CHAPTER 2: DE GIORGI-NASH-MOSER THEORY AND BEYOND FOR PARABOLIC EQUATIONS

Contents

1. Introduction

In this chapter we are mainly concerned with the parabolic equation

$$\frac{\partial f}{\partial t} = \mathrm{div}(A\nabla f) \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d, \tag{1.1}$$

on the function $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$, $d \ge 3$, with a measurable, bounded and strictly elliptic matrix A, namely A satisfies (in the sense of quadratic forms) $\nu I \le A(x) \le \nu^{-1} I$ for any $x \in \mathbb{R}^d$ and for some $\nu > 0$. The heat equation corresponds to the case $A = \nu I > 0$ to which we dedicate this introduction.

1.1. **Representation formula.** In the case $A = \frac{1}{2} I$, that is

$$\frac{\partial f}{\partial t} = \frac{1}{2}\Delta f \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d, \tag{1.2}$$

we know that

$$\gamma_t(x) := \frac{1}{(2\pi t)^{d/2}} \exp(-\frac{|x|^2}{2t}) \tag{1.3}$$

is the associated fundamental solution, that means that it is the unique solution f to equation (1.2) such that $f(t,\cdot) \rightharpoonup \delta_0$ as $t \to 0$. The coefficient 1/2 in (1.2) is just put in order to get this usual gaussian kernel γ_t (instead of a rescaled version of it). As a consequence, for any $f_0 \in L^q(\mathbb{R}^d)$, $1 \le q \le \infty$, the solution f to (1.2) satisfies $f \in C^\infty((0,\infty)\times\mathbb{R}^d)$ and more precisely the solution is given through the representation formula

$$f(t,.) = \gamma_t * f_0, \tag{1.4}$$

where $* = *_x$ stands for the convolution operator in the position variable. Let us observe that, for any $r \in [1,\infty]$,

$$\|\gamma_t\|_{L^r} = \frac{C(r,d)}{t^{\frac{d}{2}(1-\frac{1}{r})}}, \quad C(r,d) := \frac{1}{r^{\frac{d}{2r}}(2\pi)^{\frac{d}{2}(1-\frac{1}{r})}},$$

so that from the Young inequality for convolution products, we get the ultracontractivity estimate

$$\|f(t,.)\|_{L^p} \le \frac{C_{p,q}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}} \|f_0\|_{L^q}, \tag{1.5}$$

for any $t > 0$ and $p, q \in [1,\infty]$, $p \ge q$, where $C_{p,q} := C(r,d)$ and $r \in [1,\infty]$ is defined by the relation $1/p = 1/q + 1/r - 1$. When $f_0 \in L^q$, $q \in [1,\infty)$, the above estimate reveals both a instantaneous kind of smoothing effect (gain of local integrability) and a dispersion mechanism $f(t,.) \to 0$ as $t \to \infty$.

The main aim of this chapter is to recover part of these results using some techniques which are valid for a general matrix A. However, for the sake of simplicity, we will mainly consider the case $A = \nu I$, with $\nu = 1$ or $1/2$. The sequel of the introduction is dedicated to the presentation of other techniques which are specific to the heat equation or to a general diffusion equation with smooth coefficients.

1.2. **The Fourier transform.** Consider now the heat equation with source term

$$\frac{\partial f}{\partial t} = \Delta f + g \quad \text{ in } \mathbb{R} \times \mathbb{R}^d, \tag{1.6}$$

with $f, g \in L^2(\mathbb{R}^{d+1})$. We define the Fourier transform

$$\hat{h}(\tau,\xi) := \int_{\mathbb{R}^{d+1}} h(t,x) e^{-i(\tau t + x\cdot\xi)} dt dx.$$

On the Fourier side, the above heat equation is

$$i\tau \hat{f} + |\xi|^2 \hat{f} = \hat{g},$$

from what we immediately compute

$$\int_{\mathbb{R}^{d+1}} (1+\tau^2+|\xi|^4)|\hat{f}|^2 = \int_{\mathbb{R}^{d+1}} |\hat{f}|^2 + \int_{\mathbb{R}^{d+1}} \frac{\tau^2+|\xi|^4}{|i\tau+|\xi|^2|^2} |\hat{g}|^2 = \int_{\mathbb{R}^{d+1}} |\hat{f}|^2 + |\hat{g}|^2.$$

