Chap 3. Example : the Fokker-Planck equation. 1

We consider the Fokker-Planck equation.

$$\partial_t f = \mathcal{L}f := \Delta f + \mathrm{div}(xf) \quad \text{in } (0,\infty)\times\mathbb{R}^d.$$

<u>lemma 1</u> For any $f_0 \in L^2$ $\exists!\, f \in X_\infty$ var. sol. ~~There exists a semigroup.~~ <u>proof</u> $\mathcal{J}L^2$ (chap 1)

<u>lemma 2</u> $\exists$ SG $S_\mathcal{L}$ associated to the eq.

$S_\mathcal{L} \geq 0$ and $\mathcal{L}^* 1 = 0$

$$\mathcal{L}^*\varphi = \Delta\phi - x\cdot\nabla\phi.$$

<u>lemma 3</u> For many L^p_ω spaces, $S_\mathcal{L} : L^p_\omega \to L^p_\omega$, $O(e^{Kt})$.

The L^p_ω spaces are associated to

- $\omega = \langle x\rangle^k$ s.t. $L^p_\omega \subset L^1$, i.e $k > d/p'$
- $\omega = e^{a|x|^2}$ s.t. $\gamma \in L^p_\omega$, i.e $a \in (0, \tfrac12)$.

<u>Proof</u> : After several I by P :

$$\int \mathcal{L}f\, f^{p-1}\omega^p = -\theta_p \int |\nabla (f\omega)^{p/2}|^2 + \int f^p \omega^p \psi$$

with $\psi = a(p)\dfrac{|\nabla\omega|^2}{\omega^2} + b(p)\dfrac{\Delta\omega}{\omega} + \dfrac{d}{p'} - x\cdot\dfrac{\nabla\omega}{\omega}$

and we choose ω s.t. ~~$\psi \leq M 1_{B_R} - \nu$~~.

$$\psi \leq K \leq M 1_{B_R} - \nu \qquad M, K \geq 0,\ \nu > 0.$$

As a consequence

$$\frac{1}{p}\frac{d}{dt}\int f^p\omega^p = \int \mathcal{L}f\, f^{p-1}\omega^p$$
$$\leq K \int f^p\omega^p$$

and

$$\|f_t\|_{L^p_\omega} \leq e^{Kt}\|f_0\|_{L^p_\omega} \quad \text{by Gronwall lemma } \square.$$

lemma 4 $S_\mathcal{L} : L^1_\omega \to L^2_\omega \quad O\left(\frac{e^{Kt}}{t^{d/4}}\right)$

That Nash argument.

proof 1: $f := S_\mathcal{L}(t) f_0.$

$$\frac{1}{2}\frac{d}{dt}\int f^2\omega^2 \le -\int |\nabla(f\omega)|^2 + K_2 \int f^2\omega^2$$

$$\frac{d}{dt}\int f\omega \le K_1 \int f\omega$$

$$U_p(t) := e^{-K_p t}\, \|f\omega\|^p_{L^p}$$

$$\frac{d}{dt} U_1(t) \le 0 \quad \text{if } K \ge K_1.$$

$$\frac{d}{dt} U_2(t) \le -\int |\nabla(f\omega)|^2 e^{-2Kt} \quad \text{if } K \ge K_2$$

Using Nash inequality

$$\|f\omega\|_{L^2}^{1+2/d} \le \|f\omega\|_{L^1}^{2/d} \, \|\nabla(f\omega)\|_{L^2},$$

we get:

$$\frac{d}{dt} U_2 \lesssim - \frac{\|f\omega\|_{L^2}^{2(1+2/d)}\, e^{-2(1+2/d)Kt}}{\|f\omega\|_{L^1}^{4/d}\, e^{-4/d\, Kt}}$$

$$\lesssim - \frac{U_2(t)^{1+2/d}}{U_1(t)^{4/d}} \lesssim - \frac{U_2^{1+2/d}}{U_1(0)^{4/d}}.$$

We obtain

$$U_2(t) \le C\frac{U_1(0)^2}{t^{d/2}}$$

and the result with $K := \max(K_1, K_2)$ $\square$.

$(d \leq 3)$.

proof 2. We set $B := \mathcal{L} - M\chi_R$ $\quad 1_{B_R} \leq \chi_R \in C_c^2(\mathbb{R}^d)$.

We now have for $f(t) := S_B(t) f_0$

$$\frac{1}{2}\frac{d}{dt}\int f^2 \omega^2 \leq -\int |\nabla(f\omega)|^2 - \nu \int f^2 \omega^2$$

$$\frac{d}{dt}\int f\omega \leq \cancel{0} - \nu \int f \omega .$$

~~Forgetting~~ Throwing up the negative part, we get

$$\frac{1}{2}\frac{d}{dt}\int f^2 \omega^2 \leq -\int |\nabla(f\omega)|^2$$

$$\frac{d}{dt}\int f\omega \leq 0$$

and Nash proof gives

$$\|f_B \omega\|_{L^2} \leq \frac{1}{t^{d/4}} \|f_0 \omega\|_{L^1} .$$

Thanks to Duhamel formula

$$S_{\mathcal{L}} f_0 = S_B f_0 + S_B A * S_{\mathcal{L}} f_0 .$$

$$\|S_{\mathcal{L}}(t) f_0\|_{L^2_\omega} \leq \|S_B(t) f_0\|_{L^2_\omega} + \int_0^t \|S_B(t-s) A S_{\mathcal{L}} f_0\|_{L^2_\omega}$$

$$\leq \frac{C}{t^{d/4}} \|f_0\|_{L^1_\omega} + \int_0^t \frac{1}{(t-s)^{d/4}} \|A S_{\mathcal{L}} f_0\|_{L^1_\omega} ds$$

$$\leq \frac{C}{t^{d/4}} \|f_0\|_{L^1_\omega} + \underbrace{\int_0^t \frac{1}{(t-s)^{d/4}} C e^{K_1 s} ds \, \|f_0\|_{L^1_\omega}}_{e^{K_1 t} \int_0^t \frac{e^{-K_1(t-s)}}{(t-s)^{d/4}} \|f_0\|_{L^1_\omega}}$$

$$\leq \left(\frac{C}{t^{d/4}} + C e^{K_1 t} \right) \|f_0\|_{L^1_\omega} . \quad \square$$

