CHAPTER 3 - POSITIVE SEMIGROUP AND LONGTIME BEHAVIOUR

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Contents

In this chapter we make a brief presentation of the semigroup theory. We then concentrate on a particular family of positive semigroup for which an accurate analysis of the longtime asymptotic behaviour can be performed.

1. Semigroup, linear evolution equation and generator

1.1. Semigroup.

We state the definition of a continuous semigroup of linear and bounded operators.

Definition 1.1. *We say that $(S_t)_{t\ge 0}$ is a continuous semigroup of linear and bounded operators on a Banach space X, or we just say that S_t is C_0-semigroup (or a semigroup) on X, we also write $S(t) = S_t$, if the following conditions are fulfilled:*

(i) one parameter family of operators: $\forall t \ge 0$, $f \mapsto S_t f$ is linear and continuous on X;

(ii) continuity of trajectories: $\forall f \in X$, $t \mapsto S_t\, f \in C([0,\infty), X)$;

(iii) semigroup property: $S_0 = I$; $\quad \forall\, s, t \ge 0$, $S_{t+s} = S_t\, S_s$;

(iv) growth estimate: $\exists\, b \in \mathbb{R}$, $\exists\, M \ge 1$,

$$\|S_t\|_{\mathscr{B}(X)} \le M\, e^{bt} \quad \forall\, t \ge 0. \tag{1.1}$$

We then define the growth bound $\omega(S)$ by

$$\omega(S) := \limsup_{t\to\infty} \frac{1}{t} \log \|S(t)\| = \inf\{b \in \mathbb{R};\ (1.1) \text{ holds}\}.$$

We say that (S_t) is a semigroup of contractions if (1.1) holds with $b = 0$ and $M = 1$.

Remark 1.2. *The two continuity properties (i) and (ii) can be understood in the same sense of*
(a) - the strong topology of X, and we will say that S_t is a strongly continuous semigroup;
(b) - the weak $*$ topology $\sigma(X,Y)$ with $X = Y'$, Y a (separable) Banach space, and we will say that S_t is a weakly $*$ continuous semigroup.
Classical examples are the heat semigroup and the translation semigroup

$$S_t f = \gamma_t * f, \quad \gamma_t = (2\pi t)^{-1/2}\, e^{-\frac{|x|^2}{2t}}, \quad \text{and} \quad (S_t f)(x) = f(x - at),$$

in the Lebesgue space $L^p(\mathbb{R})$, $1 \le p \le \infty$, and in the space $M^1(\mathbb{R}) := (C_0(\mathbb{R}))'$ of bounded Radon measures. For $1 \le p < \infty$, the above semigroups are strongly continuous in $L^p(\mathbb{R})$. They are only weakly $*$ continuous in the spaces $L^\infty(\mathbb{R})$ and $M^1(\mathbb{R})$.

1.2. **Linear evolution equation and semigroup.** Given a linear operator Λ acting on a Banach space X (or on a subspace of X) and a initial datum g_0 belonging to X (or to a subspace of X), we consider the (abstract) linear evolution equation

$$\frac{d}{dt} g = \Lambda g \quad \text{in} \quad (0,\infty), \qquad g(0) = g_0. \tag{1.2}$$

We may associate a C_0-semigroup to the evolution equation as a mere consequence of the linearity of the equation and of the existence and uniqueness result.

Definition 1.3. *We say that the evolution equation* (1.2) *is well-posed if there exists a space* $\mathscr{E}_\infty \subset C(\mathbb{R}_+; X)$ *such that for any* $g_0 \in X$*, there exists a unique function* $g \in \mathscr{E}_\infty$ *which satisfies* (1.2) *(possibly in a weak sense), and for any* $R_0, T > 0$ *there exists* $R_T := C(T, R_0) > 0$ *such that*

$$\|g_0\|_X \le R_0 \quad \textit{implies} \quad \sup_{[0,T]} \|g(t)\|_X \le R_T. \tag{1.3}$$

Proposition 1.4. *To an evolution equation* (1.2) *which is well-posed in the sense of Definition 1.3, we may associate a continuous semigroup of linear and bounded operators* (S_t) *in the following way. For any* $g_0 \in X$ *and any* $t \ge 0$*, we set* $S(t) g_0 := g(t)$*, where* $g \in \mathscr{E}_\infty$ *is the unique weak solution to the evolution equation* (1.2) *with initial datum* g_0*.*

Corollary 1.5. *To the time autonomous parabolic equation considered in the previous chapters, we can associate a strongly continuous semigroup of linear and bounded operators.*

1.3. **Semigroup and generator.** On the other way round, in this section, starting from a given semigroup, we explain how we can associate a generator and then a solution to a differential linear equation.

Definition 1.1. *An unbounded operator* Λ *on* X *is a linear mapping defined on a linear submanifold called the domain of* Λ *and denoted by* $D(\Lambda)$ *or* $dom(\Lambda) \subset X$; $\Lambda : D(\Lambda) \to X$*. The graph of* Λ *is*

$$G(\Lambda) = \mathit{graph}(\Lambda) := \{(f, \Lambda f);\ f \in D(\Lambda)\} \subset X \times X.$$

We say that Λ *is closed if the graph* $G(\Lambda)$ *is a closed set in* $X \times X$*: for any sequence* (f_k) *such that* $f_k \in D(\Lambda)$, $\forall\, k \ge 0$, $f_k \to f$ *in* X *and* $\Lambda f_k \to g$ *in* X *then* $f \in D(\Lambda)$ *and* $g = \Lambda f$*. We denote* $\mathscr{C}(X)$ *the set of unbounded operators with closed graph and* $\mathscr{C}_D(X)$ *the set of unbounded operators which domain is dense and graph is closed.*

Definition 1.2. *For a given semigroup* (S_t) *on* X*, we define*

$$\begin{aligned} D(\Lambda) &:= \Big\{ f \in X;\ \lim_{t \searrow 0} \frac{S(t)\, f - f}{t} \ \textit{exists in}\ X \Big\}, \\ \Lambda f &:= \lim_{t \searrow 0} \frac{S(t)\, f - f}{t} \quad \textit{for any} \ \ f \in D(\Lambda). \end{aligned}$$

Clearly $D(\Lambda)$ *is a linear submanifold and* Λ *is linear:* Λ *is an unbounded operator on* X*. We call* $\Lambda : D(\Lambda) \to X$ *the (infinitesimal) generator of the semigroup* (S_t)*, and we sometimes write* $S_t = S_\Lambda(t)$*. We denote* $\mathscr{G}(X)$ *the set of operators which are the generator of a semigroup.*

We present some fundamental properties of a semigroup S and its generator Λ that one can obtain by simple differential calculus arguments from the very definitions of S and Λ.

