LECTURE 2 - TRANSPORT EQUATIONS

We first present some versions of the Gronwall lemma. The lecture next mainly addresses another simple evolution equations which is the transport equation that we solve by using the characteristics method. We finally present how the semi-group/perturbation arguments for establishing the existence of solutions to much more general equations.

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1. Topic 5. The Gronwall lemma

There are many variants of the Gronwall lemma which simplest formulation tells us that any given function $u : [0,T) \to \mathbb{R}$, $T \in (0,\infty]$, of class C^1 satisfying the differential inequality

$$u' \le au \quad \text{on} \quad (0,T), \tag{1.1}$$

for $a \in \mathbb{R}$, also satisfies the pointwise estimate

$$u(t) \le e^{at} u(0) \quad \text{on} \quad [0,T). \tag{1.2}$$

We indeed establish (1.2) by a mere time integration of the differential inequality $(u\, e^{-at})' \le 0$ that we deduce from (1.1).

We give two generalized versions of the above result.

Lemma 1.1 (classical differential version of Gronwall lemma)**.** *We assume that $u \in C([0,T);\mathbb{R})$, $T \in (0,\infty)$, satisfies the differential inequality*

$$u' \le a(t) u + b(t) \quad \text{on} \quad (0,T), \tag{1.3}$$

for some $a, b \in L^1(0,T)$. Then, u satisfies pointwise the estimate

$$u(t) \le e^{A(t)} u(0) + \int_0^t b(s) e^{A(t)-A(s)}\, ds \quad \text{on} \quad (0,T), \tag{1.4}$$

where we have defined the primitive function

$$A(t) := \int_0^t a(s)\, ds. \tag{1.5}$$

Some examples and important special cases of the Gronwall lemma are

$$u' \le a(t) u \implies u(t) \le u(0) e^{A(t)}, \tag{1.6}$$

$$u' \le au + b \implies u(t) \le u(0) e^{at} + \frac{b}{a}(e^{at} - 1), \tag{1.7}$$

$$u' \le au + b(t) \implies u(t) \le u(0) e^{at} + \int_0^t e^{a(t-s)}\, b(s)\, ds, \tag{1.8}$$

$$u' + b(t) \le a(t) u,\ a,\, b \ge 0 \implies u(t) + \int_0^t b(s)\, ds \le u(0) e^{A(t)}. \tag{1.9}$$

Proof of Lemma 1.1. We only present the proof under the stronger assumption $u \in W^{1,1}(0,T) \subset C([0,T])$. The differential inequality (1.3) means

$$-\langle u, \varphi' \rangle \le \langle au + b, \varphi \rangle \tag{1.10}$$

for any $0 \le \varphi \in \mathcal{D}(0,T)$. We set

$$v(t) := u(t)\, e^{-A(t)} - \int_0^t b(s) e^{-A(s)}\, ds,$$

and we observe that

$$v' \le 0 \ \text{ in } \ \mathcal{D}'(0,T), \qquad v \in C([0,T]).$$

Because $v \in W^{1,1}$, we immediately conclude to

$$v(t) = v(0) + \int_0^t v'(s)\, ds \le v(0) = u(0),$$

from what (1.4) follows. □

Lemma 1.2 (integral version of Gronwall lemma)**.** *We assume* $u \in \mathcal{L}^\infty(0,T;\mathbb{R})$, $T \in (0,\infty)$*, satisfies pointwise the integral inequality*

$$u(t) \le u_0 + \int_0^t a(s)u(s)\,ds + \int_0^t b(s)\,ds \quad on \quad (0,T), \tag{1.11}$$

for some $0 \le a \in L^1(0,T)$ *and* $b \in L^1(0,T)$*. Then,* u *satisfies pointwise the estimate*

$$u(t) \le u_0\, e^{A(t)} + \int_0^t b(s) e^{A(t)-A(s)}\,ds \quad on \quad (0,T). \tag{1.12}$$

