Spectral analysis of semigroups and FitzHugh-Nagumo statistical model

S. Mischler

(Paris-Dauphine & IUF)

in collaboration with C. Quininao, J. Scher, J. Touboul

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Outline of the talk

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Spectral theory for general operator and its semigroup in general (large) Banach space, without regularity ($\neq$ eventually norm continuous), without symmetry ($\neq$ Hilbert space and self-adjoint op) and without (or with) positivity (Banach lattice)

- *Spectral map Theorem* $\hookrightarrow$ $\Sigma(e^{t\Lambda}) \simeq e^{t\Sigma(\Lambda)}$ and $\omega(\Lambda) = s(\Lambda)$
- *Weyl's Theorem* $\hookrightarrow$ (quantified) compact perturbation $\Sigma_{ess}(\mathcal{A}+\mathcal{B}) \simeq \Sigma_{ess}(\mathcal{B})$
- *Small perturbation* $\hookrightarrow$ $\Sigma(\Lambda_\varepsilon) \simeq \Sigma(\Lambda)$ if $\Lambda_\varepsilon \to \Lambda$
- *Krein-Rutmann Theorem* $\hookrightarrow$ $s(\Lambda) = \sup \Re e \Sigma(\Lambda) \in \Sigma_d(\Lambda)$ when $S_\Lambda \geq 0$
- *functional space extension (enlargement and shrinkage)*
  $\hookrightarrow$ $\Sigma(L) \simeq \Sigma(\mathcal{L})$ when $L = \mathcal{L}_{|E}$
  $\hookrightarrow$ tide of spectrum phenomenon

Structure: operator which splits as

$$\Lambda = \mathcal{A} + \mathcal{B}, \quad \mathcal{A} \prec \mathcal{B}, \quad \mathcal{B} \,\text{dissipative}$$

Examples: Boltzmann, (kinetic) Fokker-Planck, Growth-Fragmentation operators and $W^{\sigma,p}(m)$ weighted Sobolev spaces

Applications / Motivations :

• (1) Convergence rate in large Banach space for linear dissipative and hypodissipative PDEs (ex: kinetic Fokker-Planck, growth-fragmentation)

• (2) Long time asymptotic for nonlinear PDEs via the spectral analysis of linearized PDEs (ex: Boltzmann, Landau, Keller-Segel) in natural φ space

• (3) Existence, uniqueness and stability of equilibrium in "small perturbation regime" in large space for nonlinear PDEs (ex: inelastic Boltzmann, Wigner-Fokker-Planck, parabolic-parabolic Keller-Segel, neural network)

Is it new?

• Simple and quantified versions, unified theory (sectorial, KR, general) which holds for the "principal" part of the spectrum

• first enlargement result in an abstract framework by Mouhot (CMP 2006)

• Unusual splitting

$$\Lambda = \underbrace{\mathcal{A}_0}_{compact} + \underbrace{\mathcal{B}_0}_{dissipative} = \underbrace{\mathcal{A}_\varepsilon}_{smooth} + \underbrace{\mathcal{A}_\varepsilon^c + \mathcal{B}_0}_{dissipative}$$

• The applications to these linear(ized) "kinetic" equations and to these nonlinear problems are clearly new

- Fredholm, Hilbert, Weyl, Stone (Functional Analysis & semigroup Hilbert framework) $\leq$ 1932
- Hyle, Yosida, Phillips, Lumer, Dyson, Dunford, Schwartz, ... (semigroup Banach framework & dissipative operator) 1940-1960
- Kato, Pazy, Voigt (analytic operator, positive operator) 1960-1975
- Engel, Nagel, Gearhart, Metz, Diekmann, Prüss, Arendt, Greiner, Blake, Mokhtar-Kharoubi, Yao, ... 1975-

Still active research field

• Semigroup school (≥ 0, bio): Arendt, Blake, Diekmann, Engel, Gearhart, Greiner, Metz, Mokhtar-Kharoubi, Nagel, Prüss, Webb, Yao, ...

• Schrodinger school / hypocoercivity and fluid mechanic: Batty, Burq, Duyckaerts, Gallay, Helffer, Hérau, Lebeau, Nier, Sjöstrand, Wayne, ...

• Probability school (diffusion equation): Bakry, Barthe, Bobkov, Cattiaux, Douc, Gozlan, Guillin, Fort, Ledoux, Roberto, Röckner, Wang, ...