Proposition 1.3. *(Differentiability property of a semigroup). Let* $f \in D(\Lambda)$*.*

(i) $S(t) f \in D(\Lambda)$ *and* $\Lambda\, S(t) f = S(t)\, \Lambda f$ *for any* $t \ge 0$*, so that the mapping* $t \mapsto S(t)\, f$ *is* $C([0,\infty); D(\Lambda))$*.*

(ii) The mapping $t \mapsto S(t) f$ *is* $C^1([0,\infty); X)$, $\frac{d}{dt} S(t) f = \Lambda S(t) f$ *for any* $t > 0$*, and then*

$$S(t) f - S(s) f = \int_s^t S(\tau)\, \Lambda f \, d\tau = \int_s^t \Lambda\, S(\tau) f \, d\tau, \qquad \forall\, t > s \ge 0.$$

Sketch of the proof of Proposition 1.3. Let $f \in D(\Lambda)$.
Proof of (i). We fix $t \ge 0$ and we compute

$$\lim_{s\to 0^+} \frac{S(s)S(t)f - S(t)f}{s} = \lim_{s\to 0^+} S(t)\frac{S(s)f - f}{s} = S(t)\Lambda f,$$

which implies $S(t)f \in D(\Lambda)$ and $\Lambda S(t)f = S(t)\Lambda f$.
Proof of (ii). We fix $t > 0$ and we compute (now) the left differential

$$\begin{aligned}&\lim_{s\to 0^-} \Big\{ \frac{S(t+s)f - S(t)f}{s} - S(t)\Lambda f \Big\} = \\ &= \lim_{s\to 0^-} \Big\{ S(t+s)\Big(\frac{S(-s)f - f}{-s} - \Lambda f\Big) + \Big(S(t+s)\Lambda f - S(t)\Lambda f\Big) \Big\} = 0,\end{aligned}$$

using that the two terms within parenthesis converge to 0 and that $\|S(t+s)\| \le M\, e^{\omega t}$ for any $s \le 0$. Together with step 1, we deduce that $t \mapsto S(t)f$ is differentiable for any $t > 0$, with derivative $\Lambda S(t)f$. We conclude to the C^1 regularity by observing that $t \mapsto S(t)\Lambda f$ is continuous. Last, we have

$$S(t)f - S(s)f = \int_s^t \frac{d}{d\tau}[S(\tau)\, f]\, d\tau = \int_s^t S(\tau)\,\Lambda f\, d\tau = \int_s^t \Lambda\, S(\tau) f\, d\tau$$

and in particular

$$\|S(t)f - S(s)f\| \le (t-s)\, M\, e^{bt}\, \|\Lambda f\|,$$

for any $t > s \ge 0$. ☐

Exercise 1.4. *For $h \in E_T := C([0,T]; D(\Lambda)) \cap C^1([0,T]; X)$ prove that $S_\Lambda h \in E_T$ and*

$$\frac{d}{dt}[S_\Lambda(t)h(t)] = S_\Lambda(t)\Lambda h(t) + S_\Lambda(t)h'(t).$$

(Hint. Write

$$\begin{aligned}\frac{S_\Lambda(t+s)h(t+s) - S_\Lambda(t)h(t)}{s} &= \frac{S_\Lambda(t+s) - S_\Lambda(t)}{s} h(t) + S_\Lambda(t+s)h'(t) \\ &\quad + S_\Lambda(t+s)\Big(\frac{h(t+s) - h(t)}{s} - h'(t)\Big)\end{aligned}$$

and pass to the limit $s \to 0$)

Definition 1.5. *Consider a Banach space X and an (unbounded) operator Λ on X. We say that $g \in C([0,\infty); X)$ is a "classical" (or Hille-Yosida) solution to the evolution equation* (1.2) *if $g \in C((0,\infty); D(\Lambda)) \cap C^1((0,\infty); X)$ so that* (1.2) *holds pointwise.*

In it worth emphasizing that Proposition 1.3 provides a "classical" solution to the evolution equation (1.2) for any initial datum $f_0 \in D(\Lambda)$ by the mean of $t \mapsto S_\Lambda(t) f_0$.

Lemma 1.6. *For any $f \in X$ and $t \ge 0$, there hold*

$$(i) \qquad \lim_{h\to 0} \frac{1}{h} \int_t^{t+h} S(s)\, f\, ds = S(t)f,$$

and

$$(ii) \quad \int_0^t S(s) f\, ds \in D(\Lambda), \qquad (iii) \quad \Lambda\Big(\int_0^t S(s) f\, ds\Big) = S(t)f - f.$$

Sketch of the proof of Lemma 1.6. The first point is just a consequence of the fact that $s \mapsto S(s)f$ is a continuous function. We then deduce

$$\begin{aligned}&\frac{1}{h}\Big\{S(h)\int_0^t S(s) f\, ds - \int_0^t S(s) f\, ds\Big) = \frac{1}{h}\Big\{\int_h^{t+h} S(s)\, f\, ds - \int_0^t S(s) f\, ds\Big\} \\ &= \frac{1}{h}\Big\{\int_t^{t+h} S(s)\, f\, ds - \int_0^h S(s) f\, ds\Big\} \underset{h\to 0}{\longrightarrow} S(t)f - f,\end{aligned}$$

which implies the two last points. ☐

In the next result we prove that $\mathscr{G}(X) \subset \mathscr{C}_D(X)$.

Definition 1.7. *We say that $\mathcal{C} \subset X$ is a core for the generator Λ of a semigroup S if*

$$\mathcal{C} \subset D(\Lambda), \quad \mathcal{C} \text{ is dense in } X \quad \text{and} \quad S(t)\,\mathcal{C} \subset \mathcal{C},\ \forall\, t \geq 0.$$

Proposition 1.8. *(Properties of the generator) Let $\Lambda \in \mathscr{G}(X)$.*
(i) The domain $D(\Lambda)$ is dense in X. In particular, $D(\Lambda)$ is a core.
(ii) Λ is a closed operator.
(iii) The mapping which associates to a semigroup its generator is injective. More precisely, if S_1 and S_2 are two semigroups with generators Λ_1 and Λ_2 and there exists a core $\mathcal{C} \subset D(\Lambda_1) \cap D(\Lambda_2)$ such that $\Lambda_{1|\mathcal{C}} = \Lambda_{2|\mathcal{C}}$, then $S_1 = S_2$. In other words, $S_1 \neq S_2$ implies $\Lambda_1 \neq \Lambda_2$.

Sketch of the proof of Proposition 1.8. For any $f \in X$ and $t > 0$, we define $f^t := t^{-1} \int_0^t S(s) f\, ds$. Thanks to Lemma 1.6-(i) & (ii), we see that $f^t \in D(\Lambda)$ and $f^t \to f$ as $t \to 0$. In other words, $D(\Lambda)$ is dense in X.
We prove (ii). Consider a sequence (f_k) of $D(\Lambda)$ such that $f_k \to f$ and $\Lambda f_k \to g$ in X. For $t > 0$, we write

$$S(t) f_k - f_k = \int_0^t S(s) \Lambda f_k\, ds,$$

and passing to the limit $k \to \infty$, we get

$$t^{-1}(S(t) f - f) = t^{-1} \int_0^t S(s) g\, ds.$$

We may now pass to the limit $t \to 0$ in the RHS term, and we obtain

$$\lim_{t \to 0} (S(t) f - f)/t = g.$$

That proves $f \in D(\Lambda)$ and $\Lambda f = g$.
We prove (iii). We observe that the mapping $t \mapsto S_i(t) f$, $i = 1, 2$, are C^1 for any $f \in \mathcal{C}$, thanks to Proposition 1.3, and

$$\begin{aligned} \frac{d}{ds} S_1(s)\, S_2(t-s) f &= \frac{dS_1(s)}{ds}\, S_2(t-s)\, f + S_1(s)\, \frac{dS_2(t-s)}{ds}\, f \\ &= S_1(s)\, \Lambda_1\, S_2(t-s)\, f - S_1(s)\, \Lambda_2\, S_2(t-s)\, f = 0. \end{aligned}$$

That implies $S_2(t) f = S_1(0) S_2(t-0) f = S_1(t) S_2(t-t) f = S_1(t)\, f$ for any $f \in \mathcal{C}$, and then $S_2 \equiv S_1$. □

1.4. **The Hille-Yosida-Lumer-Phillips' existence theory.** In a Hilbert space $X = H$, we say that a unbounded linear operator $\Lambda : D(\Lambda) \subset H \to H$ is dissipative if

$$\forall\, f \in D(\Lambda), \quad (\Lambda f, f)_H \leq 0.$$

Lemma 1.9. *Consider a semigroup S_Λ on a Hilbert space H. There is equivalence between*
(i) S_Λ is a semigroup of contractions;
(ii) Λ is dissipative.