Proof of Lemma 1.2. Step 1. We first assume $b \equiv 0$. We set $v(t) := u(t) - u_0\, e^{A(t)}$ and we compute

$$\begin{aligned} v(t) &\le \int_0^t a(s)\,u(s)\,ds + u_0\,(1 - e^{A(t)}) \\ &= \int_0^t a(s)\,(v(s) + u_0\,e^{A(s)})\,ds + u_0\,(1-e^{A(t)}) \\ &= \int_0^t a(s)\,v(s)\,ds. \end{aligned}$$

Because a is nonnegative, it yields

$$v_+(t) \le \int_0^t a(s)\,v_+(s)\,ds =: w(t). \tag{1.13}$$

The function $w \in W^{1,1}(0,T)$ satisfies

$$w'(t) = a(t)\,v_+(t) \le a(t)\,w(t) \quad \text{on} \quad (0,T),$$

and we may use Lemma 1.1 in order to deduce $w(t) \le w(0) = 0$. We next get $v(t) \le v_+(t) \le w(t) \le 0$ and the conclusion.

Step 2. We do not assume $b \equiv 0$ *anymore.* We define

$$v(t) := u(t) - u_0 e^{A(t)} - \int_0^t b(s) e^{A(t)-A(s)}\,ds.$$

We observe that we have again

$$v(t) \le \int_0^t a(s)\,v(s)\,ds,$$

and we conclude as in the first step. □

Exercise 1.3. *(1) Prove Lemma 1.1 under the additional assumptions* $a, u \ge 0$ *as a consequence of Lemma 1.2. (Hint. Pass to the limit* $\varphi \to \mathbf{1}_{[0,t]}$ *in* (1.10)*).*
(2) Prove Lemma 1.1 in full generality. (Hint. Take φ *as a primitive of* $\psi := -w + (\int_0^T w)\,\varrho$ *for arbirary* $0 \le w \in C^1_c(\varepsilon,T)$ *and* $\varrho \in C_c(0,\varepsilon)$ *a probability measure).*

Exercise 1.4. *Let* $f \in C^1((0,T)\times\mathbb{R})$ *and consider* $u, v \in C([0,T];\mathbb{R})$ *such that*

$$u' \le f(t,u), \quad v' \ge f(t,u), \quad u(0) \le v(0), \tag{1.14}$$

(in a distributional sense). Prove that $u \le v$ *on* $[0,T]$*.*

We finally present a discrete version of the Gronwall lemma.

Lemma 1.5 (discrete version of Gronwall lemma)**.** *We consider a real numbers sequence* (u_n) *such that*

$$u_{n+1} \le a_{n+1} u_n + b_{n+1}, \quad \forall\, n \ge 0, \tag{1.15}$$

where (a_n) *and* (b_n) *are two given real numbers sequences and* (a_n) *is furthermore positive. Then*

$$u_n \le A_n u_0 + \sum_{k=1}^{n} A_{k,n} b_k, \quad \forall\, n \ge 0, \tag{1.16}$$

where we have defined

$$A_n := \prod_{k=1}^{n} a_k, \quad A_{k,n} = A_n / A_k = \prod_{i=k+1}^{n} a_i.$$

Proof of Lemma 1.5. We define

$$v_n := A_n u_0 + \sum_{k=1}^{n} A_{k,n} b_k,$$

and we observe that

$$\begin{aligned} v_{n+1} &= A_{n+1} u_0 + \sum_{k=1}^{n+1} A_{k,n+1} b_k \\ &= a_{n+1} A_n u_0 + \sum_{k=1}^{n} a_{n+1} A_{k,n} b_k + b_{n+1} \\ &= a_{n+1} v_n + b_{n+1}. \end{aligned}$$

We then easily check by induction that $u_n \le v_n$ for any $n \ge 0$. ☐

A particularly interesting special case is

$$u_{n+1} + b_{n+1} \le a u_n, \ a \ge 1, b_{n+1} \ge 0 \quad \Longrightarrow \quad u_n + \sum_{k=1}^{n} b_k \le a^n u_0.$$

2. Topic 6. The characteristics method for smooth data

In this section, we consider the transport equation

$$\partial_t f + b \cdot \nabla f = 0 \quad \text{in} \quad (0, \infty) \times \mathbb{R}^d, \tag{2.1}$$

for a drift force field $b = b(t,x) : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, that we complement with an initial condition

$$f(0,x) = f_0(x) \quad \text{in} \quad \mathbb{R}^d.$$

We assume that b is C^1 and satisfies the globally Lipschitz estimate

$$|b(t,x) - b(t,y)| \le L\, |x - y|, \quad \forall\, t \ge 0, \ x, y \in \mathbb{R}^d, \tag{2.2}$$

for some constant $L \in (0, \infty)$.