We deduce

$$\|f\|^2_{L^p} \lesssim \|f\|^2_{H^1} \lesssim \|f\|^2_{L^2} + \|g\|^2_{L^2},$$

with $p := 2(d+1)/(d-1) > 2$, from the Sobolev embedding, the Fourier definition of the Sobolev space in $\mathbb{R}^{d+1}$ and the Plancherel identity. This estimate also reveals some gain of integrability of the solution to the heat equation and can be seen as a variant of (1.5).

1.3. **Energy method.** By differentiating the heat equation

$$\frac{\partial f}{\partial t} = \Delta f \quad \text{ in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{ in } \mathbb{R}^d, \tag{1.7}$$

we can easily establish some estimates on its smoothing effect. On the one hand, any solution f satisfies first the *energy identity*

$$\frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2} = \int_{\mathbb{R}^d} (\partial_t f) f = \int_{\mathbb{R}^d} (\Delta f) f = -\|\nabla f\|^2_{L^2}. \tag{1.8}$$

On the other hand, because ∇ and the elliptic operator Δ commute, we have

$$\partial_t \nabla f = \Delta \nabla f,$$

and any solution also satisfies

$$\frac{d}{dt}\|\nabla f\|^2_{L^2} = -2\|\Delta f\|^2_{L^2} = -2\|D^2 f\|^2_{L^2}.$$

Both together, we have

$$\frac{d}{dt}\Big\{\frac{1}{2}\|f\|_{L^2}^2 + t\|\nabla f\|_{L^2}^2\Big\} = -2t\|D^2 f\|_{L^2}^2 \le 0, \quad \forall\, t > 0,$$

and integrating in time this differential inequality, we readily obtain

$$\|\nabla f(t)\|_{L^2}^2 \le \frac{1}{2t}\,\|f_0\|_{L^2}^2, \quad \forall\, t > 0. \tag{1.9}$$

It is worth emphasizing that a similar result as this last estimate (1.9) is available for solutions to the general parabolic equation (1.1) when A is a smooth function, but certainly not in the case when A is only measurable. Using the Sobolev embedding in $\mathbb{R}^d$, we deduce then

$$\|f(t)\|_{L^{2^*}} \lesssim \frac{1}{t^{1/2}}\,\|f_0\|_{L^2}, \quad \forall\, t > 0,$$

with $1/2^* := 1/2 - 1/d$, recovering thus (1.5) with $p := 2^*$ and $q := 2$.

1.4. **Additional estimates.** In the chapter 1, we have already seen two more elementary qualitative properties of parabolic equations. On the one hand, integrating equation (1.7), we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} \Delta f = 0,$$

so that the *"mass"* is conserved

$$\langle f(t,\cdot)\rangle := \int_{\mathbb{R}^d} f dx = \int_{\mathbb{R}^d} f_0 dx =: \langle f_0\rangle, \quad \forall\, t \ge 0.$$

Similarly as for the energy estimate (1.8), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d} f_-^2 &= \int_{\mathbb{R}^d} \partial_t f(-f_-) = \int_{\mathbb{R}^d} \Delta f(-f_-)\\ &= \int_{\mathbb{R}^d} \nabla f \nabla f_- = -\int_{\mathbb{R}^d} |\nabla f_-|^2 \le 0,\end{aligned}$$

so that $f(t) \ge 0$ if $f_0 \ge 0$. That means that the flow associated to the heat equation preserves the positivity, or in other words, the equation (or the associated operator) satisfies a *weak maximum principle.*

More generally, for any smooth function $\beta : \mathbb{R} \to \mathbb{R}$, we have

$$\partial_t \beta(f) = \beta'(f)\partial_t f = \beta'(f)\Delta f = \Delta\beta(f) - \beta''(f)|\nabla f|^2.$$

