LECTURE 3 - PARABOLIC EQUATIONS

We present (the existence part of) the theory of variational solutions for uniformly elliptic parabolic equations. We next discuss the several approaches for dealing with the well-posedness issue of linear evolution equations.

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1. Topic 9. Introduction to the parabolic equations framework

In this lecture we will mainly focus on the parabolic equation

$$\partial_t f = \mathcal{L} f \quad \text{on} \quad (0,\infty)\times\Omega, \tag{1.1}$$

on the function $f = f(t,x)$, $t \geq 0$, $x \in \Omega \subset \mathbb{R}^d$, where $\mathcal{L}$ is the elliptic operator

$$\mathcal{L} f := \mathrm{div}(A\nabla f) + \mathrm{div}(af) + b\cdot\nabla f + cf \tag{1.2}$$

that we complement with an initial condition

$$f(0,x) = f_0(x) \quad \text{in} \quad \Omega. \tag{1.3}$$

In order to develop the variational approach for the equation (1.1)-(1.2), we assume that

$$f_0 \in L^2(\Omega) =: H, \quad \text{which is an Hilbert space,}$$

and we typically assume that the coefficients satisfy

$$A, a, b, c \in L^\infty(\Omega), \quad A \ge \nu I, \ \nu > 0. \tag{1.4}$$

We observe that for any nice function $f = f(x)$, any $\alpha \in (0,\nu)$ and any $\beta > 0$, we have

$$\begin{aligned}\langle \mathcal{L} f, f\rangle &:= \int_{\mathbb{R}^d} (\operatorname{div}(A\nabla f) + \operatorname{div}(af) + b\cdot\nabla f + c\, f) f \\ &= -\int_{\mathbb{R}^d} A\nabla f\cdot\nabla f + \int_{\mathbb{R}^d} f(b-a)\cdot\nabla f + \int_{\mathbb{R}^d} c\, f^2 \\ &\le -(\nu-\beta)\int_{\mathbb{R}^d} |\nabla f|^2 + \int_{\mathbb{R}^d} (c + \frac{|b-a|^2}{4\beta})\, f^2 \\ &\le -\alpha \|f\|^2_{H^1} + \kappa \|f\|^2_{L^2},\end{aligned}$$

with

$$\kappa := \operatorname{ess\,sup}\big(\alpha + \frac{1}{4(\nu-\alpha)}|b-a|^2 + c\big),$$

where we have used the Green-Ostrogradski divergence formula for the two first terms in the second line, the Young inequality $uv \le \beta u^2/2 + v^2/(2\beta)$, $\forall\, u, v \ge 0$, in the third line and we have particularized $\beta := \nu - \alpha$ is the last line. Now, for a (nice) solution $f = f(t,x)$ to the parabolic equation (1.1)-(1.2)-(1.3)-(1.4), we compute

$$\frac{1}{2}\frac{d}{dt}\|f(t)\|^2_{L^2} = \int_{\mathbb{R}^d} (\partial_t f) f = \langle \mathcal{L} f, f\rangle \le -\alpha\|f(t)\|^2_{H^1} + \kappa\, \|f(t)\|^2_{L^2},$$

and, thanks to the Gronwall lemma, we deduce

$$\|f(T)\|^2_{L^2} + 2\alpha\int_0^T \|f(s)\|^2_{H^1}\, ds \le e^{2\kappa T}\, \|f_0\|^2_{L^2}, \quad \forall T. \tag{1.5}$$

In other words, we have established

$$f \in L^\infty(0,T;L^2)\cap L^2(0,T;H^1). \tag{1.6}$$

It is worth emphasizing at this point that for two (nice) functions $f = f(x)$ and $g = g(x)$, we have

$$\langle \mathcal{L} f, g\rangle := \int_{\mathbb{R}^d} (\operatorname{div}(A\nabla f) + \operatorname{div}(af) + b\cdot\nabla f + c\, f) g$$

so that we may compute

$$\langle \mathcal{L} f, g\rangle = -\int_{\mathbb{R}^d} A\nabla f\cdot\nabla g - \int_{\mathbb{R}^d} f(a\cdot\nabla g) + \int_{\mathbb{R}^d} (b\cdot\nabla f) g + \int_{\mathbb{R}^d} c\, f\, g, \tag{1.7}$$