• Kinetic school ($\sim$ Boltzmann):

▷ Guo, Strain, ..., in the spirit of Hilbert, Carleman, Grad, Ukai works (Spectral analysis of the linearized (in)homogeneous Boltzmann equation, existence and convergence to the equilibrium in "small spaces")

▷ Carlen, Carvalho, Toscani, Otto, Villani, ... (log-Sobolev inequality)

▷ Desvillettes, Villani, Mouhot, Baranger, Neuman, Strain, Dolbeault, Schmeiser, ... (Poincaré inequality & hypocoercivity)

▷ Arkeryd, Esposito, Pulvirenti, Wennberg, Mouhot, ... (Spectral analysis of the linearized (in)homogeneous Boltzmann equation, existence and convergence to the equilibrium in "large spaces")

A list of related papers

- Mouhot, *Rate of convergence to equilibrium for the spatially homogeneous Boltzmann equation with hard potentials*, CMP 2006
- M., Mouhot, *Stability, convergence to self-similarity and elastic limit for the Boltzmann equation for inelastic hard spheres*, CMP 2009
- Gualdani, M., Mouhot, *Factorization for non-symmetric operators and exponential H-Theorem*, arXiv 2010
- Arnold, Gamba, Gualdani, M., Mouhot, Sparber, *The Wigner-Fokker-Planck equation: Stationary states and large time behavior*, M3AS 2012
- Cañizo, Caceres, M., *Rate of convergence to the remarkable state for fragmentation and growth-fragmentation equations*, JMPA 2011 & CAIM 2011
- Egaña, M., *Uniqueness and long time asymptotic for the Keller-Segel equation: The parabolic-elliptic case*, arXiv 2013
- M., Mouhot, *Exponential stability of slowing decaying solutions to the kinetic Fokker-Planck equation*, work in progress
- M., Scher, *Spectral analysis of semigroups and growth-fragmentation eqs*, arXiv 2013
- Carrapatoso, *Exponential convergence ... homogeneous Landau equation*, arXiv 2013
- Tristani, *Boltzmann equation for granular media with thermal force in a weakly inhomogeneous setting*, arXiv 2013
- Carrapatoso, M., *Uniqueness and long time asymptotic for the parabolic-parabolic Keller-Segel equation*, arXiv 2014
- M., Quininao, Touboul, *On a FitzHugh-Nagumo statistical model for neural networks*, work in progress

Outline of the talk

For a given operator Λ in a Banach space X, we want to prove

$$(1) \quad \Sigma(\Lambda) \cap \Delta_a = \{\xi_1\} \text{ (or } = \emptyset), \quad \xi_1 = 0$$

with $\Sigma(\Lambda)$ = spectrum of Λ, $\Delta_\alpha := \{z \in \mathbb{C}, \ \Re e \, z > \alpha\}$

$$(2) \quad \Pi_{\Lambda,\xi_1} = \text{finite rank projection}, \quad \text{i.e. } \xi_1 \in \Sigma_d(\Lambda)$$

$$(3) \quad \|S_\Lambda(I - \Pi_{\Lambda,\xi_1})\|_{X \to X} \le C_a \, e^{at}, \quad a < \Re e \xi_1$$

Definition:
We say that $L - a$ is hypodissipative iff $\|e^{tL}\|_{X \to X} \le C \, e^{at}$
$s(\Lambda) := \sup \Re e \Sigma(\Lambda)$ = spectral bound
$\omega(\Lambda) := \inf\{a \in \mathbb{R}, \text{ s.t. } L - a \text{ is hypodissipative } \}$ = growth bound

Th 1. (M., Scher) $\exists a^*$, $\exists n$
(0) $\Lambda = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded with $0 \le \zeta' < 1$,
(1) $\|S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)}\|_{X \to X} \le C_\ell \, e^{at}$, $\forall \, a > a^*$, $\forall \, \ell \ge 0$,
(2) $\|S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*n)}\|_{X \to D(\Lambda^\zeta)} \le C_n \, e^{at}$, $\forall \, a > a^*$, with $\zeta > \zeta'$,
(3) there exists a projector Π which commutes with Λ such that $\Lambda_0 := \Lambda_{|X_0}$, $X_0 := R(I - \Pi)$, $\Sigma(\Lambda_0) \cap \Delta_{a^*} = \emptyset$ and $\Lambda_1 := \Lambda_{|X_1} \in \mathcal{B}(X_1)$, $X_1 := R\Pi$

is equivalent to

(4) there exists a projector Π which commutes with Λ such that
$\Lambda_1 := \Lambda_{|X_1} \in \mathcal{B}(X_1)$, $X_1 := R\Pi$

$$\|S_\Lambda(t) \, (I - \Pi)\|_{X \to X} \le C_a \, e^{at}, \quad \forall \, a > a^*$$

In particular (spectral mapping theorem on the principal part of the spectrum)

$$\Sigma(e^{t\Lambda}) \cap \Delta_{e^{at}} = e^{t \Sigma(\Lambda) \cap \Delta_a} \quad \forall \, t \ge 0, \; a > a^*$$

and

$$\max(s(\Lambda), a^*) = \max(\omega(\Lambda), a^*)$$

We split the semigroup into invariant linear sub-manifolds (eigenspaces)

$$S_{\mathcal{L}} = \Pi\, S_{\mathcal{L}} + \Pi^{\perp}\, S_{\mathcal{L}}$$

and write the (iterated) Duhamel formula

$$S_{\mathcal{L}} = \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)} + S_{\mathcal{L}} * (\mathcal{A} S_{\mathcal{B}})^{(*N)}$$