Thanks to the Cauchy-Lipschitz theorem on ODE, we know that for any $x \in \mathbb{R}^d$ and $s \ge 0$, the equation

$$\dot x(t) = b(t, x(t)), \quad x(s) = x, \tag{2.3}$$

admits a unique solution $t \mapsto x(t) = \Phi_{t,s}(x) \in C^1(\mathbb{R}_+; \mathbb{R}^d)$. We also know that, for any $s, t \ge 0$, the vectors valued function $\Phi_{t,s} : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism which satisfies the semigroup properties $\Phi_{t,t} = \mathrm{Id}$, $\Phi_{t_3,t_2} \circ \Phi_{t_2,t_1} = \Phi_{t_3,t_1}$ for any $t_3, t_2, t_1 \ge 0$. Moreover, the mapping $[0,T] \times [0,T] \times B(0,R) \to \mathbb{R}^d$, $(s,t,x) \mapsto \Phi_{s,t}(x)$ is Lipschitz for any $T, R > 0$, and we denote by $L_{T,R}$ the associated Lipschitz constant.

The characteristics method makes possible to build a solution to the transport equation (2.1) thanks to the solutions (characteristics) of the above ODE.

Assuming first $f_0 \in C^1(\mathbb{R}^d;\mathbb{R})$, we define the function $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^d;\mathbb{R})$ by

$$\forall\, t \ge 0,\ \forall\, x \in \mathbb{R}^d, \quad f(t,x) := f_0(\Phi_t^{-1}(x)), \quad \Phi_t := \Phi_{t,0}. \tag{2.4}$$

From the associated implicit equation $f(t,\Phi_t(x)) = f_0(x)$, we deduce

$$\begin{aligned} 0 &= \frac{d}{dt}[f(t,\Phi_t(x))] = (\partial_t f)(t,\Phi_t(x)) + \dot\Phi_t(x)\cdot(\nabla_x f)(t,\Phi_t(x)) \\ &= (\partial_t f + b\cdot\nabla_x f)(t,\Phi_t(x)). \end{aligned}$$

Because the above equation holds true for any $t > 0$ and $x \in \mathbb{R}^d$ and because the function Φ_t is mapping $\mathbb{R}^d$ onto $\mathbb{R}^d$, we deduce that f satisfies the transport equation (2.1) pointwise, and thus f is a solution in the classical sense (of the differential calculus). On the other way round, for any classical solution f, thanks to the same computations, we observe that $\frac{d}{dt}[f(t,\Phi_t(x))] = 0$, so that f satisfies (2.4). In other words, a classical solution exists and is unique.

If furtheremore $f_0 \in C^1_c(\mathbb{R}^d)$, we have $f(t) \in C^1_c(\mathbb{R}^d)$ for any $t \ge 0$. Indeed, let take $R > 0$ such that $\operatorname{supp} f_0 \subset B_R$ and denote by R_t a constant such that $\Phi_t(\bar B_R) \subset B_{R_t}$, what is possible because $\Phi_t : \mathbb{R}^d \to \mathbb{R}^d$ is continuous (alternatively, one can observe that $|\Phi_t(x) - \Phi_0(x)| \le L_{t,R} t$ for any $x \in \mathbb{R}^d$ ant $t \ge 0$, so that $R_t := R + tL_{t,R}$ is suitable for any $t \ge 0$). As a consequence, $B_R \cap \Phi_t^{-1}(B^c_{R_t}) = \emptyset$, which implies that $f_0(\Phi_t^{-1}(x)) = 0$ if $x \in B^c_{R_t}$, and therefore $\operatorname{supp} f(t,\cdot) \subset B_{R_t}$. In other words, transport occurs with finite speed: that makes a great difference with the instantaneous positivity of solution (related of a "infinite speed" of propagation of particles) known for the heat equation and more generally for parabolic equations.