thanks to the Green-Ostrogradski divergence formula. Coming back to a nice solution $f = f(t,x)$ to the parabolic equation (1.1)-(1.2)-(1.3), we may multiply (1.1) by a test function $\varphi \in C^1_c([0,T)\times\mathbb{R}^d)$, and integrating by part, we have

$$\begin{aligned}-\int_{\mathbb{R}^d} f_0\varphi(0) - \int_{\mathcal{U}} f\partial_t\varphi = \int_{\mathcal{U}} \varphi\, \partial_t f = \int_{\mathcal{U}} \varphi\, \mathcal{L} f \\ = -\int_{\mathcal{U}} (A\nabla f + fa)\cdot\nabla\varphi + \int_{\mathcal{U}} (b\cdot\nabla f + c\, f)\varphi.\end{aligned}$$

That formulation gives a first meaningful (distributional) sense to a solution to the equation under the sole assumption $f \in L^2(0,T;H^1)$. Equivalently (by a density $C^1_c(\mathbb{R}^d) \subset H^1(\mathbb{R}^d)$ argument), we may write

$$-(f_0,\varphi(0)) - \int_0^T (f,\varphi')dt = \int_0^T \langle \mathcal{L} f,\varphi\rangle dt, \tag{1.8}$$

for any $\varphi \in C^1_c([0,T);H^1)$.

Definition 1.1. *For any given $f_0 \in L^2$, $T>0$, we say that*

$$f = f(t) \in L^2(0,T;H^1)$$

is a **weak solution** *to the Cauchy problem associated to the parabolic equation* (1.1)-(1.2)-(1.3) *on the time interval* $[0,T)$ *if it satisfies the weak formulation* (1.8) *for any $\varphi \in C^1_c([0,T);H^1)$. We say that f is a global weak solution if it is a weak solution on $[0,T)$ for any $T>0$.*

Theorem 1.2. *With the above definition and assumptions, for any $f_0 \in L^2$, there exists at least one global weak solution to the Cauchy problem* (1.1)-(1.2)-(1.3)-(1.4).

2. Topic 10. First proof - an implicit Euler scheme approach

In this section, we use the shorthands

$$(L^2, \|\cdot\|_{L^2}) = (H, |\cdot|), \quad (H^1, \|\cdot\|_{H^1}) = (V, \|\cdot|\|).$$

We do emphasize that in formulation (1.7) the RHS makes sense for $f,g \in V$ and more precisely

$$|\langle \mathcal{L} f, g\rangle| \le M \|f\|_V \|g\|_V,$$

for a constant $M>0$, thanks to the Cauchy-Schwarz inequality in $L^2(\mathbb{R}^d)$ and because of the hypothesis (1.4) on the coefficients. A possible choice is $M := \|A\|_{L^\infty} + \|a\|_{L^\infty} + \|b\|_{L^\infty} + \|c\|_{L^\infty}$. In other words, taking (1.7) as a definition of $\mathcal{L}$, we have

$$\mathcal{L} : V \to V', \quad V' := H^{-1}(\mathbb{R}^d),$$

is a linear and bounded operator with

$$\forall\, f \in V, \quad \|\mathcal{L} f\|_{V'} = \sup_{g \in B_V} \langle \mathcal{L} f, g\rangle \le M \|f\|_V. \tag{2.1}$$

Introducing an approximation scheme and next using a weak compactness argument in the Hilbert space $L^2(0,T;V)$, we will establish that there exists a function $f \in L^2(0,T;V)$ satisfying the weak formulation (1.8).