When furthermore β is convex, the function $\beta(f)$ is a subsolution, namely it satisfies

$$\partial_t \beta(f) \le \Delta\beta(f). \tag{1.10}$$

Integrating on $\mathbb{R}^d$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^d} \beta(f) \le \int_{\mathbb{R}^d} \Delta\beta(f) = 0,$$

and we have thus exhibited a large family of Lyapunov functional $t \mapsto \|\beta(f(t,\cdot))\|_{L^1}$, for any nonnegative convex function β. In particular, for any $p \in [1,\infty]$, the L^p-norm falls in this family, and thus

$$\|f(t,\cdot)\|_{L^p} \le \|f_0\|_{L^p}, \quad \forall\, t \ge 0. \tag{1.11}$$

Finally, for a positive solution, the dispersion/diffusion effect of the heat equation can also be brought out through the increasing of moments: we have indeed for instance

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)|x|^2\,dx = \int_{\mathbb{R}^d} f\Delta|x|^2\,dx = 2d\int_{\mathbb{R}^d} f_0, \quad \forall\, t \ge 0,$$

and when $0 \le f_0 \in L^1(\mathbb{R}^d)$, we see that the second moment grows linearly.

1.5. **Organisation of the next sections.** In the next sections, we will recover part of the properties established for the solutions to the heat equation in this introductive section. We will first focus on some robust proofs of the ultracontractivity property (1.5). We will next address the two local properties of strict positivity (Harnack inequality and strong maximum principle) and regularity (Holder continuity). We will finally explain how to establish the existence and uniqueness of fundamental solutions which behaves very similarly to the heat kernel.

2. Ultracontractivity

The aim of this section is to establish the ultracontractivity estimate (1.5) for a solution f to the parabolic equation

$$\frac{\partial f}{\partial t} = \operatorname{div}(A\nabla f) + b\cdot\nabla f \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d, \tag{2.1}$$

for a bounded strictly elliptic matrix A and a bounded vector field b. For the sake of simplicity, we will rather establish the result on the solution to the heat equation

$$\frac{\partial f}{\partial t} = \Delta f \quad \text{in } (0,\infty)\times\mathbb{R}^d, \qquad f(0,\cdot) = f_0 \quad \text{in } \mathbb{R}^d, \tag{2.2}$$

with several proofs which are not based on the representation formula (1.4). These proofs are clearly longer and more complicated to those presented in Section 1, but they are also more robust in the sense that they apply to the more general parabolic equation (2.1) and to bounded domain variants. The arguments will be presented as *a priori estimates* and we will no bother with a rigorous justification of the computations when starting from the very definition of what is a (classical or weak) solution. These generalizations and rigorous justifications are left as exercices.

2.1. **The Nash approach.** We start establishing (1.5) for $p = 2$ and $q = 1$. For that purpose, we first establish the following fundamental functional estimate.

Nash inequality. There exists a constant C_d such that for any $f \in L^1(\mathbb{R}^d) \cap H^1(\mathbb{R}^d)$, there holds

$$\|f\|_{L^2}^{1+2/d} \le C_d\,\|f\|_{L^1}^{2/d}\,\|\nabla f\|_{L^2}. \tag{2.3}$$

Proof of Nash inequality. We write, for any $R > 0$,

$$\begin{aligned} \|f\|_{L^2}^2 &= \|\hat f\|_{L^2}^2 = \int_{|\xi|\le R} |\hat f|^2 + \int_{|\xi|\ge R} |\hat f|^2 \\ &\le c_d\,R^d\,\|\hat f\|_{L^\infty}^2 + \frac{1}{R^2}\int_{|\xi|\ge R} |\xi|^2\,|\hat f|^2 \\ &\le c_d\,R^d\,\|f\|_{L^1}^2 + \frac{1}{R^2}\,\|\nabla f\|_{L^2}^2, \end{aligned}$$

where we have used the Plancherel identity in the first line and usual properties of the Fourier transform on the last line. We take the optimal choice for R by setting $R := (\|\nabla f\|_{L^2}^2/c_d\|f\|_{L^1}^2)^{\frac{1}{d+2}}$ so that the two terms at the RHS of the last line are equal and that gives (2.3). □

Alternative proofs of the Nash inequality (2.3) based on the Sobolev embedding or on the Poincaré-Wirtinger inequality are left as exercices.