Proof of Lemma 1.9. We take $f_0 \in D(\Lambda)$, we define $\mathcal{E}(t) := \|S_\Lambda(t) f_0\|_H^2$ which is a C^1 function and we compute

$$\frac{d}{dt} \mathcal{E}(t) = 2(\Lambda S_\Lambda(t) f_0, S_\Lambda(t) f_0)_H.$$

The statement just says that $\mathcal{E}$ is decreasing if and only if $\mathcal{E}'$ is nonpositive. □

We say that an (unbounded) operator Λ is maximal if there exists $x_0 > 0$ such that

$$R(x_0 - \Lambda) = X. \tag{1.1}$$

We say that Λ is m-dissipative if Λ is dissipative and maximal.

We present now the Lumer-Phillips' version of the Hille-Yosida Theorem which establishes the link between semigroup of contractions and dissipative operator.

Theorem 1.10 (Hille-Yosida, Lumer-Phillips). *Consider $\Lambda \in \mathscr{C}_D(X)$. The two following assertions are equivalent:*
(a) Λ is the generator of a semigroup of contractions;
(b) Λ is dissipative and maximal.

Elements of proof of Theorem 1.10. We just give the proof of the easy part *(a) implies (b)*, but not of the hard part *(b) implies (a)*. We assume (a). From the above discussion, we only have to prove that Λ is maximal. From Lemma 1.6-(iii) applied to $S := S_\Lambda(t)e^{-x_0 t}$, $x_0 > 0$, we have

$$(x_0 - \Lambda)\left(\int_0^t S_\Lambda(s)e^{-x_0 s} f\,ds\right) = f - S_\Lambda(t)e^{-x_0 t},$$

for any $f \in X$ and $t > 0$. Because $\|S_\Lambda(s)e^{-x_0\tau} f\| \le e^{-x_0\tau}\|f\|$, we may pass to the limit $t \to \infty$ in both sides of the equation, and we get

$$(x_0 - \Lambda)g = f, \quad g := \int_0^\infty S_\Lambda(s)e^{-x_0 s} f\,ds \in D(\Lambda),$$

what is nothing but the maximality property for Λ. Notice that the property $g \in D(\Lambda)$ comes from the hypothesis that Λ is closed. □

2. Duhamel formula and mild solution

Consider the evolution equation

$$\frac{d}{dt}g = \Lambda g + G \text{ on } (0,T), \quad g(0) = g_0, \tag{2.1}$$

for an unbounded operator Λ on X, an initial datum $g_0 \in X$ and a source term $G : (0,T) \to X$, $T \in (0,\infty)$. For $G \in C((0,T);X)$, a classical solution g is a function

$$g \in X_T := C([0,T);X) \cap C^1((0,T);X) \cap C((0,T);D(\Lambda)) \tag{2.2}$$

which satisfies (2.1) pointwise. For $U \in L^1(0,T;\mathscr{B}(\mathcal{X}_1,\mathcal{X}_2))$ and $V \in L^1(0,T;\mathscr{B}(\mathcal{X}_2,\mathcal{X}_3))$, we define the time convolution $V * U \in L^1(0,T;\mathscr{B}(\mathcal{X}_1,\mathcal{X}_3))$ by setting

$$(V * U)(t) := \int_0^t V(t-s)\,U(s)\,ds = \int_0^t V(s)\,U(t-s)\,ds, \quad \text{for a.e. } t \in (0,T).$$

Lemma 2.1 (Variation of parameters formula). *Consider the generator Λ of a semigroup S_Λ on X. For $G \in C((0,T);X) \cap L^1(0,T;X)$, $\forall T > 0$, there exists at most one classical solution $g \in X_T$ to* (2.1) *and this one is given by*

$$g = S_\Lambda g_0 + S_\Lambda * G. \tag{2.3}$$

Proof of Lemma 2.1. Assume that $g \in X_T$ satisfies (2.1). For any fixed $t \in (0,T)$, we define $s \mapsto u(s) := S_\Lambda(t-s)g(s) \in C^1((0,t);X) \cap C([0,t];X)$. On the one hand, we compute

$$u'(s) = -\Lambda S_\Lambda(t-s)g(s) + S_\Lambda(t-s)g'(s) = S_\Lambda(t-s)G(s),$$

for any $s \in (0,t)$, so that $u' \in L^1(0,T;X)$. On the other hand, we have

$$g(t) - S_\Lambda(t)g_0 = u(t) - u(0) = \int_0^t u'(s)\,ds.$$

We conclude by putting together the two identities. □

When $G \in C((0,T);X) \cap L^1(0,T;D(\Lambda))$ and $g_0 \in D(\Lambda)$, we observe that $\bar g := S_\Lambda g_0 + S_\Lambda * G$ belongs to X_T and

$$\frac{d}{dt}\bar g(t) = \Lambda S_\Lambda(t)g_0 + \Lambda(S_\Lambda * G)(t) + S_\Lambda(0)G(t) = \Lambda \bar g(t) + G(t),$$

so that $\bar g$ is a classical solution to the evolution equation (2.1).

When $G \in L^1(0,T;X)$ and $g_0 \in X$, we observe that $\bar g \in C([0,T];X)$, $\bar g(0) = g_0$ and it is the limit of classical solutions by a density argument. We say that $\bar g$ is a mild solution to the evolution equation (2.1).

Lemma 2.2 (Duhamel formula)**.** *Consider two semigroups S_Λ and $S_\mathcal{B}$ on the same Banach space X, assume that $D(\Lambda) = D(\mathcal{B})$ and define $\mathcal{A} := \Lambda - \mathcal{B}$. If $\mathcal{A}S_\mathcal{B}, S_\mathcal{B}\mathcal{A} \in L^1(0,T;\mathscr{B}(X))$ for any $T \in (0,\infty)$, then*

$$S_\Lambda = S_\mathcal{B} + S_\Lambda * \mathcal{A}S_\mathcal{B} = S_\mathcal{B} + S_\mathcal{B}\mathcal{A} * S_\Lambda \quad in \quad \mathscr{B}(X).$$

Proof of Lemma 2.1. Take $f \in D(\Lambda) = D(\mathcal{B})$, $t > 0$, and define $s \mapsto u(s) := S_\Lambda(s)S_\mathcal{B}(t-s)f \in C^1([0,t];X) \cap C([0,t];D(\Lambda))$. We observe that

$$\begin{aligned} u'(s) &= S_\Lambda(s)\Lambda S_\mathcal{B}(t-s)f - S_\Lambda(s)\mathcal{B}S_\mathcal{B}(t-s)f \\ &= S_\Lambda(t-s)\mathcal{A}S_\mathcal{B}(s)f, \end{aligned}$$

for any $s \in (0,t)$, from which we deduce

$$S_\Lambda(t)f - S_\mathcal{B}(t)f = \int_0^t u'(s)\,ds = \int_0^t S_\Lambda(t-s)\mathcal{A}S_\mathcal{B}(s)f\,ds.$$

By density and continuity, we deduce that the same holds for any $f \in X$, and that establishes the first version of the Duhamel formula. The second version follows by reversing the role of S_Λ and $S_\mathcal{B}$. □

Assume as in Lemma 2.2 that Λ splits as $\Lambda = \mathcal{A} + \mathcal{B}$. From the above second version of Duhamel formula, we observe that for any $g_0 \in D(\Lambda)$, the function $\bar g(t) := S_\Lambda(t)g_0 \in X_T$ is a classical solution to the evolution equation (1.2) and satisfies the following functional equation

$$g = S_\mathcal{B} g_0 + S_\mathcal{B}\mathcal{A} * g. \tag{2.4}$$

On the other way round, we observe that if $g \in X_T$ is a solution to the functional equation (2.4), then

$$\begin{aligned} g'(t) &= \mathcal{B}S_\mathcal{B}(t)g_0 + \mathcal{B}(S_\mathcal{B}\mathcal{A} * g)(t) + S_\mathcal{B}(0)\mathcal{A}g(t) \\ &= \mathcal{B}g(t) + \mathcal{A}g(t) = \Lambda g(t), \end{aligned}$$

so that g is a classical solution to the evolution equation (1.2). [Here we need $S_\mathcal{B}\mathcal{A} * g \in D(\Lambda)$ or define the object by duality, see below]. More generally, when $S_\mathcal{B}\mathcal{A} \in L^1(0,T;\mathscr{B}(X))$, we say that $g \in C([0,T];X)$ is a mild solution to the evolution equation (1.2) if g is a solution to the functional equation (2.4).