These two identities together

$$S_{\mathcal{L}} \quad = \quad \Pi\, S_{\mathcal{L}} + \Pi^{\perp} \{ \sum_{\ell=0}^{N-1} S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*\ell)} \} + \Pi^{\perp}\, S_{\mathcal{L}} \, * (\mathcal{A} S_{\mathcal{B}})^{(*N)}$$

For the last term, we use the inverse Laplace transform formula

$$\Pi^{\perp} S_{\mathcal{L}} \, * (\mathcal{A} S_{\mathcal{B}})^{(*N)}(t) = \lim_{M \to \infty} \frac{i}{2\pi} \int_{a-iM}^{a+iM} e^{zt}\, \Pi^{\perp} R_{\Lambda}(z) (\mathcal{A} R_{\mathcal{B}}(z))^{N}\, dz$$

and we conclude by showing

$$\| R_{\Lambda}(z) (\mathcal{A} R_{\mathcal{B}}(z))^{N} \| \le C/|y|^2, \quad \forall\, z = a + iy,\, |y| \ge M,\, a > a_*$$

Th 2. (M., Scher) $\exists a^*$, $\exists n$
(0) $\Lambda = \mathcal{A} + \mathcal{B}$, where $\mathcal{A}$ is $\mathcal{B}^{\zeta'}$-bounded with $0 \le \zeta' < 1$,
(1) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*\ell)}\|_{X \to X} \le C_\ell \, e^{at}$, $\forall \, a > a^*$, $\forall \, \ell \ge 0$,
(2) $\|S_\mathcal{B} * (\mathcal{A} S_\mathcal{B})^{(*n)}\|_{X \to D(\Lambda^\zeta)} \le C_n \, e^{at}$, $\forall \, a > a^*$, with $\zeta > \zeta'$,
(3') $\int_0^\infty \|(\mathcal{A} S_\mathcal{B})^{(*n+1)}\|_{X \to Y} \, e^{-at} \, dt < \infty$, $\forall \, a > a^*$, with $Y \subset\subset X$,
is equivalent to

(4') there exist $\xi_1, ..., \xi_J \in \bar\Delta_a$, there exist $\Pi_1, ..., \Pi_J$ some finite rank projectors, there exist $T_j \in \mathcal{B}(R\Pi_j)$ such that $\Lambda \Pi_j = \Pi_j \Lambda = T_j \Pi_j$, $\Sigma(T_j) = \{\xi_j\}$, in particular

$$\Sigma(\Lambda) \cap \bar\Delta_a = \{\xi_1, ..., \xi_J\} \subset \Sigma_d(\Sigma)$$

and there exists a constant C_a such that

$$\|S_\Lambda(t) - \sum_{j=1}^J e^{t T_j} \Pi_j\|_{X \to X} \le C_a \, e^{at}, \quad \forall \, a > a^*$$

Th 4. (M. & Scher) On a "Banach lattice of functions" X,
(1) Λ such as in Weyl's Theorem for some $a^* \in \mathbb{R}$;
(2) $\exists b > a^*$ and $\psi \in D(\Lambda^*) \cap X'_+ \backslash \{0\}$ such that $\Lambda^* \psi \geq b\, \psi$;
(3) S_Λ is positive (Λ satisfies Kato's inequalities/weak maximum principle);
(4) Λ satisfies a strong maximum principle.
Defining $\lambda := s(\Lambda)$, there holds

$$a^* < \lambda = \omega(\Lambda) \quad \text{and} \quad \lambda \in \Sigma_d(\Lambda),$$

and there exists $0 < f_\infty \in D(\Lambda)$ and $0 < \phi \in D(\Lambda^*)$ such that

$$\Lambda f_\infty = \lambda\, f_\infty, \quad \Lambda^* \phi = \lambda\, \phi, \quad R\Pi_{\Lambda,\lambda} = \text{Vect}(f_\infty),$$

and then

$$\Pi_{\Lambda,\lambda} f = \langle f, \phi \rangle\, f_\infty \quad \forall\, f \in X.$$

Moreover, there exist $\alpha \in (a^*, \lambda)$ and $C > 0$ such that for any $f_0 \in X$

$$\| S_\Lambda(t) f_0 - e^{\lambda t}\, \Pi_{\Lambda,\lambda} f_0 \|_X \leq C\, e^{\alpha t}\, \| f_0 - \Pi_{\Lambda,\lambda} f_0 \|_X \qquad \forall\, t \geq 0.$$

Th 5. (M. & Mouhot; Tristani)
Assume
(0) $\Lambda_\varepsilon = \mathcal{A}_\varepsilon + \mathcal{B}_\varepsilon$ in X_i, $X_{-1} \subset\subset X_0 = X \subset\subset X_1$, $\mathcal{A}_\varepsilon \prec \mathcal{B}_\varepsilon$,
(1) $\|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*\ell)}\|_{X_i \to X_i} \le C_\ell \, e^{at}$, $\forall \, a > a^*$, $\forall \, \ell \ge 0$, $i = 0, \pm 1$,
(2) $\|S_{\mathcal{B}_\varepsilon} * (\mathcal{A}_\varepsilon S_{\mathcal{B}_\varepsilon})^{(*n)}\|_{X_i \to X_{i+1}} \le C_n \, e^{at}$, $\forall \, a > a^*$, $i = 0, -1$,
(3) $X_{i+1} \subset D(\mathcal{B}_{\varepsilon|X_i}), D(\mathcal{A}_{\varepsilon|X_i})$ for $i = -1, 0$ and

$$\|\mathcal{A}_\varepsilon - \mathcal{A}_0\|_{X_i \to X_{i-1}} + \|\mathcal{B}_\varepsilon - \mathcal{B}_0\|_{X_i \to X_{i-1}} \le \eta_1(\varepsilon) \to 0, \quad i = 0, 1,$$