Exercise 2.1. *Make explicit the construction and formulas in the three following cases:*
(1) $b(x) = b \in \mathbb{R}^d$ is a constant vector. (Hint. One must find $f(t,x) = f_0(x - bt)$).
(2) $b(x) = x$. (Hint. One must find $f(t,x) = f_0(e^{-t}x)$).
(3) $b(x,v) = v$, $f_0 = f_0(x,v) \in C^1(\mathbb{R}^d \times \mathbb{R}^d)$ and look for a solution $f = f(t,x,v) \in C^1((0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d)$. (Hint. One must find $f(t,x,v) = f_0(x - vt, v)$).
(4) Assume that $b = b(x)$ and prove that (S_t) is a group on $C(\mathbb{R}^d)$, where

$$\forall\, f_0 \in C(\mathbb{R}^d),\ \forall\, t \in \mathbb{R},\ \forall\, x \in \mathbb{R}^d \quad (S_t f_0)(x) = f(t,x) := f_0(\Phi_t^{-1}(x)). \tag{2.5}$$

Exercise 2.2. *(1) Show that*

$$f(t,x,v) := f_0(x - vt, v)e^{-t}, \quad t \ge 0,\ x \in \mathbb{R}d,\ v \in \mathbb{R}^d,$$

is a solution to the dampted free transport equation

$$\partial_t f + v\cdot\nabla_x f = -f, \quad f(0,\cdot) = f_0.$$

(2) Show that

$$f(t,x) := f_0(\Phi_{0,t}(x))\, e^{-\int_0^t c(\tau,\Phi_{\tau,t}(x))\,d\tau} + \int_0^t G(s,\Phi_{s,t}(x))\, e^{-\int_s^t c(\tau,\Phi_{\tau,t}(x))\,d\tau}\, ds$$

is a solution to the transport equation with source term

$$\partial_t f + b \cdot \nabla f + c\, f = G, \quad f(0) = f_0, \tag{2.6}$$

with $b = b(t,x)$, $c = c(t,x)$ *and* $G = G(t,x)$ *smooth functions. (Hint. Compute the time derivative of* $f(t,\Phi_t(x)) \exp \int_0^t c(s,\Phi_s(x))\, ds$*).*

Exercise 2.3. *1) Consider the transport equation with vanishing boundary condition*

$$\begin{cases} \partial_t f + \partial_x f = 0 \\ f(t,0) = 0, \quad f(0,x) = f_0(x), \end{cases} \tag{2.7}$$

where $f = f(t,x)$, $t \ge 0$, $x \ge 0$. *Assume* $f_0 \in C^1_c(]0,\infty[)$. *Establish that* $\bar f(t,x) := f_0(x-t)$ *provides a solution to equation (2.7).*

2) Consider the transport equation with boundary condition

$$\begin{cases} \partial_t f + \partial_x f + a f = 0 \\ f(t,0) = b(t), \quad f(0,x) = f_0(x), \end{cases} \tag{2.8}$$

where $f = f(t,x)$, $t \ge 0$, $x \ge 0$. *Assume* $a \in L^\infty(\mathbb{R}_+)$, $f_0 \in C^1_c(]0,\infty[)$ *and* $b \in C^1_c(]0,T[)$. *Show that the characteristics method provides a unique smooth solution* f *given by* $f = \bar f$, *with*

$$\bar f(t,x) := e^{A(x-t)-A(x)} f_0(x-t) \mathbf{1}_{x>t} + e^{-A(x)} b(t-x) \mathbf{1}_{t>x}, \quad A(x) := \int_0^x a(u)\, du.$$