3. Dual semigroup and weak solution

Consider a Banach space X and an operator $A \in \mathscr{C}_D(X)$, with X endowed with the topology norm, and we denote $Y = X'$ in that case, or with $X = Y'$ endowed with the weak $*$ topology $\sigma(X,Y)$ for a separable Banach space Y. We define the subspace

$$D(A^*) := \big\{\varphi \in Y;\ \exists C \ge 0,\ \forall f \in D(A),\ |\langle \varphi, Af\rangle| \le C\,\|f\|_X\big\}$$

and next the adjoint operator A^* on Y by

$$\langle A^*\varphi, f\rangle = \langle \varphi, Af\rangle, \quad \forall \varphi \in D(A^*),\ f \in D(A).$$

Because $D(A) \subset X$ is dense, the operator A^* is well and uniquely defined and it is obviously linear. Because A has a closed graph, the operator A^* has also a closed graph. When A is a bounded operator, then A^* is also a bounded operator. When X is reflexive, then the domain $D(A^*)$ is always dense into X', so that $A^* \in \mathscr{C}_D(X')$. For a general Banach space X and a general operator A, then $D(A^*)$ is dense into X' for the weak $*\sigma(X',X)$ topology, but it happens that $D(A^*)$ is not dense into X' for the strong topology.

Consider now a semigroup S with generator Λ and $f_0 \in D(\Lambda)$. Multiplying by $\varphi \in C^1_c([0,T);D(\Lambda^*))$ the equation (1.2) satisfied by $g(t) := S(t)f_0$ and integrating in time, we get

$$\langle f_0, \varphi(0)\rangle_{X,X'} + \int_0^T \langle S(t)f_0, \varphi'(t) + \Lambda^*\varphi(t)\rangle_{X,X'}\,dt = 0.$$

Because the mapping $f_0 \mapsto S(t)f_0$ is continuous in X and the inclusion $D(\Lambda) \subset X$ is dense from Proposition 1.8, we see that the above formula is also true for any $f_0 \in X$. In other words, the semigroup $S(t)$ provides a weak solution (in the above sense) to the evolution equation (1.2) for any $f_0 \in X$.

We aim to show now that the semigroup theory provides an answer to the well-posedness issue of weak solutions to that equation for any generator Λ. More precisely, given a semigroup, we introduce its dual semigroup and we then establish that the initial semigroup provides the unique weak solution to the associated homogeneous and inhomogeneous evolution equations.

Proposition 3.1. *Consider a strongly continuous semigroup $S = S_\Lambda$ on a Banach space X with generator Λ and the dual semigroup S^* as the one-parameter family $S^*(t) := S(t)^*$ for any $t \ge 0$. Then the following hold:*
(1) S^* is a weakly $*$ continuous semigroup on X' with same growth bound as S.
(2) The generator of S^* is Λ^*. In other words, $(S_\Lambda)^* = S_{\Lambda^*}$.
(3) The mapping $t \mapsto S^*(t)\varphi$ is $C([0,\infty); X')$ (for the strong topology) for any $\varphi \in D(\Lambda^*)$. Similarly, $t \mapsto S^*(t)\varphi$ is $C^1([0,\infty); X') \cap C([0,\infty); D(\Lambda^*))$ for any $\varphi \in D(\Lambda^{*2})$.

Proof of Proposition 3.1. (1) We just write

$$\langle S^*(t)\varphi, f\rangle = \langle \varphi, S(t) f\rangle =: T_f(t,\varphi) \quad \forall\, t \ge 0,\ f \in X,\ \varphi \in X',$$

and we see that $(t,\varphi) \mapsto T_f(t,\varphi)$ is continuous for any $f \in X$.
(2) Denoting by $D(L)$ and L the domain and generator of S^* as defined as in section 1.3, for any $\varphi \in D(L)$ and $f \in D(\Lambda)$ we have

$$\begin{aligned} \langle L\varphi, f\rangle &:= \lim_{t\to 0} \Big\langle \frac{1}{t}(S(t)^*\varphi - \varphi), f\Big\rangle \\ &= \lim_{t\to 0} \Big\langle \varphi, \frac{1}{t}(S(t) f - f)\Big\rangle = \langle \varphi, \Lambda f\rangle, \end{aligned}$$

from which we immediately deduce that $D(L) \subset D(\Lambda^*)$ and $L = \Lambda^*|_{D(L)}$. To conclude, we use that L is closed. More precisely, for a given $\varphi \in D(\Lambda^*)$, we associate the sequence (φ^ε) defined through

$$\varphi^\varepsilon := \frac{1}{\varepsilon}\int_0^\varepsilon S(t)^*\varphi\, dt.$$

We have $\varphi^\varepsilon \rightharpoonup \varphi$ in the weak $*\sigma(X',X)$ sense, $\varphi^\varepsilon \in D(L)$ and, for any $f \in D(\Lambda)$,

$$\begin{aligned} \langle L\varphi^\varepsilon, f\rangle &= \langle \Lambda^*\varphi^\varepsilon, f\rangle = \langle \varphi^\varepsilon, \Lambda f\rangle \\ &= \langle \varphi, \frac{1}{\varepsilon}\int_0^\varepsilon S(t)\Lambda f\, dt\rangle \to \langle \varphi, \Lambda f\rangle, \end{aligned}$$

so that $L\varphi^\varepsilon \rightharpoonup \Lambda^*\varphi$ in the weak $*\sigma(X',X)$ sense. The graph $G(L)$ of L being closed, we have $(\varphi, \Lambda^*\varphi) \in G(L)$, which in turns implies $\varphi \in D(L)$ and finally $L = \Lambda^*$.
(3) From Proposition 1.3, we have

$$\|S^*(t)\varphi - S^*(s)\varphi\|_{X'} = \Big\|\int_s^t S^*(\tau)\Lambda^*\varphi\, d\tau\Big\|_{X'} \le M e^{bt}(t-s)\|\Lambda^*\varphi\|_{X'}$$

for any $t > s \ge 0$ and $\varphi \in D(\Lambda^*)$, so that $t \mapsto S^*(t)\varphi$ is Lipschitz continuous from $[0,\infty)$ into X' endowed with the strong topology. ☐

Proposition 3.2. *Consider a weakly $*$ continuous semigroup $T = S_{\mathcal{L}}$ on a Banach space $X = Y'$ with generator $\mathcal{L}$, and the dual semigroup T^* as the one-parameter family $T^*(t) := T(t)^*$ of bounded operator on Y for any $t \ge 0$. Then the following hold:*
(1) $S = T^*$ is a strongly continuous semigroup on Y with same growth bound as T.
(2) The generator Λ of S satisfies $\mathcal{L} = \Lambda^*$.

Proof of Proposition 3.2. Just as in the proof of Proposition 3.1, we have $(t,f) \mapsto \langle \varphi, S(t) f\rangle$ is continuous for any $\varphi \in X'$. That means that $S(t)$ is a weakly $\sigma(X,X')$ continuous semigroup in X and therefore a strongly continuous semigroup in X thanks to Theorem **??**. The rest of the proof is unchanged with respect to the proof of Proposition 3.1. ☐

For any $g_0 \in X$ and $G \in L^1(0,T;X)$, we say that $g \in C([0,T];X)$ is a weak solution to the inhomogeneous initial value problem (2.1) if

$$\langle \varphi(T), g(T)\rangle - \langle \varphi(0), g_0\rangle = \int_0^T \big\{\langle \varphi' + \Lambda^*\varphi, g\rangle + \langle \varphi, G\rangle\big\}\, dt, \tag{3.1}$$

for any $\varphi \in C^1([0,T];X') \cap C([0,T];D(\Lambda^*))$.