The cornerstone $L^1 - L^2$ estimate. We consider now a solution f to the heat equation (2.2) and we recall that

$$\frac{d}{dt}\int_{\mathbb{R}^d} f(t,x)^2\,dx = -2\int_{\mathbb{R}^d} |\nabla f(t,x)|^2\,dx, \quad \forall\, t \ge 0, \tag{2.4}$$

from (1.8), and

$$\|f(t,\cdot)\|_{L^1} \le \|f_0\|_{L^1}, \quad \forall\, t \ge 0, \tag{2.5}$$

from (1.11) with $p = 1$. Putting together that two last equations and the Nash inequality, we obtain the following ordinary differential inequality

$$\frac{d}{dt}\int_{\mathbb{R}^d} f_t^2\,dx \le -C_d\|f_0\|_{L^1}^{-\frac{2}{\alpha}}\Big(\int_{\mathbb{R}^d} f_t^2\,dx\Big)^{1+\frac{1}{\alpha}}, \quad \alpha := d/2.$$

Recalling that $\bar u := (\alpha^{-1}Kt)^{-\alpha}$ is a supersolution to the ordinary differential inequality

$$u' \le -K\,u^{1+\alpha}, \quad \alpha = 2/d > 0,$$

we deduce from a maximum principle (Gronwall lemma) that $u \le \bar u$ and thus in our case

$$\int_{\mathbb{R}^d} f_t^2\,dx \le \Big(\frac{\alpha}{C_d}\|f_0\|_{L^2}^{\frac{2}{\alpha}}\Big)^{\alpha}\frac{1}{t^\alpha} = \Big(\frac{d}{2C_d}\Big)^{\frac{d}{2}}\frac{\|f_0\|_{L^1}^2}{t^{d/2}}, \quad \forall\, t > 0. \tag{2.6}$$

That is nothing but the announced estimate (1.5) for $p = 2$ and $q = 1$.

Extension to $L^q - L^p$ estimates. In order to prove the estimate for the full range of exponents, we use some duality, semigroup and interpolation elementary arguments as follow.

• We first use a duality argument. For given $T > 0$ and ϕ_T, we consider the solution ϕ to the backward heat equation

$$-\partial_t\phi = \Delta\phi, \quad \phi(T) = \phi_T,$$

and we observe that if f still denotes the solution to the forward heat equation (2.2), we have by performing two integrations by parts in the last line

$$\begin{aligned}\frac{d}{dt}\int f\phi &= \int(\partial_t f)\phi + \int f\partial_t\phi\\ &= \int(\Delta f)\phi + \int f(-\Delta\phi)\\ &= -\int\nabla f\cdot\nabla\phi + \int\nabla f\cdot\nabla\phi = 0,\end{aligned}$$

so that

$$\int f(T)\phi_T = \int f_0\phi(0). \tag{2.7}$$

We also observe that $\psi(t) := \phi(T-t)$ is a solution to the forward heat equation (2.2) with initial datum $\psi(0) = \phi_T$, so that we may use the already proved estimate (1.5) and we get

$$\|\phi(0)\|_{L^2} = \|\psi(T)\|_{L^2} \le \frac{C}{T^{d/4}} \|\psi(0)\|_{L^1} = \frac{C}{T^{d/4}} \|\phi_T\|_{L^1}.$$

Combining the Riesz representation theorem, the duality identity (2.7), the Cauchy-Schwarz inequality and that last estimate on the dual problem, we have

$$\begin{aligned}\|f(T)\|_{L^\infty} &= \sup_{\|\phi_T\|_{L^1}\le 1} \int f(T)\phi_T = \sup_{\|\phi_T\|_{L^1}\le 1} \int f_0 \phi(0) \\ &\le \sup_{\|\phi_T\|_{L^1}\le 1} \|f_0\|_{L^2} \|\phi(0)\|_{L^2} \le \frac{C}{T^{d/4}} \|f_0\|_{L^2},\end{aligned}$$

which is nothing but estimate (1.5) for $p = \infty$ and $q = 2$.

