,g\}$, this implies in particular:

$$\left| \int_{\Gamma_i} \overline{H_j} \cdot \vec{\mathrm{d}x} \right| \leqslant C(\Omega)\varepsilon_2, \quad \forall i \in \{1,\dots,g\}. \tag{170}$$

Now we consider g points $x_1,\dots,x_g$ in $\mathbb{R}^2 \backslash \overline{\Omega}$ respectively in the interior of Γ_i for $i \in \{1,\dots,g\}$ (if Γ_i is the external curve, we fix x_i more precisely in the interior of Γ_0).

Then we consider the modified functions:

$$\tilde{H}_j(x) = H_j(x) - \sum_{i=1}^{g} \frac{\int_{\Gamma_i} \overline{H_j} \cdot \vec{\mathrm{d}x}}{2\mathrm{i}\pi(x - x_i)}. \tag{171}$$

Then $\tilde{H}_j$ is holomorphic on Ω, regular up to the boundary. The circulation of its conjugate around any Γ_m for $m \in \{1,\dots,g\}$ is null; hence the circulation of $\overline{\tilde{H}_j}$ is also null around Γ_0.

Consequently, the circulation of $\overline{\tilde{H}_j}$ around any inner connected component of the boundary is 0. Moreover, H_j is holomorphic in Ω. Hence, the function $\overline{\tilde{H}_j}$ is the gradient of a harmonic function:

$$\overline{\tilde{H}_j} = \nabla\tilde{\theta}_j = \frac{\partial\tilde{\theta}_j}{\partial x_1} + \mathrm{i}\frac{\partial\tilde{\theta}_j}{\partial x_2}, \tag{172}$$

for some $\tilde{\theta}_j \in C^\infty(\overline{\Omega};\mathbb{R})$, with

$$\Delta\tilde{\theta}_j = 0 \quad \text{in } \Omega. \tag{173}$$

We have left to modify a little $\tilde{\theta}_j$ in order to obtain (118). For that, for $j \in \{1, \ldots, \kappa\}$, one defines a function $\mathcal{G}_j$ of class C^∞ on $\partial\Omega$ such that

$$\begin{aligned} &\mathcal{G}_j = \partial_\nu \tilde{\theta}_j \quad \text{on } \partial\Omega\setminus\Sigma, \\ &\|\mathcal{G}_j\|_{C^{1,\alpha}(\partial\Omega)} \leqslant C(\Omega,\Sigma)\|\partial_\nu\tilde{\theta}_j\|_{C^{1,\alpha}(\partial\Omega\setminus\Sigma)}, \\ &\int_{\partial\Omega} \mathcal{G}_j = 0, \end{aligned} \tag{174}$$

for a given $\alpha \in (0,1)$.

Then one introduces for $j \in \{1, \ldots, \kappa\}$ the function $\hat{\theta}_j$ in $C^\infty(\overline{\Omega};\mathbb{R})$ such that

$$\begin{aligned} &\Delta\hat{\theta}_j = 0 \quad \text{in } \Omega, \\ &\partial_\nu\hat{\theta}_j = \mathcal{G}_j \quad \text{on } \partial\Omega, \\ &\int_{\Omega} \hat{\theta}_j = 0. \end{aligned} \tag{175}$$

Then one finally defines:

$$\check{\theta}(x,t) := \sum_{j=1}^{\kappa} \rho_j(t)\big(\tilde{\theta}_j(x) - \hat{\theta}_j(x)\big). \tag{176}$$

Then (116) follows from the fact that the v_i are of compact support in $(0,1)$. Relation (117) follows from (173), (175) and (176). One easily deduces from (174), (175) and (176) that (118) holds.

There remains to prove (119).