Step 1. For a given $f_0 \in H$ and $\varepsilon > 0$, we seek $f_1 \in V$ such that

$$f_1 - \varepsilon \mathcal{L} f_1 = f_0. \tag{2.2}$$

We introduce the bilinear form $\mathfrak{a} : V \times V \to \mathbb{R}$ defined by

$$\mathfrak{a}(u,v) := (u,v) - \varepsilon\, \langle \mathcal{L} u, v\rangle.$$

Thanks to the assumptions made on $\mathcal{L}$, we have

$$|\mathfrak{a}(u,v)| \le |u|\,|v| + \varepsilon\, M\, \|u\|\,\|v\|,$$

and

$$\mathfrak{a}(u,u) \ge |u|^2 + \varepsilon\,\alpha\,\|u\|^2 - \varepsilon\,\kappa\,|u|^2 \ge \varepsilon\,\alpha\,\|u\|^2, \tag{2.3}$$

whenever $\varepsilon\,\kappa < 1$, what we assume from now on. On the other hand, the mapping $v \in V \mapsto (f_0, v)$ is a linear and continuous form. We may thus apply the Lax-Milgram theorem which implies

$$\exists!\, f_1 \in V, \qquad (f_1, v) - \varepsilon\langle \mathcal{L} f_1, v\rangle = (f_0, v), \quad \forall\, v \in V.$$

Step 2. We fix $\varepsilon > 0$ such that $\varepsilon\kappa < 1/2$ and we build by induction the sequence (f_k) in $V \subset H$ defined by the family of equations (implicit Euler scheme)

$$\frac{f_{k+1} - f_k}{\varepsilon} = \mathcal{L}\, f_{k+1}, \qquad \forall\, k \ge 0. \tag{2.4}$$

From the identity

$$(f_{k+1}, f_{k+1}) - \varepsilon\,\langle \mathcal{L} f_{k+1}, f_{k+1}\rangle = (f_k, f_{k+1}),$$

and (2.3) again, we deduce

$$|f_{k+1}|^2 + \varepsilon\,\alpha\,\|f_{k+1}\|^2 - \varepsilon\,\kappa\,|f_{k+1}|^2 \le |f_k|\,|f_{k+1}| \le \frac{1}{2}|f_k|^2 + \frac{1}{2}|f_{k+1}|^2,$$

and then

$$|f_{k+1}|^2 + 2\varepsilon\alpha\,\|f_{k+1}\|^2 \le (1 - 2\varepsilon\kappa)^{-1}\,|f_k|^2, \quad \forall\, k \ge 0.$$

Thanks to the discrete version of the Gronwall lemma, we get

$$|f_n|^2 + 2\alpha \sum_{k=1}^{n} \varepsilon \|f_k\|^2 \le (1 - 2\varepsilon\kappa)^{-n} |f_0| \le e^{2\kappa\varepsilon n}\,|f_0|, \quad \forall\, n \ge 1.$$

We now fix $T > 0$, $n \in \mathbb{N}^*$, and we define

$$\varepsilon := T/n, \quad t_k = k\,\varepsilon, \quad f^\varepsilon(t) := f_{k+1} \text{ on } [t_k, t_{k+1}).$$

The last estimate writes then

$$2\alpha \int_0^T \|f^\varepsilon\|^2\,dt \le e^{2\kappa T}|f_0|^2. \tag{2.5}$$

Step 3. Consider a test function $\varphi \in C^1_c([0,T);V)$ and define $\varphi_k := \varphi(t_k)$, so that $\varphi_n = \varphi(T) = 0$. Multiplying the equation (2.4) by φ_k and summing up from $k = 0$ to $k = n-1$, we get

$$-(\varphi_0, f_0) - \sum_{k=0}^{n-1} (\varphi_{k+1} - \varphi_k, f_{k+1}) = \sum_{k=0}^{n-1} \varepsilon\,\langle \mathcal{L} f_{k+1}, \varphi_k\rangle.$$

Introducing the two functions $\varphi^\varepsilon, \varphi_\varepsilon : [0,T) \to V$ defined by

$$\varphi^\varepsilon(t) := \varphi_k \quad \text{and} \quad \varphi_\varepsilon(t) := \frac{t_{k+1} - t}{\varepsilon}\,\varphi_k + \frac{t - t_k}{\varepsilon}\,\varphi_{k+1} \qquad \text{for} \quad t \in [t_k, t_{k+1}),$$

in such a way that

$$\varphi_\varepsilon'(t) = \frac{\varphi_{k+1} - \varphi_k}{\varepsilon} \qquad \text{for} \quad t \in (t_k, t_{k+1}),$$

the above equation also writes

$$-(\varphi(0), f_0) - \int_0^T (\varphi_\varepsilon', f^\varepsilon)\,dt = \int_0^T \langle \mathcal{L} f^\varepsilon, \varphi^\varepsilon\rangle\,dt. \tag{2.6}$$