(Hint. When $f \in C^1([0,T]\times\mathbb{R}_+)$, *observe that both*

$$\frac{d}{dt}(e^{A(t+x)} f(t,t+x)) = 0, \quad \frac{d}{dx}(e^{A(x)} f(t+x,x)) = 0, \quad A(x) := \int_0^x a(u)\, du,$$

and then $f = \bar f$. *Also observe that* $\bar f \in C^1([0,T]\times\mathbb{R}_+)$ *in that case and conclude).*

3. Topic 7. The characteristics method for non-smooth data

As a second step, we want to generalize the construction of solutions to a wider class of initial data. For $p \in [1,\infty)$, we observe that, at least formally, the following computation holds for a given (positive) solution f of the transport equation (2.1):

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}^d} f^p\, dx &= \int_{\mathbb{R}^d} \partial_t f^p\, dx = \int_{\mathbb{R}^d} p f^{p-1}\, \partial_t f\, dx \\ &= -\int_{\mathbb{R}^d} p f^{p-1}\, b \cdot \nabla_x f\, dx = -\int_{\mathbb{R}^d} b \cdot \nabla_x f^p\, dx \\ &= \int_{\mathbb{R}^d} (\mathrm{div}_x b) f^p\, dx \le \|\mathrm{div}_x b\|_{L^\infty} \int_{\mathbb{R}^d} f^p\, dx. \end{aligned}$$

With the help of the Gronwall lemma, we learn from that differential inequality that the following (still formal) estimate holds

$$\|f(t)\|_{L^p} \le e^{Bt/p}\, \|f_0\|_{L^p} \quad \forall\, t \ge 0, \tag{3.1}$$

with $B := \|\mathrm{div}_x b\|_{L^\infty_{tx}}$.

We recall/accept the Liouville theorem which tells us that the Jacobian function $J := \det D\Phi_t(y)$ satisfies the ODE

$$\frac{d}{dt} J = (\mathrm{div} b(t,\Phi_t(y)))J, \quad J(0,y) = 1,$$

so that

$$\det D\Phi_t(y) = e^{\int_0^t (\operatorname{div} b(s,\Phi_s(y)))ds}. \tag{3.2}$$

Proposition 3.1. *Under the standard assumptions on the vector field b and the usual definition on the associated flow Φ_t, for any $f_0 \in L^p(\mathbb{R}^d)$, with $p \in [1,\infty)$, the function*

$$\bar f(t,x) := f_0(\Phi_{-t}(x)) \tag{3.3}$$

belongs to $C([0,T);L^p(\mathbb{R}^d))$, satisfies (3.1) and is a solution to the transport equation

$$\partial_t f + b\cdot\nabla_x f = 0, \quad f(0) = f_0,$$

in the distributional sense.

Proof of Proposition 3.1. Thanks to the dominated convergence theorem of Lebesgue, we clearly have the continuity property. On the other hand, we compute

$$\begin{aligned}\int_{\mathbb{R}^d} |\bar f(t,x)|^p dx &= \int_{\mathbb{R}^d} |f_0(\Phi_{-t}(x))|^p dx \\ &= \int_{\mathbb{R}^d} |f_0(y)|^p e^{\int_0^t (\operatorname{div} b(s,\Phi_s(y)))ds} dy \\ &\le e^{Bt} \int_{\mathbb{R}^d} |f_0(y)|^p dy,\end{aligned}$$

where we have used the Liouville theorem (3.2) in the second line. The estimate (3.1) follows. Finally, from the above definition and the group property of the flow, for a.e. $y \in \mathbb{R}^d$ and for any $t \in (0,\infty)$, we observe that