(4) the limit operator satisfies (in both spaces X_0 and X_1)

$$\Sigma(\Lambda_0) \cap \Delta_a = \{0\}, \quad 0 \text{ simple}$$

Then

$$\Sigma(\Lambda_\varepsilon) \cap \Delta_a = \{\xi_1^\varepsilon\}, \quad \xi_1^\varepsilon \text{ simple}, \quad \xi_1^\varepsilon \to 0$$

Discussion / perspective

• With Theorem 1 at hand, the growth analysis of the semigroup S_Λ reduces to the spectral analysis (spectrum and eigenspace) for its generator Λ

• In Theorems 1, 2, 3, 4 one can take $n = 1$ in the simplest situations (most of space homogeneous equations in dimension $d \leq 3$), but one need to take $n \geq 2$ for the space inhomogeneous Boltzmann equation and the kinetic Fokker-Planck equation

• Open problem: Beyond the "dissipative case"?
▷ example of the Fokker-Planck equation for "soft confinement potential" and relation with "weak Poincaré inequality" by Röckner-Wang
▷ Links with semi-uniform stability by Lebeau & co-authors, Burq, Liu-R, Bátkal-E-P-S, Batty-D, ...
▷ applications to the Boltzmann and Landau equations associated with "soft potential"
▷ spectral mapping theorem, Krein-Rutman theorem, extension theorem

Outline of the talk

A FitzHugh-Nagumo statistical model

$$\partial_t f = \mathcal{Q}_\varepsilon(\mathcal{J}_f, f) = \partial_x(Af) + \partial_v(Bf) + \partial^2_{vv} f \quad \text{on } (0,\infty)\times\mathbb{R}^2$$

complemented withy an initial condition

$$f(0,.) = f_0 \geq 0 \quad \text{in } \mathbb{R}^2.$$

where

$$\begin{cases} A = A(x,v) = ax - bv, \quad B = B_\varepsilon[\mathcal{J}_f] = B(x,v;\mathcal{J}_f) \\ B(x,v;\mu) = v^3 - v + x + \varepsilon\,(v-\mu), \quad \mathcal{J}_f := \int_{\mathbb{R}^2} v\, f(x,v)\, dvdx \end{cases}$$

- $t \geq 0$ is the time variable, $v \in \mathbb{R}$ is the membrane potential of one neuron, $x \in \mathbb{R}$ is an auxiliary variable
- $f = f(t,x,v) \geq 0$ is the time-dependent density of neurons in state $(x,v) \in \mathbb{R}^2$
- a, b, ε are positive parameters and ε is the connectivity of the network

The equation being in divergence form the number of neurons is a constant along time (that's better!):

$$\int_{\mathbb{R}^2} f(t,x,v) dxdv = \int_{\mathbb{R}^2} f_0 dxdv \equiv 1.$$

• As a simplification of the Hodgin-Huxley 4d ODE, FitzHugh-Nagumo 2d ODE describes the electric activity of one neuron and writes

$$\begin{aligned} \dot v &= v - v^3 - x + I_{ext} = -B_0 + I_{ext} \\ \dot x &= bv - ax = -A, \end{aligned}$$

with $I_{ext} = i(t) + \sigma \dot W$ exterior input split as a deterministic part + a stochastic noise. We assume $i(t) \equiv 0$.

• For a network of N coupled neurons, the associated model writes for the state $\mathcal{Z}^i_t := (\mathcal{X}^i_t, \mathcal{V}^i_t)$ of the neuron labeled $i \in \{1, ..., N\}$

$$\begin{aligned} d\mathcal{V}^i &= [-B_0(\mathcal{X}^i, \mathcal{V}^i) - \sum_{j=1}^N \varepsilon_{ij} \, (\mathcal{V}^i - \mathcal{V}^j)]dt + \sigma \, d\mathcal{W}^i \\ d\mathcal{X}^i &= -A(\mathcal{X}^i, \mathcal{V}^i)dt \end{aligned}$$

where $\varepsilon_{ij} > 0$ corresponds to the connectivity between the two neurons labeled i and j. The model takes into account an intrinsic deterministic dynamic + mean field interaction + stochastic noise.