On the one hand, from (2.5) and the fact that $L^2(0,T;V)$ is a Hilbert space, we know that up to the extraction of a subsequence, there exists $f \in L^2(0,T;V)$ such that $f^\varepsilon \rightharpoonup f$ weakly in $L^2(0,T;V)$ and thus $\mathcal{L} f^\varepsilon \rightharpoonup \mathcal{L} f$ weakly in $L^2(0,T;V')$. On the other hand, from the above construction, we have $\varphi'_\varepsilon \to \varphi'$ and $\varphi_\varepsilon \to \varphi$ both uniformly in $L^\infty(0,T;V)$ (using that φ and φ' belong to $C([0,T];V)$ and thus are uniformly continuous). We may then pass to the limit as $\varepsilon \to 0$ in (2.6) and we get (1.8). More concretely, we are just saying that

$$f^\varepsilon \rightharpoonup f, \quad \nabla f^\varepsilon \rightharpoonup \nabla f \quad \text{weakly in} \quad L^2(\mathscr{U}),$$
$$\varphi'_\varepsilon \to \varphi', \quad \varphi^\varepsilon \to \varphi \quad \nabla\varphi^\varepsilon \rightharpoonup \nabla\varphi \quad \text{strongly in} \quad L^2(\mathscr{U}),$$

and we may pass to the limit $\varepsilon \to 0$ in both integrals

$$\int_0^T (\varphi'_\varepsilon, f^\varepsilon)\,dt = \int_{\mathscr{U}} \varphi'_\varepsilon f^\varepsilon$$

and

$$\int_0^T \langle \mathcal{L} f^\varepsilon, \varphi^\varepsilon\rangle\,dt = -\int_{\mathscr{U}} \nabla f^\varepsilon\cdot\nabla\varphi_\varepsilon + \int_{\mathscr{U}} (b\cdot\nabla f^\varepsilon + c f^\varepsilon)\varphi_\varepsilon.$$

Exercise 2.1. *Establish the same existence result under the assumptions*

$$a, b \in L^d(\Omega), \quad c \in L^1_{\rm loc}(\Omega), \quad c_+ \in L^{d/2}(\Omega).$$

3. Topic 11. Second proof of the existence part - a variational approach

3.1. **A variant of the Lax-Milgram theorem.** We consider a Hilbert space $\mathscr{H}$ endowed with a scalar product $(\cdot,\cdot)$ and the associated norm $|\cdot|$. We consider next a subspace $\Phi \subset \mathscr{H}$ endowed with a pre-Hilbertian scalar product $(\!(\cdot,\cdot)\!)$ and the associated norm $\|\cdot\|$ such that

$$|\varphi| \le C\|\varphi\|, \quad \forall\,\varphi \in \Phi. \tag{3.1}$$

We finally consider a bilinear form $\mathcal{E} : \mathscr{H}\times\Phi \to \mathbb{R}$ such that

$$\forall\,\varphi\in\Phi,\ \exists\, C_\varphi \ge 0, \quad |\mathcal{E}(f,\varphi)| \le C_\varphi |f|, \quad \forall\, f\in\mathscr{H}, \tag{3.2}$$

$$\exists\,\alpha > 0, \qquad \mathcal{E}(\varphi,\varphi) \ge \alpha\|\varphi\|^2, \quad \forall\,\varphi\in\Phi. \tag{3.3}$$

Theorem 3.1. *For any linear and continuous form $\ell : \Phi \to \mathbb{R}$, meaning that*

$$|\ell(\varphi)| \le C\|\varphi\|, \quad \forall\,\varphi\in\Phi, \tag{3.4}$$

there exists at least one $f \in \mathscr{H}$ such that

$$\mathcal{E}(f,\varphi) = \ell(\varphi), \quad \forall\,\varphi\in\Phi. \tag{3.5}$$