Proposition 3.3. *Assume that Λ generates a semigroup S on X. For any $g_0 \in X$ and $G \in L^1(0,T;X)$, there exists a unique weak solution to equation (2.1), which is nothing but the mild solution*

$$\bar g = S_\Lambda g_0 + S_\Lambda * G. \tag{3.2}$$

Proof of Proposition 3.3. We define

$$\bar g(t) = \bar g_t := S(t) g_0 + \int_0^t S(t-s) G(s)\, ds \in C([0,T];X).$$

For any $\varphi = \varphi_t \in C^1([0,T];X') \cap C([0,T];D(\Lambda^*))$, we have

$$\langle \varphi_t, \bar g_t\rangle = \langle S^*_t \varphi_t, g_0\rangle + \int_0^t \langle S^*_{t-s}\varphi_t, G_s\rangle\, ds \in C^1([0,T])$$

and then

$$\begin{aligned} \frac{d}{dt}\langle \varphi_t, \bar g_t\rangle &= \langle S^*_t(\Lambda^*\varphi_t + \varphi_t'), g_0\rangle + \int_0^t \langle S^*_{t-s}(\Lambda^*\varphi_t + \varphi_t'), G_s\rangle\, ds + \langle G_t, \varphi_t\rangle \\ &= \langle \Lambda^*\varphi_t + \varphi_t', \bar g_t\rangle + \langle \varphi_t, G_t\rangle, \end{aligned}$$

from which we deduce that $\bar g$ is a weak solution to the inhomogeneous initial value problem (2.1) in the weak sense of equation (3.1). Now, if g is another weak solution, the function $f := g - \bar g$ is then a weak solution to the homogeneous initial value problem with vanishing initial datum, namely

$$\langle \varphi(T), f(T)\rangle = \int_0^T \langle \varphi' + \Lambda^*\varphi, f\rangle\, dt, \quad \forall \varphi \in C^1([0,T];X') \cap C([0,T];D(\Lambda^*)).$$

A first way to conclude is to define

$$\varphi(s) := \int_s^T S^*(\tau - s)\, \psi(\tau)\, d\tau,$$

for any given $\psi \in C^1_c((0,T);D(\Lambda^*))$, and to observe that $\varphi \in C^1([0,T];X') \cap C([0,T];D(\Lambda^*))$ is a (backward) solution to the dual problem

$$-\varphi' = \Lambda^*\varphi + \psi \ \text{ on } \ (0,T), \quad \varphi(T) = 0.$$

For that choice of test function, we get

$$0 = \int_0^T \langle \psi, f\rangle\, dt, \quad \forall \psi \in C^1_c((0,T);D(\Lambda^*)),$$

and thus $g = \bar g$.
An alternative way to get the uniqueness result is to define $\varphi(t) := S^*(T-t)\psi$ for a given $\psi \in D(\Lambda^*)$. Observing that φ is a (backward) solution to the dual problem

$$-\varphi' = \Lambda^*\varphi, \quad \varphi(T) = \psi, \tag{3.3}$$

that choice of test function leads to

$$\langle \psi, f(T)\rangle = 0 \qquad \forall \psi \in D(\Lambda^*),\ \forall T > 0,$$

and thus again $g = \bar g$. □

Exercise 3.4. *Consider a Banach space X and an unbounded operator Λ on X. We assume that $X = Y'$ for a Banach space Y and that the dual operator Λ^* generates a strongly continuous semigroup T on Y. Prove that $S := T^*$ is a (at least) weakly $*\,\sigma(X,Y)$ continuous semigroup on X with generator Λ and that it provides the unique weak solution to the associated evolution equation.*

Lemma 3.5. *Consider a semigroup $S = S_{\mathcal L}$ on a Banach space X. Consider a Banach space $Y \subset X'$ which is dense and assume that there exists a linear and bounded mapping $\Lambda : Y \to X'$ such that*

$$\langle S(t) f_0, \psi\rangle = \langle f_0, \psi\rangle + \int_0^t \langle S(\tau) f_0, \Lambda\psi\rangle\, d\tau,$$

for any $t \ge 0$, $f \in X$ and $\psi \in Y$. Then $Y \subset D(\mathcal L^*)$ and $\mathcal L = \bar\Lambda^*$.

Proof of Lemma 3.5. For $f_0 \in X$ and $\psi \in Y$, we write

$$\big\langle \frac{S(t) f_0 - f_0}{t}, \psi \big\rangle = \left\langle \frac{1}{t} \int_0^t S(\tau) f_0 d\tau, \Lambda \psi \right\rangle.$$

When furthermore $f_0 \in D(\mathcal{L})$, we may pass to the limit $t \to 0$, en we deduce

$$\langle \mathcal{L} f_0, \psi \rangle = \langle f_0, \Lambda \psi \rangle .$$

From the very definitions of $D(\mathcal{L}^*)$ and $\mathcal{L}^*$, we deduce that $Y \subset D(\mathcal{L}^*)$ and $\Lambda = \mathcal{L}^*_{|Y}$. □

4. A perturbation trick

We give a very efficient result for proving the existence of a semigroup associated to a generator which is a mild perturbation of the generator of a semigroup.

Theorem 4.1. *Consider $S_\mathcal{B}$ a semigroup satisfying the growth estimate $\|S_\mathcal{B}(t)\|_{\mathscr{B}(X)} \le M\, e^{bt}$ and $\mathcal{A}$ a bounded operator. Then, $\Lambda := \mathcal{A} + \mathcal{B}$ is the generator of a semigroup which satisfies the growth estimate $\|S_\Lambda(t)\|_{\mathscr{B}(X)} \le M\, e^{b' t}$, with $b' = b + M\|\mathcal{A}\|$.*

Proof of Theorem 4.1. Step 1. Existence. Take $g_0 \in X$. We fix $T > 0$ and for $T^* \in (0, T)$, we define

$$\mathscr{E} := C([0, T^*]; X), \quad \|g\|_\mathscr{E} := \sup_{t \in [0, T^*]} \|g(s)\|_X,$$

as well as for any $g \in \mathscr{E}$, the function

$$f(t) := S_\mathcal{B}(t) g_0 + (S_\mathcal{B} \mathcal{A} * g)(t).$$

We clearly have $f \in \mathscr{E}$, so that we have defined a mapping $\Phi : \mathscr{E} \to \mathscr{E}$, $g \mapsto \Phi(g) := f$. For two given functions $g_1, g_2 \in \mathscr{E}$, the associated images f_1, f_2 satisfy

$$\begin{aligned} \|f_2(t) - f_1(t)\|_X &= \Big\| \int_0^t S_\mathcal{B}(s) \mathcal{A}(g_2(t-s) - g_1(t-s))\, ds \Big\| \\ &\le \int_0^t M e^{bs} \|\mathcal{A}\| \|g_2 - g_1\|_\mathscr{E}\, ds, \end{aligned}$$

for any $t \in [0, T^*]$, so that

$$\|f_2 - f_1\|_\mathscr{E} \le T^* M e^{bT} \|\mathcal{A}\| \, \| \|g_2 - g_1\|_\mathscr{E}.$$

Choosing $T^* \in (0, T)$ small enough, in such a way that $T^* M e^{bT} \|\mathcal{A}\| < 1$, we see that Φ is then a contraction on $\mathscr{E}$. From the Banach fixed point theorem, there exists a unique fixed point to the mapping Φ. In other words, there exists $g \in \mathscr{E}$ such that

$$g(t) = S_\mathcal{B}(t) g_0 + (S_\mathcal{B} \mathcal{A} * g)(t), \quad \forall\, t \in [0, T^*]. \tag{4.1}$$