From (170), one deduces that the correcting term $\tilde{H}_j - H_j$ satisfies:

$$\|\tilde{H}_j - H_j\|_{C^1(\overline{\Omega})} \leqslant C(\Omega,\Sigma,x_i)\varepsilon_2. \tag{177}$$

Furthermore, as the v_i are tangent on $\partial\Omega\setminus\Sigma$, the normal part of $\overline{\tilde{H}_j}$ on $\partial\Omega\setminus\Sigma$ is less (in norm $C^{1,\alpha}$) than a factor of ε_2. Consequently with (174) and (175), one obtains:

$$\|\hat{\theta}_j\|_{C^{2,\alpha}(\overline{\Omega})} \leqslant C(\Omega,\Sigma,x_i)\varepsilon_2. \tag{178}$$

Finally, (164), (166), (169), (176), (177) and (178) lead to:

$$\|v - \nabla\check{\theta}\|_{C^0([0,1],C^1(\overline{\Omega}))} \leqslant 2\varepsilon_2 + C(\Omega,\Sigma,x_i)\varepsilon_2. \tag{179}$$

It follows from Gronwall's lemma that

$$\big\|\phi^v(0,1,\cdot) - \phi^{\nabla\check{\theta}}(0,1,\cdot)\big\|_{C^1(\Gamma_i)} < K\|v - \nabla\check{\theta}\|_{C^0([0,1],C^1(\overline{\Omega}))}, \tag{180}$$

where K is a constant depending on the second derivatives of v (and of the domain, of Σ and of the choice of the x_i).

So finally, by setting ε_2 small enough (v being fixed for a given ε), one deduces (119).

7.4. Proof of Lemma 6

As Γ_1 is connected in the boundary of a regular domain, and as it is disjoint from Γ_2, it is one of the Jordan curves composing the boundary.

Then, one can observe that it is equivalent to prove this density in the higher-derivatives Hölder spaces $C^{k,\alpha}$. Let us hence prove this later density.

Consider g in $C^\infty(\Gamma_1)$ (we can trivially restrict ourselves to the case when g has this regularity). We want to approximate it by a $\partial_\nu y(u)$. For that, we consider the vector field $\hat{g}:\Gamma_1 \to \mathbb{R}^2$ such that $\hat{g}\cdot\nu = g$ and $\hat{g}\cdot\tau = 0$ on Γ_1.

By Proposition 4, one can find $\phi \in H(\Omega)\cap C^\infty(\overline{\Omega};\mathbb{C})$ such that

$$\|\overline{\hat{g}} - \phi\|_{C^{k+1}(\Gamma_1)} < \varepsilon. \tag{181}$$

(Here $\overline{\hat{g}}$ is the complex conjugate of $\hat{g}$.)

As for (171) we consider a modified function ϕ_2 such that the circulation of its complex conjugate around any connected component of the boundary is 0.

As the circulation of $\hat{g}$ around the connected components of Γ_1 is of order ε, the distance between ϕ and ϕ_2 (in norm $C^{k,\alpha}(\overline{\Omega})$) is of order ε.

Now, because the circulation of $\overline{\phi_2}$ around any connected component of the boundary is 0, and because ϕ_2 is holomorphic in Ω, one gets that $\overline{\phi_2}$ is the gradient of a harmonic function (regular up to the boundary), say

$$\overline{\phi_2} = \nabla h^{\phi_2} \quad \text{in } \Omega.$$

Since h^{ϕ_2} is defined up to a constant, one can moreover require that is satisfies

$$\int_{\Gamma_1} h^{\phi_2}\,\mathrm{d}x = 0. \tag{182}$$

Now we have to modify h^{ϕ_2} in order to have a corresponding harmonic function whose trace is exactly zero along Γ_1. Precisely we define h^3 by:

$$\begin{aligned} &\Delta h^3 = 0 \quad \text{in } \Omega,\\ &h^3 = h^{\phi_2} \quad \text{on } \Gamma_1,\\ &h^3 = 0 \quad \text{on } \Gamma_2. \end{aligned}$$

With (181), (182) and $\hat{g}.\tau = 0$, one gets that $\|h^3\|_{C^{k+1,\alpha}(\overline{\Omega})}$ is of order ε. So $h^{\phi_2} - h^3$ solves the problem raised by Lemma 6.

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