Proof of Theorem 3.1. For a fixed $\varphi\in\Phi$, the mapping $f \mapsto \mathcal{E}(f,\varphi)$ is a linear and continuous form on $\mathscr{H}$, so that, from the Riesz-Fréchet representation theorem in $\mathscr{H}$, there exists $\mathcal{A}\varphi \in \mathscr{H}$ such that

$$\mathcal{E}(f,\varphi) = (f, \mathcal{A}\varphi), \quad \forall\, f\in\mathscr{H},\ \varphi\in\Phi, \tag{3.6}$$

and $\mathcal{A} : \Phi \to \mathscr{H}$ is a linear mapping. Because of (3.3), $\mathcal{A}$ is one-to-one (injection). On the linear subspace $\mathscr{G} := \mathcal{A}\Phi \subset \mathscr{H}$, we may then define the inverse linear mapping $\mathcal{B} := \mathcal{A}^{-1} : \mathscr{G} \to \Phi$. Using (3.6), (3.3) and (3.1), for any $g\in\mathscr{G}$, we have

$$\alpha\|\mathcal{B}g\|^2 \le \mathcal{E}(\mathcal{B}g,\mathcal{B}g) = (\mathcal{B}g, g) \le |\mathcal{B}g||g| \le C\|\mathcal{B}g\||g|,$$

from what we immediately deduce that $\mathcal{B}$ is bounded with norm $\|\mathcal{B}\| \le C/\alpha$. Defining $\overline{\mathscr{G}}$ the closure of $\mathscr{G}$ in $\mathscr{H}$ (for the norm $|\cdot|$) and $\hat\Phi$ the completion of Φ for the norm $\|\cdot\|$, we may uniquely extend $\mathcal{B}$ as $\bar{\mathcal{B}} : \bar{\mathscr{G}} \to \hat\Phi$, $\bar{\mathcal{B}}_{|\mathscr{G}} = B$. We may also uniquely extend ℓ as a linear and continuous form $\bar\ell$ on $\hat\Phi$. The equation (3.5) becomes

$$(f, \mathcal{A}\varphi) = \bar\ell(\varphi), \quad \forall\, \varphi \in \Phi,$$

or equivalently

$$(f, \psi) = \bar\ell(\bar{\mathcal{B}}\psi), \quad \forall\, \psi \in \bar{\mathscr{G}}. \tag{3.7}$$

From the Riesz-Fréchet representation theorem in $\bar{\mathscr{G}}$ and because $\bar\ell \circ \bar{\mathcal{B}}$ is a linear and continuous mapping on $\bar{\mathscr{G}}$, there exists a unique $f \in \bar{\mathscr{G}}$ solution to (3.7), and this one provides a solution to (3.5). When $\bar{\mathscr{G}} \neq \mathscr{H}$, the problem (3.5) has a family of solutions given by $\{f\} + \bar{\mathscr{G}}^\perp$. □

3.2. **An alternative proof of Theorem 1.2.** We consider the parabolic equation (1.1)-(1.2)-(1.3)-(1.4) with same notations, with $A := I$ and $a := 0$ for simplicity and we additionally assume

$$\sup c + \frac{1}{2}|b|^2 \le -\frac{1}{2}. \tag{3.8}$$

This additional assumption will be removed in the next section. We define the Hilbert space $\mathscr{H} := L^2(0,T; H^1(\mathbb{R}^d))$ endowed with its usual norm and the pre-Hilbert space $\Phi := C^1_c([0,T) \times \mathbb{R}^d)$ endowed with the norm $\|\cdot\|$ defined by

$$\|\varphi\|^2 := \int_0^T \|\varphi(t,\cdot)\|^2_{H^1(\mathbb{R}^d)} dt + \|\varphi(0,\cdot)\|^2_{L^2(\mathbb{R}^d)}.$$