Furthermore, from (4.1), the continuous function $u_t := e^{-bt} \sup_{s \in [0,t]} \|g_s\|_X$ satisfies

$$u_t \le M \|g_0\|_X + M \|\mathcal{A}\| \int_0^t u_s\, ds,$$

and the Gronwall lemma implies $u_t \le M\, \|g_0\|_X\, e^{M\, \|\mathcal{A}\|\, t}$, so that

$$\|g_t\|_X \le M\, \|g_0\|_X e^{b' t}, \quad \forall\, t \in [0, T^*]. \tag{4.2}$$

Step 2. Weak solution. We fix $\varphi = \varphi_t \in C^1(\mathbb{R}_+; X) \cap C(\mathbb{R}_+; D(\mathcal{B}^*))$. Denoting $g_t := g(t)$, $S^*_t := S_{\mathcal{B}^*}(t)$, we define

$$\lambda(t) = \langle \varphi_t, g_t \rangle = \langle S^*_t \varphi_t, g_0 \rangle + \int_0^t \langle S^*_{t-s} \varphi_t, \mathcal{A} g_s \rangle\, ds.$$

We clearly have $\lambda \in C^1([0,T^*])$ and

$$\begin{aligned}\lambda'(t) &= \langle S_t^* \mathcal{B}^* \varphi_t + S_t^* \varphi_t', g_0 \rangle + \langle \varphi_t, \mathcal{A} g_t \rangle + \int_0^t \langle S_{t-s}^* \mathcal{B}^* \varphi_t + S_{t-s}^* \varphi_t', \mathcal{A} g_s \rangle \, ds. \\ &= \Big\langle \mathcal{B}^* \varphi_t + \varphi_t', \Big(S_\mathcal{B}(t) + \int_0^t S_{t-s} \mathcal{A} U_s \Big) g_0 \Big\rangle + \langle \mathcal{A}^* \varphi_t, g_t \rangle \\ &= \Big\langle \Lambda^* \varphi_t + \varphi_t', g_t \Big\rangle.\end{aligned}$$

By writing

$$\langle \varphi_t, g_t \rangle - \langle \varphi_0, g_0 \rangle = \int_0^t \lambda'(s) \, ds,$$

we conclude with

$$\langle \varphi_t, g_t \rangle_{X',X} - \langle \varphi_0, g_0 \rangle_{X',X} = \int_0^t \langle \varphi_s' + \Lambda^* \varphi_s, g_s \rangle_{X',X} \, ds, \quad \forall\, t \in (0,T^*].$$

Because $\Lambda - \mathcal{B} =: \mathcal{A} \in \mathcal{B}(X)$, we see that $D(\Lambda) = D(\mathcal{B})$ and thus $D(\Lambda^*) = D(\mathcal{B}^*)$, and this precisely means that $g \in \mathscr{E}$ is a weak solution to the evolution equation (1.2) on the interval of time $[0,T^*]$ and associated to the initial datum g_0. Repeating the construction on any $[kT^*,(k+1)T^*]$, we get a solution on $[0,T]$, and next on $\mathbb{R}_+$, since $T > 0$ is arbitrary. In other words, we have been able to prove the existence of a global weak solution $g \in C(\mathbb{R}_+;X)$ to the evolution equation (1.2) associated to the initial datum g_0.

Step 3. Regularity. We now consider $g_0 \in D(\Lambda)$ and $T > 0$. For $T^* \in (0,T)$, we define

$$\mathscr{F} := C^1([0,T^*];X), \quad \|g\|_\mathscr{F} := \|g\|_\mathscr{E} + \|g'\|_\mathscr{E}$$

as well as for any $g \in \mathscr{F}$, the function

$$f_t := S_\mathcal{B}(t) g_0 + (S_\mathcal{B} \mathcal{A} * g)(t).$$

We observe that

$$\begin{aligned}\frac{1}{h}(f_{t+h} - f_t) &= \frac{1}{h}[S_\mathcal{B}(t+h) g_0 - S_\mathcal{B}(t) g_0] + \frac{1}{h} \int_t^{t+h} S_\mathcal{B}(s) \mathcal{A} g_{t+h-s} ds \\ &\quad + \frac{1}{h} \int_0^t S_\mathcal{B}(s) \mathcal{A} [g_{t+h-s} - g_{t-s}] \, ds \\ &\to S_\mathcal{B}(t) \mathcal{B} g_0 + S_\mathcal{B}(t) \mathcal{A} g_0 + \int_0^t S_\mathcal{B}(s) \mathcal{A} g_{t-s}' \, ds = f_t',\end{aligned}$$

as $h \to 0$, where the limit term belongs to $\mathscr{E}$, so that $f \in \mathscr{F}$. From the computations made in Step 1 and the one made just above, for two given functions $g_1, g_2 \in \mathscr{F}$, the associated images f_1, f_2 satisfy

$$\begin{aligned}\|f_2 - f_1\|_\mathscr{F} &= \sup_{[0,T^*]} \Big\| \int_0^t S_\mathcal{B}(t-s) \mathcal{A} [g_2(s) - g_1(s) + g_2'(s) - g_1'(s)] \, ds \Big\| \\ &\le \|\mathcal{A}\| \, M T_* e^{bT} \, \|g_2 - g_1\|_\mathscr{F}.\end{aligned}$$

Arguing as in Step 1, and from the Banach fixed point theorem again, there exists $g \in \mathscr{F}$ which satisfies the functional equation (4.1). From (4.1), we observe that

$$\begin{aligned}\frac{1}{h}(S_\mathcal{B}(h) g_t - g_t) &= \frac{1}{h}(g_{t+h} - g_t) - \frac{1}{h} \int_t^{t+h} S_\mathcal{B}(t+h-s) \mathcal{A} g_s \, ds \\ &\to g_t' - \mathcal{A} g_t,\end{aligned}$$

in X as $h \to 0$, so that $g \in C([0,T^*];D(\mathcal{B}))$ and g is a classical solution to the evolution equation (1.2) on the interval of time $[0,T^*]$ and associated to the initial datum g_0. Repeating the argument, we build in that way a global classical solution $g \in C^1(\mathbb{R}_+,X) \cap C(\mathbb{R}_+,D(\Lambda))$.

Step 4. The backward dual problem and the conclusion. Exactly in the same way, for any $\psi_0 \in D(\Lambda^*)$, we may build a global classical solution $\psi \in C^1(\mathbb{R}_+,X') \cap C(\mathbb{R}_+,D(\Lambda))$ to the dual equation

$$\frac{d}{dt}\psi = \Lambda^* \psi, \quad \psi(0) = \psi_0.$$

Then, for a given $T > 0$ and a given $\varphi_T \in D(\Lambda^*)$, taking $\psi_0 := \varphi_T$, next defining φ as above and finally setting $\varphi(t) := \psi(T-t)$, we build a function $\varphi \in C^1([0,T],X') \cap C([0,T],D(\Lambda^*))$ which is a classical solution to the backward dual problem

$$\frac{d}{dt}\varphi = -\Lambda^*\varphi, \quad \varphi(T) = \varphi_T. \tag{4.3}$$

In order to conclude, we proceed exactly as in the proof of Proposition 3.3. We consider two global weak solutions $g_1, g_2 \in C(\mathbb{R}_+; X)$ to the evolution equation (1.2) associated to the same initial datum g_0. The difference $g := g_2 - g_0$, then satisfies

$$\langle \varphi_T, g_T \rangle_{X',X} = \int_0^T \langle \varphi_s' + \Lambda^* \varphi_s, g_s \rangle_{X',X}\, ds,$$

for any $\varphi \in C([0,T]; D(\Lambda^*)) \cap C^1([0,T]; X')$. For any $\varphi_T \in D(\mathcal{B}^*)$, choosing the function φ satisfying (4.3), we have

$$\langle \varphi_T, g_T \rangle_{X',X} = 0,$$

and thus $g_T = 0$. That establishes the uniqueness of the weak solution to the evolution equation (1.2). As a consequence of Proposition 1.3, we immediately deduce that Λ generates a semigroup and this one satisfies the announced growth estimate thanks to (4.2). □

5. Doblin-Harris Theorem in a Banach lattice

We formulate a general abstract constructive Doblin-Harris theorem.