Lemma 5 Lyapunov condition $(d \leq 3)$.

$$\|S_{\mathcal{L}}(t) f\|^2_{L^2_\omega} \leq e^{-\nu t} \|f_0\|^2_{L^2_\omega} + K \|f_0\|_{L^1}.$$

proof. Repeating the first proof of Lemma 4, we have

$$\|S_B(t) f_0\|^2_{L^2(\omega)} \lesssim e^{-\nu t} \|f_0\|^2_{L^2_\omega} \qquad \nu > 0.$$

$$\|S_B(t) f_0\|_{L^1_\omega} \leq e^{-\nu t} \|f_0\|_{L^1_\omega}$$

$$\|S_B(t) f_0\|^2_{L^2_\omega} \leq \frac{e^{-\nu t}}{t^{d/4}} \|f_0\|_{L^1_\omega}.$$

We write again the Duhamel formula

$$S_{\mathcal{L}}(t) = S_B + S_B A * S_{\mathcal{L}}$$

For the second term we have

$$\|S_B A * S_{\mathcal{L}} f_0\|^2_{L^2_\omega} \leq \int_0^t \|S_B(t-s) A S_{\mathcal{L}}(s) f_0\|^2_{L^2_\omega} ds$$

$$\lesssim \int_0^t \frac{e^{-\nu(t-s)}}{(t-s)^{d/4}} \|A S_{\mathcal{L}}(s) f_0\|_{L^1_\omega} ds$$

$$\lesssim \int_0^t \frac{e^{-\nu(t-s)}}{(t-s)^{d/4}} \|S_{\mathcal{L}}(s) f_0\|_{L^1} ds$$

$$\lesssim \underbrace{\int_0^\infty \frac{e^{-\nu \tau}}{\tau^{d/4}} d\tau}_{=: K} \|f_0\|_{L^1}$$

□.

Lemma 6. (Doblin-Harris).

$\forall T_0\ \forall R_0, \exists R_1, \exists \eta_1 \quad S_T f_0 \geq \eta_1 \mathbb{1}_{B_{R_1}} \int_{B_{R_0}} f_0, \quad \forall f_0 \geq 0 \in L^1_\omega.$

proof. By the maximum principle, we may assume $\operatorname{supp} f_0 \subset B_{R_0}$ and thus $f_0 \in L^1_\omega$, two weight function. We fix (for instance) $\omega(x) = \langle x \rangle$.

We know that

$$\|f_t\|_{L^1_\omega} \leq C e^{\kappa_1 t} \|f_0\|_{L^1_\omega}.$$

We fix $T > 0$. For any $t \in (0, T)$, we write

$$\int_{B_R} f_t = \int_{\mathbb{R}^d} f_t - \int_{B_R^c} f_t$$

$$\geq \int_{\mathbb{R}^d} f_0 - \frac{1}{\omega(R)} \int f_t\, \omega$$

$$\geq \int_{B_{R_0}} f_0 - \frac{C e^{\kappa_1 T}}{\omega(R)} \int f_0\, \omega$$

$$\geq \left(1 - C e^{\kappa_1 T} \frac{\omega(R_0)}{\omega(R)}\right) \int_{B_{R_0}} f_0.$$

$$\geq \frac{1}{2} \int_{B_{R_0}} f_0$$

by choosing $R = R_1$ large enough.

We deduce that f satisfies

$$\|f\|_{L^1(Q_0)} \geq C \int_{B_{R_0}} f_0$$

with $Q_0 := (T_0, T) \times B_{R_1}$, $T_0 := T/2$.

On the other hand, because $f \in L^2(Q)$ is a var sol on $Q \supset\supset Q_0$, we may apply the Harnack ineq ~~on Q_1~~ and we get

$$\inf_{Q_1} f \gtrsim \sup_{Q_0} f \gtrsim \cancel{\|f\|_{L^1(Q_0)}}$$

with $Q_1 := (T, T+1) \times B_{R_1}$.

We deduce

$$f(T) \geq 1_{B_{R_1}} \inf_{Q_1} f \gtrsim 1_{B_{R_1}} \sup_{Q_0} f \gtrsim 1_{B_{R_1}} \|f\|_{L^1(Q_0)}$$

$$\gtrsim 1_{B_{R_1}} \frac{1}{2} \int_{B_{R_0}} f_0 . \quad \square$$

As a conclusion. ~~$\exists \lambda > 0$~~

<u>Theorem</u> $\exists \lambda > 0$, $C \geq 1$, s.t.

$$\| f - \int f_0 \, \underset{\underset{M(x)}{\|}}{\underset{\gamma(x)}{\overset{\|}{G(x)}}} \|_{L^2_\omega} \leq C e^{-\lambda t} \| f_0 - \int f_0 \, G \|_{L^2_\omega}.$$