We also define the bilinear form

$$\mathcal{E}(f,\varphi) := \int_{\mathscr{U}} (\nabla f \cdot \nabla \varphi - (b \cdot \nabla f + cf)\varphi - f \partial_t \varphi)\, dxdt,$$

with always $\mathscr{U} := (0,T) \times \mathbb{R}^d$, and the linear form

$$\ell(\varphi) := \int_{\mathbb{R}^d} \varphi(0,\cdot) f_0 \, dx.$$

We observe that

$$\mathcal{E}(\varphi,\varphi) \;=\; \int_{\mathscr{U}} (|\nabla\varphi|^2 - \nabla\varphi \cdot b\,\varphi - c\varphi^2) dxdt + \frac{1}{2}\int_{\mathbb{R}^d} \varphi(0,x)^2 dx \ge \frac{1}{2}\|\varphi\|^2,$$

where we have used the Young inequality and the condition (3.8) in order to get the last inequality, that $\mathcal{E}$ also satisfies (3.2) and that ℓ satisfies (3.4). From Theorem 3.1, we know that there exists $f \in \mathscr{H}$ satisfying (3.5), or in other words

$$\int_{\mathscr{U}} (\nabla f \cdot \nabla \varphi - (b \cdot \nabla f + cf)\varphi - f \partial_t \varphi)\, dxdt = \int_{\mathbb{R}^d} \varphi(0,\cdot) f_0 \, dx,$$

for any $\varphi \in C^1_c([0,T) \times \mathbb{R}^d)$. Because $C^1_c([0,T) \times \mathbb{R}^d) \subset C^1_c([0,T); H^1(\mathbb{R}^d))$ with dense embedding, we deduce that f is in fact a weak-solution in the sense of Definition 1.1.

3.3. **A time dependent variant of Theorem 1.2.** We consider the parabolic equation

$$\partial_t f = \mathcal{L} f := \mathrm{div}(A\,\nabla f) + \mathrm{div}(af) + b\cdot\nabla f + cf + \mathfrak{F}, \tag{3.9}$$

where A_{ij}, a_i, b_i and c are possible time dependent coefficients and where A_{ij} is uniformly elliptic in the sense that

$$\forall\, t\in(0,T),\ \forall\, x\in\mathbb{R}^d,\ \forall\,\xi\in\mathbb{R}^d\quad A_{ij}(t,x)\,\xi_i\xi_j \ge \nu\,|\xi|^2,\quad \nu>0. \tag{3.10}$$

Theorem 3.2 (J.-L. Lions)**.** *Assume that*

$$A,\ a,\ b,\ c\in L^\infty((0,T)\times\mathbb{R}^d) \tag{3.11}$$

and that A satisfies the uniformly elliptic condition (3.10)*. For any* $f_0\in L^2(\mathbb{R}^d)$ *and* $\mathfrak{F} := F_0 + \mathrm{div} F$, $F_i\in L^2(\mathscr{U})$*, there exists at least a weak solution* $f\in L^2(0,T;H^1)$ *to the Cauchy problem associated to* (3.9) *in the sense that*

$$\begin{aligned}\int_{\mathbb{R}^d} f(t)\varphi(t)\,dx &= \int_{\mathbb{R}^d} f_0\varphi(0)\,dx + \int_0^t\!\!\int_{\mathbb{R}^d}(\mathfrak{F}\varphi + f\partial_t\varphi)\,dxds\\ &+\int_0^t\!\!\int_{\mathbb{R}^d}\{(b\cdot\nabla f + cf)\,\varphi - (A\nabla f + af)\cdot\nabla\varphi\}\,dxds,\end{aligned} \tag{3.12}$$

for any$\varphi\in C^1_c([0$

Proof of Theorem 3.2. Step 1. We proceed similarly as in the alternative proof of Theorem 1.2 in Section 3.2 and in particular we define $\mathscr{H}$ and Φ in the same way. We now define the bilinear form on $\mathscr{H}\times\Phi$ by

$$\mathcal{E}(f,\varphi) := \int_{\mathscr{U}}((A\nabla f + af)\cdot\nabla\varphi - (b\cdot\nabla f + cf)\varphi - f\partial_t\varphi)\,dxdt$$

and the linear form on Φ by

$$\ell(\varphi) := \int_{\mathscr{U}}(F_0\varphi - F\cdot\nabla\varphi)\,dxdt + \int_{\mathbb{R}^d}\varphi(0,\cdot)f_0\,dx.$$