We assume $\varepsilon_{ij} := \varepsilon/N$, $(\mathcal{Z}_0^1 ..., \mathcal{Z}_0^N)$ are i.i.d. random variables with same law f_0 and we pass to the limit $N \to \infty$.
We get that $(\mathcal{Z}_t^1, ..., \mathcal{Z}_t^N)$ is chaotic which means that any two neurons $\mathcal{Z}_t^i$ and $\mathcal{Z}_t^j$ are asymptotically independent and $\mathcal{Z}_t^i \to \bar{\mathcal{Z}}_t = (\bar{\mathcal{X}}_t, \bar{\mathcal{Y}}_t)$ which is a solution to the nonlinear ODS

$$\bar{\mathcal{V}} = [-B_0(\bar{\mathcal{X}}, \bar{\mathcal{V}}) - \varepsilon\,(\bar{\mathcal{V}} - \mathbb{E}(\bar{\mathcal{V}}))]dt + \sigma\, d\mathcal{W}$$
$$\bar{\mathcal{X}} = -A(\bar{\mathcal{X}}, \bar{\mathcal{V}})dt.$$

From Ito calculus we immediately see that the law $f(t,.) := \mathcal{L}(\bar{\mathcal{X}}_t, \bar{\mathcal{V}}_t)$ satisfies the associated backward Kolmogrov equation which is nothing but the FHN nonlinear statistical equation (here and below we make the choice $\sigma := \sqrt{2}$ for the sake of simplification of notations).

We introduce the weight function $m_0 = m_0(x,v) := 1 + x^2/2 + v^2/2$ and the weighted Lebesgue spaces $L^p(m)$ associated to the norm

$$\|f\|_{L^p(m)} = \|fm\|_{L^p}, \quad \|f\|_{W^{1,p}(m)} = \|f\|_{L^p(m)} + \|\nabla f\|_{L^p(m)},$$

and the shorthand $L^p_k := L^p(m_0^{k/2})$.

Th 8. M., Quininao, Touboul

For any $f_0 \in \mathcal{E}_0 := L^1_4 \cap L^1 \log L^1 \cap \mathbb{P}(\mathbb{R}^2)$ there exists a unique global solution $f \in C([0,\infty); L^1 \cap \mathbb{P})$ to the FHN statistical equation. It also satisfies

$$\int f_t m \le \max(C_m, \int f_0 m), \quad \|f_t\|_{\mathcal{H}^1(m)} \le \max(C_2, \|f_0\|_{\mathcal{H}^1(m)}).$$

It depends continuously in the initial datum: $f_{n,t} \to f_t$ in L^1_2 for any time $t \ge 0$ if $f_{n,0} \to f_0$ in L^1_2 and $\|f_{n,0}\|_{L^1_4} + H(f_{n,0}) \le C$.
For any $\tau > 0$ there exists C_τ such that

$$\sup_{t \ge \tau} \|f_t\|_{H^1} \le C_\tau.$$

Th 9. M., Quininao, Touboul

There exists at least one stationary solution G to the FHN statistical equation:

$$\exists\, G \in H^1(m) \cap \mathbb{P}(\mathbb{R}^2), \quad 0 = \partial_x(AG) + \partial_v(B_\varepsilon[\mu_G]G) + \partial^2_{vv} G \quad \text{in } \mathbb{R}^2.$$

Th 10. M., Quininao, Touboul

There exists $\varepsilon^* > 0$ such that in the small connectivity regime $\varepsilon \in (0, \varepsilon^*)$ the stationary solution is unique and exponentially stable: there exist $\eta^*_\varepsilon > 0$, $a < 0$ such that $\eta^*_\varepsilon \to \infty$ when $\varepsilon \to 0$ and

$$\forall\, f_0 \in L^1_2 \cap \mathbb{P}, \quad \|f_0 - G\|_{L^1_2} \le \eta^*_\varepsilon \text{ there holds } \|f(t) - G\|_{L^1_2} \le C\, e^{at}\ \forall\, t \ge 0$$

• We follow a strategy introduced in M., Mouhot (CMP 2009) for the inelastic homogenous Boltzmann equation and improved in Tristani (arXiv 2013) in a weakly inhomogeneous setting.
• But we fundamentally use the fact that the limit equation (for $\varepsilon = 0$) is positive and it is then exponentially asymptotically stable thanks to the Krein-Rutmann theorem (Theorem 4)
• We also use some "hypocoercive" calculus tricks developed by Hérau and Villani for the kinetic Fokker-Planck equation

Proof - L^1_k estimate

The vector field (A, B) does not derive from a potential (even in the case $\varepsilon = 0$) but has the following "confinement property"

$$\begin{aligned} -x\,A - v\,B &= -ax^2 + bxv - v^4 - (1+\varepsilon)xv + \varepsilon\mu x \\ &\le C(a,b,\varepsilon) - \frac{a}{2}x^2 - \frac{1}{2}v^4 + 2\frac{\varepsilon^2}{a}\mu^2. \end{aligned}$$

Also observe (Cauchy-Schwarz inequality)

$$\mathcal{J}_f^2 \le \int f\, v^2 dxdv \quad \forall f \in \mathbb{P}(\mathbb{R}^2).$$

Lemma (uniform in time L^1_k estimate, $k \ge 2$)