• We next use a semigroup like argument. Gathering estimates (1.5) for $(p,q) = (2,1)$ and $(p,q) = (\infty, 2)$, we have

$$\|f(t)\|_{L^\infty} \le \frac{C_{2\infty}}{(t/2)^{d/4}} \|f(t/2)\|_{L^2} \le \frac{C_{2\infty}}{(t/2)^{d/4}} \frac{C_{12}}{(t/2)^{d/4}} \|f_0\|_{L^1},$$

which is nothing but the estimate (1.5) for $p = \infty$ and $q = 1$.

• We finally use some elementary interpolation arguments and more precisely repeatedly the Holder inequality. For $p \in (1,\infty)$, using the L^1 estimate (2.5) and the already probed Nash estimate for $(p,q) = (\infty, 1)$, we have

$$\|f(t)\|_{L^p} \le \|f(t)\|_{L^\infty}^{1-\frac{1}{p}} \|f(t)\|_{L^1}^{\frac{1}{p}} \le (C_{1\infty} t^{-d/2} \|f_0\|_{L^1})^{1-\frac{1}{p}} \|f_0\|_{L^1}^{\frac{1}{p}}$$

which is nothing but the estimate (1.5) for p and $q = 1$. For $q \in (1,\infty)$, using that last estimate with $p := q'$ on the dual backward problem and repeating the duality argument, we have

$$\|f(T)\|_{L^\infty} \le \sup_{\|\phi_T\|_{L^1}\le 1} \|f_0\|_{L^q} \|\phi(0)\|_{L^{q'}} \le C_{1q'} T^{-\frac{d}{2}(1-\frac{1}{q'})} \|f_0\|_{L^q},$$

what is nothing but the estimate (1.5) for $p = \infty$ and q. Finally, for $1 < q < p < \infty$, we use the already proved estimates and the Holder inequality in order to get

$$\|f(t)\|_{L^p} \le \|f(t)\|_{L^\infty}^{1-\frac{q}{p}} \|f(t)\|_{L^q}^{\frac{q}{p}} \le (C_{q\infty} t^{-\frac{d}{2}\frac{1}{q}})^{1-\frac{q}{p}} \|f_0\|_{L^q},$$

what is nothing but the estimate (1.5).

2.2. **An alternative Nash proof.** Anticipating with the most classical De Giorgi-Moser approach that we will develop in the next sections, we present here an alternative proof which mixes some arguments coming from Nash argument (the non expansion estimate in any L^r spaces) and one coming from the De Giorgi-Moser approach (the use of the Sobolev inequality). For simplicity, we assume here $d \ge 3$ (in order to be able to use the simplest version of the Sobolev inequality).

Multiplying the equation (2.4) by φ^2, with $0 \le \varphi \in C^1([0,T])$, $\varphi(0) = \varphi(T) = 0$, and integration in time, we find

$$2\int_0^T \varphi^2 \int |\nabla f|^2 = \int_0^T (\varphi^2)' \int f^2.$$

Using the Sobolev inequality

$$\|f\|_{L^{2^*}} \le C_S \|\nabla f\|_{L^2}, \quad \frac{1}{2^*} = \frac{1}{2} - \frac{1}{d}, \tag{2.8}$$

we deduce

$$\|\varphi f\|^2_{L^2(0,T;L^{2^*})} \lesssim \int_0^T \varphi (\varphi')_+ \|f\|^2_{L^2} dt. \tag{2.9}$$

We observe that

$$\begin{aligned} \int_0^T \|f\|^2_{L^2} \varphi \varphi'_+ \, dt &\le \int_0^T \|f\|^{2(1-\theta)}_{L^1} \varphi'_+ \varphi^{1-2\theta} \|f\|^{2\theta}_{L^{2^*}} \varphi^{2\theta} dt \\ &\le \Big(\int_0^T \|f\|^2_{L^1} (\varphi'_+ \varphi^{1-2\theta})^{\frac{1}{1-\theta}} dt \Big)^{1-\theta} \Big(\int_0^T \|f\|^2_{L^{2^*}} \varphi^2 dt \Big)^{\theta}, \end{aligned}$$

where we have used the interpolation inequality with $1/2 = 1 - \theta + \theta/2^*$ in the first line and the Holder inequality in the second line. Using (2.9) in order to bound the last term and simplifying both sides of the inequality, we obtain

$$\int_0^T \|f\|^2_{L^2} \varphi \varphi'_+ \, dt \lesssim \int_0^T \|f\|^2_{L^1} (\varphi'_+ \varphi^{1-2\theta})^{\frac{1}{1-\theta}} dt.$$