We additionally first assume that

$$\sup c \le -\min(\frac{1}{2},\frac{\nu}{2}) - \frac{1}{2\nu}\|a-b\|^2_{L^\infty}. \tag{3.13}$$

In that case, we may observe that

$$\begin{aligned}\mathcal{E}(\varphi,\varphi) &= \int_{\mathscr{U}}(A\nabla\varphi\cdot\nabla\varphi + \nabla\varphi\cdot(a-b)\,\varphi - c\varphi^2)dxdt + \frac{1}{2}\int_{\mathbb{R}^d}\varphi(0,x)^2dx\\ &\ge \min(\frac{1}{2},\frac{\nu}{2})\|\varphi\|^2,\end{aligned}$$

that $\mathcal{E}$ also satisfies (3.2) and that ℓ satisfies (3.4). Exactly as in Section 3.2, we deduce the existence of a weak solution $f\in\mathscr{H}$ to the parabolic equation (3.9) with the help of Theorem 3.1.

Step 2. We do not assume anymore (3.13). We define $c_\lambda := c-\lambda$, with $\lambda>0$ large enough in such a way that c_λ satisfies the additional condition (3.13), and we set $\mathfrak{F}_\lambda := e^{-\lambda t}\mathfrak{F}$. We may apply the first step with the choice of functions A, a, b, c_λ, f_0, $\mathfrak{F}_\lambda$, and we thus obtain the existence of a variational solution $g\in\mathscr{H}$ to the modified equation

$$\partial_t g + \lambda g = \mathrm{div}(A\,\nabla g) + \mathrm{div}(ag) + b\cdot\nabla g + cg + e^{-\lambda t}\mathfrak{F}\ \text{ in }\ \mathscr{U}, \tag{3.14}$$

with initial condition $g(0,\cdot) = f_0$. For any $\varphi \in C^1_c([0,T);H^1(\mathbb{R}^d))$, choosing $\phi := e^{\lambda t}\varphi \in C^1_c([0,T);H^1(\mathbb{R}^d))$ as a test function in the variational formulation of (3.14), we immediately deduce that $f := e^{\lambda t} g \in \mathscr{H}$ satisfies (3.12). □

Exercise 3.3. *Consider the transport equation*

$$\partial_t f = \mathrm{div}(af) + b\cdot\nabla f + cf, \quad f(0) = f_0,$$

with

$$a,b,c \in L^\infty((0,T)\times\mathbb{R}^d), \quad f_0 \in L^2(\mathbb{R}^d),$$

and prove the existence of a weak solution $f \in L^2((0,T)\times\mathbb{R}^d)$ thanks to the variational method.

4. Topic 12. Generalities about evolution PDEs

• From well-posed evolution equation to semigroup.
We consider an evolution equation

$$\partial_t f = \mathcal{L} f, \quad f(0) = f_0. \tag{4.1}$$

For two Banach spaces X and $\mathscr{X} \subset C(\mathbb{R}_+;X)$, we assume that for any $f_0 \in X$, there exists a unique function $f \in \mathscr{X}$ which is a solution to the evolution equation (possibly in a weak sense) and that for any $T,R>0$ there exists $C_{T,R}$ such that

$$\sup_{[0,T} \|f(t)\|_X \le C_{T,R} \quad \text{if} \quad \|f_0\| \le R.$$

Then, there exists a semigroup S on X such that the above solution is given by $f = S_t f_0$. We recall the definition of a semigroup:

We say that $S = (S_t)_{t\ge0}$ is a continuous semigroup of linear and bounded operators on a Banach space X, or we just say that S_t is C_0-semigroup (or a semigroup) on X, if the following conditions are fulfilled:

(i) one parameter family of operators: $\forall t\ge0$, $f\mapsto S_t f$ is linear and continuous on X;

(ii) continuity of trajectories: $\forall f\in X$, $t\mapsto S_t\, f \in C([0,\infty),X)$;

(iii) semigroup property: $S_0 = I$; $\quad \forall\, s,\, t\ge0$, $S_{t+s} = S_t\, S_s$;

(iv) growth estimate: $\exists\,\kappa\in\mathbb{R}$, $\exists\, M\ge1$,

$$\|S_t\|_{\mathscr{B}(X)} \le M\, e^{\kappa t} \quad \forall\, t\ge0. \tag{4.2}$$