For $m_0 := 1 + x^2/2 + v^2/2$ and any $f \in \mathbb{P}(\mathbb{R}^2)$, there holds for $C_i > 0$

$$\int \mathcal{Q}_\varepsilon[\mu, f] m_0 \le C_1\,(1+\mu^2) - C_2 \int f\,(1 + x^2 + v^4).$$

As a consequence, for any $f \in \mathbb{P}(\mathbb{R}^2)$

$$\int \mathcal{Q}_\varepsilon[\mathcal{J}_f, f] m_0 \le C_3 - C_2 \int f\, m_0,$$

and for any $f_0 \in \mathbb{P}(\mathbb{R}^2)$

$$\mathcal{J}_{f(t)}^2 \le \int f_t\, m_0 \le \max\Big(\frac{C_3}{C_2}, \int f_0\, m_0\Big) \quad \forall t \ge 0, \qquad m = m_0^{k/2}.$$

Proof - $\mathcal{H}^1$ estimate

In the same way for $m = e^{\kappa\, m_0}$

$$\frac{d}{dt}\int f^2 m^2 = 2\int \mathcal{Q}_\varepsilon[\mu, f]\, f\, m^2 \le C_1 \int f^2 - C_2 \int f^2\, m^2 m_0 - \int |\partial_v f|^2 m^2,$$

but we do not know how to conclude (in order to get uniform in times bound) !?

We introduce the (equivalent) twisted norm (reminiscent of hypocoercivity theory)

$$\|f\|^2_{\mathcal{H}^1(m)} := \|f\|^2_{L^2(m)} + \|\nabla_x f\|^2_{L^2(m)} + \alpha^{5/6}(\nabla_x f, \nabla_v f)_{L^2(m)} + \alpha\|\nabla_v f\|^2_{L^2(m)}$$

for $\alpha > 0$ small enough. For the associated scalar product $\langle\cdot,\cdot\rangle$

$$\begin{aligned}\langle Q_\varepsilon[\mu, f], f\rangle &\le C_1 \int f^2 - C_2 \int f^2\, m^2 m_0 - \alpha \int |\partial_v f|^2 m^2 - \alpha^{5/6}\int |\partial_x f|^2 m^2 \\ &\le K_1\|f\|_{\mathcal{H}^1} - K_2\|f\|^2_{\mathcal{H}^1}\end{aligned}$$

by using Nash inequality

$$\|f\|^2_{L^2} \le C\|f\|_{L^1}\|f\|_{H^1}.$$

Lemma uniform in times $\mathcal{H}^1$ estimate)

For any $f_0 \in \mathbb{P}(\mathbb{R}^2)$

$$\|f_t\|_{\mathcal{H}^1(m)} \le \max\Big(\frac{K_1}{K_2}, \|f_0\|_{\mathcal{H}^1(m)}\Big) \quad \forall\, t \ge 0.$$

Proof - local a priori bound and existence

We compute

$$\begin{aligned}\frac{d}{dt}\int f\log f &= \int(\partial_{vv}f)\log f + \int(\partial_x(Af)+\partial_v(Bf))\log f \\ &= -\int\frac{(\partial_v f)^2}{f} + \int(\partial_x A + \partial_v B)f \\ &\le -\mathcal{I}_v(f) + \int m_0 f, \qquad \mathcal{I}_v(f) := \int\frac{(\partial_v f)^2}{f}.\end{aligned}$$

We conclude by standard (weak L^1 compacteness) argument to the existence of a solution $f \in C([0,\infty); L^1)$ such that

$$\sup_{[0,T]}\int f\,(m_0^2 + \log f) + \int_0^T \mathcal{I}_v(f)\,dt \le C_T \quad \forall\, T > 0$$

for any $f_0 \in L^1_4 \cap \mathbb{P} \cap L^1\log L^1$.

For any two solutions f_1 et f_2 to the FHN equation

$$\partial_t f_i = \partial_x(A f_i) + \partial_v(\mathcal{B}_i f_i) + \partial^2_{vv} f_i$$

with $\mathcal{B}_i := B_0 + \varepsilon(v - \mathcal{J}_{f_i})$, the difference $f = f_2 - f_1$ satisfies

$$\partial_t f = \partial_x(A f) + \partial_v(\mathcal{B}_1 f) + \varepsilon\, \mathcal{J}_f \partial_v f_2 + \partial^2_{vv} f.$$

As a consequence, by Kato's inequality

$$\partial_t |f| \le \partial_x(A|f|) + \partial_v(\mathcal{B}_1 |f|) + \varepsilon\, |\partial_v f_2|\, |\mathcal{J}_f| + \partial^2_{vv}|f|.$$

Using the inequality

$$\int |\partial_v f_2| m_0 \le \Big(\int f_2\, m_0^2 \Big)^{1/2} \Big(\int \frac{|\partial_v f_2|^2}{f_2} \Big)^{1/2} \le C\, \mathcal{I}_v(f_2)^{1/2}$$

we get

$$\begin{aligned} \frac{d}{dt} \int |f| m_0 \quad &\le \quad \int |f| (-A \partial_x m_0 - B \partial_v m_0) + \varepsilon\, C\, \mathcal{I}_v(f_2)^{1/2} \int |f| m_0 + \int |f| \\ &\le \quad (C + \varepsilon \mathcal{I}_v(f_2)) \int |f| m_0. \end{aligned}$$

We conclude to the uniqueness by Gronwall lemma.