$$\bar f(t+s,\Phi_s(y)) = \bar f(t,y), \quad \forall\, s \ge 0. \tag{3.4}$$

Let us then fix $\varphi \in \mathcal{D}((0,T)\times\mathbb{R}^d)$. We compute

$$\begin{aligned} 0 &= \frac{d}{ds}\int_0^T\int_{\mathbb{R}^d} \bar f(t,y)\varphi(t,y)\,dydt \\ &= \frac{d}{ds}\int_0^T\int_{\mathbb{R}^d} \bar f(t+s,\Phi_s(y))\varphi(t,y)\,dydt \\ &= \frac{d}{ds}\int_0^T\int_{\mathbb{R}^d} \bar f(t,x)\varphi(t-s,\Phi_{-s}(x))e^{-\int_0^s (\operatorname{div} b)(\Phi_\tau(x))d\tau}\,dxdt \\ &= \int_0^T\int_{\mathbb{R}^d} \bar f(t,x)\frac{d}{ds}[\varphi(t-s,\Phi_{-s}(x))e^{-\int_0^s \operatorname{div} b(\Phi_\tau(x))ds}]\,dxdt \\ &= \int_0^T\int_{\mathbb{R}^d} \bar f(t,x)[-\partial_t\varphi - b\cdot\nabla\varphi - \operatorname{div} b\varphi](t-s,\Phi_{-s}(x))e^{-\int_0^s \operatorname{div} a(\Phi_\tau(x))ds}\,dxdt,\end{aligned}$$

where we have used the relation (3.4) in the second line and the change of variables $x = \Phi_s(y)$ together with the Liouville theorem (3.2) in the third line. Taking $s = 0$, we get

$$0 = \int_0^T\int_{\mathbb{R}^d} \bar f(t,x)[-\partial_t\varphi - a\cdot\nabla\varphi - (\operatorname{div} a)\varphi](t,x)\,dxdt,$$

which exactly means that $\bar f$ is a solution to equation (3.3) in the distributional sense. □

Exercise 3.2. *Prove that* (3.3) *does not depend of the choice of the function* $f_0 \in \mathcal{L}^p(\mathbb{R}^d)$ *in the class* $\{f_0\} \in L^p(\mathbb{R}^d)$. *(Hint. Take* $g_0 \in \{f_0\}$ *and compute* $\|f_0 \circ \Phi_{-t} - g_0 \circ \Phi_{-t}\|_{L^p}$*).*

Exercise 3.3. *(1) For any matrix* $B \in M_d(\mathbb{R})$ *and* $h \in \mathbb{R}$, *prove that*

$$\det(I + hB) = 1 + h\,\mathrm{tr}B + \mathcal{O}(h^2).$$

(2) Consider $A, B \in C^1((0,T); M_d(\mathbb{R}))$ *which satisfy*

$$\frac{d}{dt}A(t) = B(t)A(t),$$

and prove that

$$\frac{d}{dt}(\det A(t)) = (\mathrm{tr}\,B(t))(\det A(t)).$$

(3) Establish the Liouville theorem (3.2).

Exercise 3.4. *Consider the dampted free transport equation with source term*

$$\left\{ \begin{array}{l} \partial_t f + v \cdot \nabla_x + f = G \\ \quad f(0,\cdot) = f_0, \end{array} \right. \tag{3.5}$$

where $f = f(t,x,v)$, $t \geq 0$, $x, v \in \mathbb{R}^d$, $f_0 \in L^1(\mathbb{R}^{2d})$ *and* $G \in L^1((0,T) \times \mathbb{R}^{2d})$. *Establish that*

$$f(t,x,v) := f_0(x - vt, v)e^{-t} + \int_0^t G(s, x + (s-t)v, v)e^{s-t}ds \tag{3.6}$$

belongs to $C([0,T]; L^1(\mathbb{R}^{2d}))$ *and provides a weak solution.*

Exercise 3.5. *Consider the transport equation with boundary condition*

$$\left\{ \begin{array}{l} \partial_t f + \partial_x f + af = 0 \\ f(t,0) = b(t), \quad f(0,x) = f_0(x), \end{array} \right. \tag{3.7}$$

where $f = f(t,x)$, $t \geq 0$, $x \geq 0$, $a \in L^\infty(\mathbb{R}_+)$, $f_0 \in L^1(\mathbb{R}_+)$ *and* $b \in L^1([0,T])$.
(a) Establish the a priori estimate

$$\sup_{[0,T]} \|f(t,\cdot)\|_{L^1} \ \leq (\|b\|_{L^1(0,T)} + \|f_0\|_{L^1})e^{t\|a\|_{L^\infty}}, \quad \forall\, t \geq 0.$$

(Hint. Use the Gronwall lemma).
(b) Establish the existence of a weak solution $f \in C([0,T]; L^1(\mathbb{R}_+))$.