We consider a Banach lattice X, which means that X is a Banach space endowed with a closed positive cone X_+ (we write $f \ge 0$ if $f \in X_+$ and we recall that $f = f_+ - f_-$ with $f_\pm \in X_+$ for any $f \in X$. We also denote $|f| := f_+ + f_-$). We assume that X is in duality with another Banach lattice Y, with closed positive cone Y_+, so that the bracket $\langle \phi, f \rangle$ is well defined for any $f \in X$, $\phi \in Y$, and that $f \in X_+$ (resp. $\phi \ge 0$) iff $\langle \psi, f \rangle \ge 0$ for any $\psi \in Y_+$ (resp. iff $\langle \phi, g \rangle \ge 0$ for any $g \in X_+$), typically $X = Y'$ or $Y = X'$. We write $\psi \in Y_{++}$ if $\psi \in Y$ satisfies $\langle \psi, f \rangle > 0$ for any $f \in X_+ \backslash \{0\}$.

Example 5.2. *The typical case (and unique example) we have in mind is* $X := L^p_\omega$, *for* $p \in [1,\infty]$ *and a weight function* $\omega : \mathbb{R}^d \to \mathbb{R}$, *where*

$$L^p_\omega := \{ f \in L^1_{\rm loc}(\mathbb{R}^d);\ \|f\|_{L^p_\omega} := \|f\omega\|_{L^p} < \infty \},$$

and $Y := L^{p'}_{\omega^{-1}}$.

We consider a positive and conservative (or stochastic) semigroup $S = (S_t) = (S(t))$ on X, that means that (S_t) is a semigroup on X such that

- $S_t : X_+ \to X_+$ for any $t \ge 0$,
- there exist $\phi_1 \in Y_{++}$, $\|\phi_1\| = 1$, and a dual semigroup $S^* = S^*_t = S^*(t)$ on Y such that $S^*_t \phi_1 = \phi_1$ for any $t \ge 0$. More precisely, we assume that S^*_t is a bounded linear mapping on Y such that $\langle S_t f, \phi \rangle = \langle f, S^*_t \phi \rangle$, for any $f \in X$, $\phi \in Y$ and $t \ge 0$, and thus in particular $S^*_t : Y_+ \to Y_+$ for any $t \ge 0$.

Example 5.3. *For the linear McKean equation associated to the operator* $\mathcal{L} f := \Delta f + \mathrm{div}(af)$ *defined on (a subspace of)* $X := L^p_\omega \subset L^1$, *the function* $\phi_1 := 1 \in L^\infty \subset Y$ *fulfills the second condition (conservative property).*

We denote by $\mathcal{L}$ the generator of S with domain $D(\mathcal{L})$. For $\psi \in Y_+$, we define the seminorm

$$[f]_\psi := \langle |f|, \psi \rangle, \ \forall f \in X.$$

Proposition 5.4. *A positive and conservative semigroup* S *on a Banach lattice* X *is a semigroup of contraction for the seminorm associated to the conservation* ϕ_1, *in other words*

$$[S(t)f]_{\phi_1} \le [f]_{\phi_1}, \quad \forall t \ge 0,\ \forall f \in X. \tag{5.4}$$

Proof of Proposition 5.4. For $f \in X$, we may write $f = f_+ - f_-$, $f_\pm \in X_+$, and then compute

$$\begin{aligned} |S_t f| &\le |S_t f_+| + |S_t f_-| \\ &= S_t f_+ + S_t f_- = S_t |f|, \end{aligned}$$

where we have used the positivity property of S_t in the second line. We deduce

$$[S_t f]_{\phi_1} \le \langle S_t |f|, \phi_1 \rangle = \langle |f|, S_t^* \phi_1 \rangle$$

and thus (5.4), because of the stationarity property of ϕ_1. □

In order to obtain a very accurate and constructive description of the longtime asymptotic behaviour of the semigroup S, we introduce additional assumptions.

• We first make the strong dissipativity assumption

$$\|S(t)f\| \le C_0 e^{\lambda t} \|f\| + C_1 \int_0^t e^{\lambda(t-s)} [S(s)f]_{\phi_1} ds, \tag{5.5}$$

for any $f \in X$ and $t \ge 0$, where $\lambda < 0$ and $C_i \in (0, \infty)$.

• Next, we make the Doblin-Harris positivity assumption

$$S_T f \ge \eta_{\varepsilon,T} g_\varepsilon [f]_{\psi_\varepsilon}, \quad \forall f \in X_+, \tag{5.6}$$

for any $T \ge T_1 > 0$ and $\varepsilon > 0$, where $\eta_{\varepsilon,T} > 0$, $g_\varepsilon \in X_+ \backslash \{0\}$ and (ψ_ε) is a bounded and decreasing family of $Y_+ \backslash \{0\}$.

• We finally assume the following compatibility condition of family of interpolation inequalities

$$[f]_{\phi_1} \le \xi_\varepsilon \|f\| + \Xi_\varepsilon [f]_{\psi_\varepsilon}, \ \forall f \in X, \ \varepsilon \in (0,1], \tag{5.7}$$

for two positive real numbers families (ξ_ε) and (Ξ_ε) such that $\xi_\varepsilon \searrow 0$ as $\varepsilon \searrow 0$.

Example 5.5. *When $X := L^p_\omega \subset L^1$ (what is equivalent to $\omega^{-1} \in L^{p'}$) with $\omega \to \infty$ as $|x| \to \infty$ and $\phi_1 = 1 \in L^\infty \subset Y$, such an interpolation family holds with $\psi_\varepsilon := \mathbf{1}_{B_R}$, $R := \varepsilon^{-1}$. We indeed have*

$$\int_{\mathbb{R}^d} |f| = \int_{B_R^c} |f| + \int_{B_R} |f| \le \|\omega \mathbf{1}_{B_R^c}\|_{L^{p'}} \|f\|_{L^p_\omega} + [f]_{\psi_\varepsilon},$$

with $\xi_\varepsilon := \|\omega \mathbf{1}_{B_R^c}\|_{L^{p'}} \to 0$ as $\varepsilon \to 0$.

Theorem 5.6. *Consider a semigroup S on a Banach lattice X which satisfies the above conditions. Then, there exists a unique normalized positive stationary state $f_1 \in D(\mathcal{L})$, that is*

$$\mathcal{L} f_1 = 0, \quad f_1 \ge 0, \quad \langle \phi_1, f_1 \rangle = 1.$$

Furthermore, there exist some constructive constants $C \ge 1$ and $\lambda_2 < 0$ such that

$$\|S(t)f - \langle f, \phi_1 \rangle f_1\| \le C e^{\lambda_2 t} \|f - \langle f, \phi_1 \rangle f_1\| \tag{5.8}$$

for any $f \in X$ and $t \ge 0$.

Sketch of the proof of Theorem 5.6.