By using the decay of the L^p norms (1.11) with $p = 2$ and $p = 1$, we have

$$\|f_T\|_{L^2} \le \|f_t\|_{L^2}, \quad \|f_t\|_{L^1} \le \|f_0\|_{L^1}, \quad \forall\, t \in [0, T],$$

and together with the last estimate, we obtain

$$A_\varphi(T) \|f_T\|^2_{L^2} \le B_\varphi(T) \|f_0\|^2_{L^1},$$

where

$$A_\varphi(T) := \int_0^T \varphi \varphi'_+ \, dt, \quad B_\varphi(T) := \int_0^T (\varphi'_+ \varphi^{1-2\theta})^{\frac{1}{1-\theta}} dt.$$

Choosing $\varphi(t) := \varphi_0(t/T)$ and performing the change of variable $s := t/T$, we easily compute

$$A_\varphi(T) \;=\; \int_0^1 \varphi_0 \varphi'_{0+} ds.$$

Using that $\theta = d/(d+2)$, we also have

$$\begin{aligned} B_\varphi(T) &= \int_0^T (\varphi'_+)^{\frac{d+2}{2}} \varphi^{\frac{2-d}{d}} dt. \\ &= T^{-d/2} \int_0^1 (\varphi'_0)_+^{\frac{d+2}{2}} \varphi_0^{\frac{2-d}{d}} dt. \end{aligned}$$

The last term is finite when $\varphi_0(s) = s^a(1-s)^a$, with $s^{(a-1)\frac{d+2}{2} + a\frac{2-d}{d}} \in L^1(0,1)$ what is the case when $a > (2/d)/(d/2 + 2/d)$. All together, we have established

$$\|f_T\|^2_{L^2} \lesssim T^{-d/2} \|f_0\|^2_{L^1}.$$

That is again the decay estimate (1.5) with $p = 2$ and $q = 1$. Taking advantage of that last estimate, we are able to obtain (1.5) for the full range of exponents $1 \le q < p \le \infty$ by proceeding exactly as in Section **??**.

2.3. **Energy estimate and Moser iterative argument.** Let us consider again a solution f to the heat equation (2.2). We integrate in time equation (1.8) in order to get

$$\frac{1}{2}\int f_t^2 + \int_s^t \int |\nabla f|^2 = \frac{1}{2}\int f_s^2,$$

for any $0 < s < t$. We fix $0 < t_0 < t_1 < t < T$ and we integrate in $s \in (t_0, t_1)$ the above equation. We obtain

$$(t_1 - t_0)\int f_t^2 + 2(t_1 - t_0)\int_{t_1}^t \int |\nabla f|^2 \le \int_{t_0}^T \int f^2.$$

Taking the supremum in $t \in (t_1, T)$ of both terms at the RHS, we deduce

$$\sup_{[t_1,T]} \int f_t^2 + 2\int_{t_1}^T \int |\nabla f|^2 \le \frac{2}{t_1 - t_0}\int_{t_0}^T \int f^2, \tag{2.10}$$

for any $0 \le t_0 < t_1 \le T$. Using the Sobolev inequality (2.8), we have proved

$$\|f\|^2_{L^\infty(I_1;L^2)} \le \frac{1}{t_1 - t_0}\|f\|^2_{L^2(I_0;L^2)},$$
$$\|f\|^2_{L^2(I_1;L^{2^*})} \le \frac{C_S^2}{2}\frac{1}{t_1 - t_0}\|f\|^2_{L^2(I_0;L^2)},$$

where $I_i := [t_i, T]$. We now recall the interpolation inequality

$$\|g\|_{L^{q_\theta}L^{r_\theta}} \le \|g\|^\theta_{L^{q_0}L^{r_0}}\|g\|^{1-\theta}_{L^{q_1}L^{r_1}}, \tag{2.11}$$