4. Topic 8. Duhamel formula and perturbation argument (bis)

In this section, we explain how the previous analysis and a perturbation argument make possible to tackle some more general evolution equations. We consider the relaxation equation

$$\partial_t f + v \cdot \nabla_x f = \rho_f \mathscr{M} - f \quad \text{in} \quad (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d, \tag{4.1}$$

where we define

$$\rho_f := \int f dv, \quad \mathscr{M} := \frac{1}{(2\pi)^{d/2}} e^{-|v|^2/2}.$$

We complement that equation with an initial datum $f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d)$. We denote by (S_t) the semigroup defined by

$$(S_t f_0)(x,v) := f_0(x - vt, v)e^{-t}$$

and associated to the evolution equation

$$\partial_t f + v \cdot \nabla_x f = -f \quad \text{in} \quad (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

Because of the variation of parameters formula (3.6), we may look for a function $f \in \mathcal{X} := C([0,T]; L^1(\mathbb{R}^{2d}))$, $T > 0$, which satisfies the equation in the mild sense

$$f_t = S_t f_0 + \int_0^t S_{t-s}[\rho_{f_s} \mathscr{M}] ds$$

(that is the Duhamel formula again). This one will automatically satisfies (4.1) as a consequence of Exercise 3.4. For a given function $g \in \mathcal{X}$, we define

$$h_t := S_t f_0 + \int_0^t S_{t-s}[\rho_{g_s} \mathscr{M}] ds, \quad \forall\, t \in (0,T),$$

and we denote $g \mapsto \mathcal{U} g := h$ this mapping. We aim to prove that $\mathcal{U} : \mathcal{X} \to \mathcal{X}$ and that there exists a unique fixed point $f \in \mathcal{X}$ such that $f = \mathcal{U} f$.

On the one hand, because $\rho_g \mathscr{M} \in L^1((0,T) \times \mathbb{R}^{2d})$ and Exercise 3.4, we have $h \in \mathcal{X}$. Now, for $g_1, g_2 \in \mathcal{X}$ and denoting $h_1 := \mathcal{U} g_1$, $h_2 := \mathcal{U} g_2$, $h := h_2 - h_1$, $g := g_2 - g_1$, we have

$$h_t \;=\; \int_0^t S_{t-s}[\rho_{g_s} \mathscr{M}] ds.$$

We compute

$$\begin{aligned} \|h_t\|_{L^1} &\leq \int_0^t \|S_{t-s}[\rho_{g_s} \mathscr{M}]\|_{L^1} ds \\ &\leq \int_0^t e^{-(t-s)} \|\rho_{g_s} \mathscr{M}\|_{L^1} ds \\ &\leq \int_0^t e^{-(t-s)} \|g_s\|_{L^1} \|\mathscr{M}\|_{L^1} ds \\ &\leq (1 - e^{-T}) \sup_{[0,T]} \|g_s\|_{L^1}, \end{aligned}$$

so that

$$\|h\|_{\mathcal{X}} \leq \alpha_T \|g\|_{\mathcal{X}}, \quad \alpha_T := 1 - e^{-T}.$$

For any $T > 0$, we have $\alpha_T < 1$, and the Banach fixed point theorem tells us that there exists a unique fixed point $f \in \mathcal{X}$ to the mapping $\mathcal{U}$.

Exercise 4.1. *Consider the renewal equation*

$$\left\{ \begin{array}{l} \partial_t f + \partial_x f + a f = 0 \\ f(t,0) = \rho_{f(t)}, \quad f(0,x) = f_0(x), \end{array} \right. \tag{4.2}$$

where $f = f(t,x)$, $t \geq 0$, $x \geq 0$, and

$$\rho_g := \int_0^\infty g(y)\, a(y)\, dy.$$

Assume $a \in L^\infty(\mathbb{R}_+)$ and $f_0 \in L^1(\mathbb{R}_+)$. Establish that there exists a unique mild solution $f \in C([0,T]; L^1(\mathbb{R}_+))$ to equation (4.2).