Step 1. The Lyapunov condition. From (5.5) and (5.4), we have

$$\|S_t f\| \le C_0 e^{\lambda t} \|f\| + C_1 \int_0^t e^{\lambda(t-s)} [f]_{\phi_1} ds,$$

and we may thus choose $T \ge T_1$ large enough in such a way that

$$\|S_T f\| \le \gamma_L \|f\| + K [f]_{\phi_1}, \tag{5.9}$$

with

$$\gamma_L := C_0 e^{\lambda T} \in (0,1), \quad K := C_1/\lambda.$$

Step 2. The conditional Doblin-Harris estimate. Take $f \ge 0$ such that $\|f\| \le A[f]_{\phi_1}$ with $A > K/(1-\gamma_L)$. We have thus

$$\|f\| \le A(\xi_\varepsilon \|f\| + \Xi_\varepsilon [f]_{\psi_\varepsilon}),$$

for any $\varepsilon > 0$, thanks to the interpolation inequality (5.7). Choosing $\varepsilon > 0$ small enough, we immediately obtain

$$\|f\| \le 2 \Xi_\varepsilon [f]_{\psi_\varepsilon}.$$

Together with $[f]_{\phi_1} \le \|f\|$ and the Doblin-Harris positivity condition (5.6), we conclude to the conditional Doblin-Harris positivity estimate

$$S_T f \ge c g_\varepsilon [f]_{\phi_1}$$

for all $T \ge T_1$, with $c^{-1} = c_A^{-1} := 2A \Xi_\varepsilon \eta_{\varepsilon,T}^{-1}$.

Step 3. The conditional coupling property. We may now improve the non-expensive estimate (5.4) on the set $\mathcal{N} := \{f \in X; \langle \phi_1, f \rangle = 0\}$. Take indeed $f \in \mathcal{N}$ such that $\|f\| \le A[f]_{\phi_1}$. Observing that $[f_\pm]_{\phi_1} = [f]_{\phi_1}/2$ and thus

$$\|f_\pm\| \le \|f\| \le 2A[f_\pm]_{\phi_1},$$

the previous estimate tells us that

$$S_T f_\pm \ge \varrho g_\varepsilon \quad \varrho := c_{2A}[f]_{\phi_1}.$$

Slightly modifying the arguments of Proposition 5.4, we compute now

$$\begin{aligned} |S_T f| &\le |S_T f_+ - \varrho g_\varepsilon| + |S_T f_- - \varrho g_\varepsilon| \\ &= S_T |f| - 2\varrho g_\varepsilon. \end{aligned}$$

We deduce

$$[S_T f]_{\phi_1} \le \langle |f|, \phi_1 \rangle - 2\varrho \langle \phi_1, g_\varepsilon \rangle,$$

and thus conclude to the *conditional coupling estimate*

$$[S_T f]_{\phi_1} \le \gamma_H [f]_{\phi_1}, \tag{5.10}$$

with $\gamma_H := 1 - 2c_{2A} \langle \phi_1, g_\varepsilon \rangle \in (0,1)$.

Step 4. We introduce a new equivalent norm $|\!|\!| \cdot |\!|\!|$ on X defined by

$$|\!|\!| f |\!|\!| := [f]_{\phi_1} + \beta \|f\|. \tag{5.11}$$

Using the three properties (5.4), (5.9) and (5.10), we may prove that there exist $\beta > 0$ small enough and $\gamma \in (0,1)$ such that

$$|\!|\!| S_T f_0 |\!|\!| \le \gamma |\!|\!| f_0 |\!|\!|, \quad \text{for any } f_0 \in \mathcal{N}. \tag{5.12}$$

We observe that we have the alternative

$$A\,[f_0]_{\phi_1} \ge \|f_0\| \quad \text{or} \quad A\,[f_0]_{\phi_1} < \|f_0\|.$$

In the first case of the alternative, using the Lyapunov estimate (5.9) and the coupling estimate (5.10), we have

$$\begin{aligned} |\!|\!| S_T f_0 |\!|\!| &= [S_T f_0]_{\phi_1} + \beta \|S_T f_0\| \\ &\le (\gamma_H + \beta K)[f_0]_{\phi_1} + \beta \gamma_L \|f_0\| \\ &\le \gamma_1 |\!|\!| f_0 |\!|\!|, \end{aligned}$$

with $\gamma_1 := \max(\gamma_H + \beta K, \gamma_L) < 1$, by fixing from now on $\beta > 0$ small enough. In the second case of the alternative, using the Lyapunov estimate (5.9) and the non expansion estimate (5.4), we have

$$\begin{aligned} |\!|\!| S_T f_0 |\!|\!| &= [S_T f_0]_{\phi_1} + \beta \|S_T f_0\| \\ &\le (1 + \beta K - \beta\delta)[f_0]_{\phi_1} + \beta(\gamma_L + \delta/A)\|f_0\| \\ &\le \gamma_2 |\!|\!| f_0 |\!|\!|, \end{aligned}$$

with $\gamma_2 := \max(1 + \beta K - \beta\delta, \gamma_L + \delta/A)$ for any $0 < \beta\delta < 1 + \beta K$. We take $\delta := K + \vartheta$, $\vartheta > 0$, so that we get

$$\gamma_2 = \max(1 - \beta\vartheta, (\gamma_L + K/A) + \vartheta/A) < 1,$$

by choosing $\vartheta > 0$ small enough and by recalling from the very definition of A that $\gamma_L + K/A < 1$. In any cases, we have thus established (5.12) with $\gamma := \max(\gamma_1, \gamma_2) < 1$.

Step 5. In order to prove the existence and uniqueness of the stationary state $f_1 \in X_1$, we fix $g_0 \in \mathcal{M} := \{g \in X_1, g \ge 0, \langle g, \phi_1 \rangle = 1\}$, and we define recursively $g_k := S_T g_{k-1}$ for any $k \ge 1$. Thanks to (5.12), we get

$$\sum_{k=1}^{\infty} |\!|\!| g_k - g_{k-1} |\!|\!| \le \sum_{k=0}^{\infty} \gamma^k |\!|\!| g_1 - g_0 |\!|\!| < \infty,$$

so that (g_k) is a Cauchy sequence in $\mathcal{M}$. We set $f_1 := \lim g_k \in \mathcal{M}$ which is a stationary state for the mapping S_T, as seen by passing to the limit in the recursive equations defining (g_k). From (5.12) again, this is the unique stationary state for this mapping in $\mathcal{M}$. From the semigroup property, we have $S_t f_1 = S_t S_T f_1 = S_T(S_t f_1)$ for any $t > 0$, so that $S_t f_1$ is also a stationary state in $\mathcal{M}$, and thus $S_t f_1 = f_1$ for any $t > 0$, by uniqueness.

Step 6. For $f \in X$, we see that $h := f - \langle f, \phi_1 \rangle \phi_1 \in \mathcal{N}$, and using recursively (5.12), we deduce

$$|||S_{nT} h||| \le \gamma^n |||h|||, \quad \forall\, n \ge 0.$$

The estimate (5.8) then follows by using the equivalence of the norms $\|\cdot\|$ and $|||\cdot|||$, the semigroup property and the growth estimate (1.1) for dealing with intermediate times $t \in (nT, (n+1)T)$. For $t \ge 0$ and $t = nT + \tau$, $n \ge 0$, $\tau \in [0, T)$, we may indeed write

$$\begin{aligned}
\|S(t)h\| &\le \beta^{-1} |||S(T)^n S(\tau) h||| \\
&\le \beta^{-1} \gamma^n |||S(\tau) h||| \\
&\le \beta^{-1} e^{n \log \gamma} (1+\beta) \|S(\tau) h\| \\
&\le e^{(t/T - \tau/T) \log \gamma} (1 + \beta^{-1}) M e^{bT} \ \|h\|,
\end{aligned}$$

where in the last line we have used (1.1) with $b \ge 0$ (what we may always assume and it is also imposed in our case by the conservation hypothesis), from what we conclude to (5.8) with $\lambda_2 := T^{-1} \log \gamma < 0$ and $C := \gamma^{-1}(1 + \beta^{-1}) M e^{bT}$. □