Proof - existence of stationary solution

Define

$$\mathcal{Z} := \{f \in \mathcal{H}^1(m) \cap \mathbb{P}; \quad \|f\|_{\mathcal{H}^1(m)} \le K_1/K_2\}$$

and

$$S = (S_t) \text{ by } S_t f_0 := f_t \text{ solution of the FHN equation.}$$

• $\mathcal{Z}$ is a convex and strongly compact subset of the Banach space L^1_2;
• S leaves $\mathcal{Z}$ invariant and it is a L^1_2-continuous semigroup.

A direct application of the Schauder fixed point theorem implies

$$\exists\, G \in \mathcal{Z} \text{ such that } S_t G = G \quad \forall\, t \ge 0$$

or equivalently

G is a stationary solution to the FHN equation (for any given $a, b, \varepsilon > 0$).

We may simplify that existence part by working in the space of symmetric solutions $\mathcal{S}$ (i.e. $f \in \mathcal{S}$ iff $f(-x,-v) = f(x,v)$) in which space the FHN equation is linear.

For any stationary state $G_\varepsilon \in \mathcal{Z}$, we define the linearized operator

$$\mathcal{L}_\varepsilon h := \partial_x(Ah) + \partial_v((B_0 + \varepsilon(v - \mu_{G_\varepsilon}))h) - \varepsilon\mu_h\partial_v G_\varepsilon + \partial^2_{vv} h$$

We write

$$\mathcal{L}_\varepsilon = \mathcal{A} + \mathcal{B}_\varepsilon, \quad \mathcal{A}h := M\chi_R(x,v)\, h$$

and we have

$$(1) \quad \|S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*k)}\|_{\mathbf{B}(X)} \le C_k\, e^{-t}$$

and

$$(2) \quad \|S_\mathcal{B} * (\mathcal{A}S_\mathcal{B})^{(*n)}\|_{\mathbf{B}(L^1(m),H^1(mm_0))} \le C_n\, e^{-t}$$

As a consequence, the Weyl theorem (Theorem 2) implies

$$\Sigma(\mathcal{L}) \cap \Delta_{-1} = \text{finite} \subset \Sigma_d(\mathcal{L}).$$

Proof of estimates (1) and (2)

• the estimate

$$(1) \quad \|S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*k)}\|_{\mathbf{B}(X)} \le C_k \, e^{-t}$$

is a consequence of the fact that
▷ $\mathcal{A} \in \mathbf{B}(X)$, $X = L^1(m)$, $L^2(m)$, $\mathcal{H}^1(m)$;
▷ $\mathcal{B}$ is -1-dissipative in $X = L^1(m)$, $L^2(m)$, $\mathcal{H}^1(m)$ as a consequence of the already established estimates

$$\int \mathcal{Q}_\varepsilon[\mu, f] f^{p-1} m^p \le C_1 \int f^p - C_2 \int f^p m^p m_0$$

and the similar estimate in $\mathcal{H}^1(m)$.

• the estimate

$$(2) \quad \|S_{\mathcal{B}} * (\mathcal{A} S_{\mathcal{B}})^{(*n)}\|_{\mathbf{B}(L^1(m), H^1(m m_0))} \le C_n \, e^{-t}$$

is similar to the Nash argument in the proof of the stability of $\mathcal{Z}$. More precisely, introducing

$$\mathcal{F}(t,h) := \|h\|^2_{L^1(m)} + t^\bullet \|h\|^2_{L^2(m)} + t^\bullet \|\nabla_v h\|^2_{L^2(m)} + t^\bullet (\nabla_v h, \nabla_x h)_{L^2(m)} + t^\bullet \|\nabla_x h\|^2_{L^2(m)}$$

we are able to prove (for convenient exponents $\bullet \ge 1$)

$$\frac{d}{dt} \mathcal{F}(t, S_{\mathcal{B}}(t) h) \le 0, \quad \forall \, t \in [0, T].$$

We observe that in $X = L^p(m)$

$$\mathcal{L}_0 h = \partial_x(Ah) + \partial_v((B_0 h) + \partial^2_{vv} h$$

is such that
(1) $\mathcal{L} = \mathcal{A} + \mathcal{B}_0$ as above with $a^* = -1$;
(2) $\exists G_0 \in \mathcal{Z}$, $\mathcal{L}_0 G_0 = 0$ and $\mathcal{L}_0^* 1 = 0$;
(3) $\mathcal{L}_0$ is strongly positive, in the sense that
$\triangleright$ $S_{\mathcal{L}_0}$ is a positive semigroup : $f_0 \ge 0$ implies $S_{\mathcal{L}_0}(t) f_0 \ge 0$;
$\triangleright$ $\mathcal{L}_0$ satisfies a weak maximum principle: $(\mathcal{L}_0 - a) f \le 0$ and a large imply $f \ge 0$;
$\triangleright$ $\mathcal{L}_0$ satisfies Kato inequality : $\mathcal{L}_0 \theta(f) \ge \theta'(f) \mathcal{L}_0 f$, $\theta(s) = |s|, s_+$;
$\triangleright$ $\mathcal{L}_0$ satisfies a strong maximum principle: $(\mathcal{L}_0 - \mu) f \le 0$ and $f \in X_+ \backslash \{0\}$ imply $f > 0$.