We say that S is a semigroup of contractions if (4.2) *holds with $M=1$ and $\kappa=0$.*

• From semigroup to evolution equation. On the other way round, for a given semigroup S, we may associate its generator in the following way. We define the domain

$$D(\mathcal{L}) := \Big\{ f\in X;\ \lim_{t\searrow0} \frac{S_t\, f - f}{t} \text{ exists in } X \Big\},$$

and next the generator

$$\mathcal{L}\, f \ := \lim_{t\searrow0} \frac{S_t\, f - f}{t} \quad \text{for any} \quad f\in D(\mathcal{L}).$$

It turns out that for any $f_0\in D(\mathcal{L})$ (resp. $f_0\in X$) the flow $f := S_t f_0$ provides a strong (resp. weak) solution to the evolution equation (4.1) associated to its generator $\mathcal{L}$.

• **Explicit semigroup.** They are some (few) evolution PDEs for which we may build explicitly the solutions through a representation formula (among them are the heat equation and the transport equation). That provides in the same time the solution and the associated semigroup.

• **Spectral analysis and evolution equation.** They are some evolution PDEs associated to an integro-differential operator $\mathcal{L}$ acting in some Hilbert space $\mathcal{H}$ for which we may establish the existence of spectral basis. That means that there exists a sequence (ϕ_k, λ_k) of $\mathcal{H} \times \mathbb{R}$ such that the space generated by (ϕ_k) is dense in $\mathcal{H}$ and

$$(\phi_k, \phi_\ell) = \delta_{k\ell}, \quad \mathcal{L}\phi_k = \lambda\phi_k, \quad \forall\, k, \ell \geq 1.$$

For any $f_0 \in \mathcal{H}$, the evolution equation (4.1) is equivalent to

$$f_k' = \lambda_k f_k, \quad f_k(0) = (f_0, \phi_k)_{\mathcal{H}}.$$

We thus obtain that the function

$$f(t) := \sum_{k=1}^{\infty} e^{\lambda_k t} (f_0, \phi_k)_{\mathcal{H}} \phi_k$$

is a solution to (4.1).

• **Perturbation / Duhamel formula.** Consider a semigroup $S_\mathcal{B}$ with generator $\mathcal{B}$ and an operator $\mathcal{A}$ which is bounded by $\mathcal{B}$ (in a sense to specify). We may then build a (mild) solution to the evolution equation associated to the operator $\mathcal{L} := \mathcal{B} + \mathcal{A}$ through one of the two Duhamel formulas

$$S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{B}\mathcal{A} * S_\mathcal{L} = S_\mathcal{B} + S_\mathcal{L} * \mathcal{A}S_\mathcal{B},$$

that we establish to be true using the Banach-Picard point Theorem exactly as we have done for perturbing the heat equation (in the first lecture) and the free transport equation (in the second lecture).

• **The variational approach.** In a Hilbert space framework, the variational approach of J.-L. Lions provides an efficient tools for proving the existence of solutions for a large class of evolution PDE, including parabolic equations and transport equations.

• **The Hille-Yosida theory**. Any semigroup is a semigroup of contractions in a convenient equivalent Banach space. Thanks to the Hille-Yosida-Lumer-Phillips theorem, we may characterize the class of operators which are the generator of semigroups of contractions: they are the operator with dense domain, closed graph and which are maximal dissipative. In a Hilbert space, we say that an operator $\mathcal{L}$ is maximal dissipative if

$$\exists\, x_0 \in \mathbb{R},\ \forall\, x \geq x_0,\ R(x - \mathcal{L}) = \mathcal{H} \quad \text{and} \quad \forall\, f \in D(\mathcal{L}),\ (\mathcal{L}f, f)_\mathcal{H} \leq 0$$

and it has closed graph if $\{(f, \mathcal{L}f);\ f \in \mathcal{H}\}$ is closed in $\mathcal{H} \times \mathcal{H}$. We may then build a solution to the evolution equation associated to $\mathcal{L}$ by just using the Euler implicit scheme (2.4).