The Peron-Frobenius-Krein-Rutman theorem asserts

$$G_0 \in \mathbb{P} \quad \mathcal{L}_0 G_0 = 0, \quad G_0 \text{ is unique and stable.}$$

More precisely
(1) $\exists a < 0$ such that $\Sigma(\mathcal{L}_0) \cap \Delta_a = \{0\}$;
(2) 0 is simple and $\ker \mathcal{L}_0 = \mathrm{vect}\, G_0$;
(3) $\Pi_0 h = \langle h \rangle G_0$ and $\mathcal{L}_0$ is invertible from $R(I - \Pi_0)$ onto X.

From the the Krein-Rutman theorem, for any solution $\mathcal{L}_0 f = g \in L^2(m)$ with $\langle g \rangle = 0$

$$\|f\|_{L^2(m)} \le C\, \|g\|_{L^2(m)}.$$

Using the additional estimate

$$\forall f \quad \int (\mathcal{L}_0 f) f\, m_0 m^2 \le C_1 \int f^2 m^2 - \kappa_1 \int f^2 m_0^2 m^2 - \kappa_1 \int (\partial_v f)^2 m_0 m^2,$$

we deduce the stronger bound

$$\|f\|_{\mathcal{V}} := \|f\|_{L^2(mH)} + \|\nabla_v f\|_{L^2(mH^{1/2})} \le C\, \|g\|_{L^2(m)}.$$

For any two stationary solutions, we now write

$$\begin{aligned} G_\varepsilon - F_\varepsilon &= \mathcal{L}_0^{-1} \Big[\mathcal{L}_0\, G_\varepsilon - \mathcal{L}_0\, F_\varepsilon \Big] \\ &= \varepsilon\, \mathcal{L}_0^{-1} \Big[\partial_v \Big((v - \mathcal{J}(F_\varepsilon)) F_\varepsilon - (v - \mathcal{J}(G_\varepsilon)) G_\varepsilon \Big) \Big] \end{aligned}$$

and then

$$\begin{aligned} \|F_\varepsilon - G_\varepsilon\|_{\mathcal{V}} &\le \varepsilon\, C \Big\| \partial_v \Big((v - \mathcal{J}(F_\varepsilon))(F_\varepsilon - G_\varepsilon) + (\mathcal{J}(F_\varepsilon) - \mathcal{J}(G_\varepsilon)) G_\varepsilon \Big) \Big\|_{L^2(m)} \\ &\le \varepsilon\, C\, \|F_\varepsilon - G_\varepsilon\|_{\mathcal{V}}. \end{aligned}$$

which in turn implies that necessarily $\|F_\varepsilon - G_\varepsilon\|_{\mathcal{V}} = 0$ for $\varepsilon > 0$ small enough.

The above Krein-Rutman theorem on $\mathcal{L}_0$ and the following properties on $\mathcal{L}_\varepsilon$

$$\mathcal{L}_\varepsilon \to \mathcal{L}_0 \quad \text{and} \quad \mathcal{L}_\varepsilon^* 1 = 0$$

imply (thanks to Theorem 5)

$$\Sigma(\mathcal{L}_\varepsilon) \cap \Delta_a = \{0\}, \quad a < 0, \ \varepsilon \text{ small } > 0.$$

For any solution f the function $h := f - G_\varepsilon$ satisfies

$$\partial_t h = \mathcal{L}_\varepsilon h - \varepsilon \partial_v [\mu_h \, h].$$

From the spectral mapping theorem, we may compute (rigorously at the level of the Duhamel formulation)

$$\begin{aligned} \frac{d}{dt} \|h\|_{L^2}^2 &\leq 2a\|h\|_{L^2}^2 + 2a\|\partial_v h\|_{L^2}^2 + \varepsilon \, |\mu_h| \, \|h\|_{L^2} \, \|\partial_v h\|_{L^2} \\ &\leq 2a\|h\|_{L^2}^2 + C \, \|h\|_{L^2}^4. \end{aligned}$$

As a consequence, the set $\mathcal{C} := \{\|h\|_{L^2}^2 \leq |a|/C\}$ is stable. Then for any $h_0 \in \mathcal{C}$, we get

$$\|h(t)\|_{L^2} \leq C \, e^{at}.$$

Open questions

• What about the "large" connectivity regime: ε is not small?

$\triangleright$ unstability of "the" steady state?

$\triangleright$ periodic solutions? local stability of one of them?

• What about a Hodgin-Huxley statistical model based on the Hodgin-Huxley 4d ODE sytem?

• What about elapsed time (with delay) type model?