where

$$\frac{1}{q_\theta} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}, \quad \frac{1}{r_\theta} = \frac{\theta}{r_0} + \frac{1-\theta}{r_1}, \quad \theta \in [0,1],$$

which proof is left as an exercise. Using this interpolation inequality with θ such that

$$\frac{1}{p} := \frac{1-\theta}{2} = \frac{\theta}{2} + \frac{1-\theta}{2^*},$$

we deduce

$$\|f\|^2_{L^p(I_1;L^p)} \le \frac{C}{t_1 - t_0}\|f\|^2_{L^2(I_0;L^2)}, \quad p := 2(1 + 2/d). \tag{2.12}$$

In fact, for a subsolution $g \ge 0$, we may repeat the above argument, and we get in the same manner

$$\|g\|^2_{L^p(\mathcal{U}_{k+1})} \le C\frac{1}{t_{k+1} - t_k}\|g\|^2_{L^2(\mathcal{U}_k)}, \tag{2.13}$$

with $\mathcal{U}_k := I_k \times \mathbb{R}^d$, $I_k := (t_k, T]$ and $0 \le t_k < t_{k+1} < T$.
We consider now a solution $f \ge 0$ to the heat equation and we define

$$t_k := \frac{T}{2} - \frac{T}{2^k}, \quad k \ge 1, \quad p_{k+1} := (1 + 2/d)p_k, \quad k \ge 1, \quad p_1 := 2.$$

Because $p_k/2 \ge 1$, and thus $s \mapsto |s|^{p_k/2}$ is convex, and because of (1.10), the function $g := f^{p_k/2}$ is a subsolution. Applying (2.13) to this function g, we obtain

$$\begin{aligned} \|f\|_{L^{p_{k+1}}(\mathcal{U}_{k+1})} &= \|f^{p_k/2}\|^{2/p_k}_{L^p(\mathcal{U}_{k+1})} \\ &\le \big(C\frac{2^k}{T}\|f^{p_k/2}\|^2_{L^2(\mathcal{U}_k)}\big)^{1/p_k} = \big(C\frac{2^k}{T}\big)^{1/p_k}\|f\|_{L^{p_k}(\mathcal{U}_k)}. \end{aligned}$$

Observing that

$$\sum_{k=1}^{\infty} \frac{1}{p_k} = \frac{1}{2} \sum_{j=0}^{\infty} \frac{1}{(1+2/d)^j} = \frac{1}{2} + \frac{d}{4},$$

we deduce that

$$\prod_{k=1}^{\infty} \big(C \frac{2^k}{T}\big)^{1/p_k} \lesssim T^{-1/2-d/4}.$$

As a consequence, we have

$$\begin{aligned} \|f\|_{L^\infty(\mathcal{U}_\infty)} &\le \liminf_{k\to\infty} \|f\|_{L^{p_k}(\mathcal{U}_k)} \\ &\le \liminf_{k\to\infty} \prod_{j=1}^{k} \big(C \frac{2^j}{T}\big)^{1/p_j} \|f\|_{L^{p_1}(\mathcal{U}_1)} \end{aligned}$$

and thus

$$\|f\|_{L^\infty(\mathcal{U}_\infty)} \lesssim T^{-1/2-d/4} \|f\|_{L^2(\mathcal{U}_1)}. \tag{2.14}$$

Finally, together with the decay of the L^2 norm (1.8) which implies

$$\|f\|_{L^2(\mathcal{U}_1)} \le T^{1/2} \|f_0\|_{L^2},$$

we have thus established

$$\|f_T\|_{L^\infty} \lesssim \frac{1}{T^{d/4}} \|f_0\|_{L^2}. \tag{2.15}$$

Estimate (2.15) is the dual estimate of (2.6). We may thus end the proof of the full range estimate (1.5) by arguing by duality and interpolation exactly as in Section **??**.

2.4. **Other methods.** We may prove the same kind of estimates by considering the evolution of $\beta(f)$ with other choices of the convex (or concave) function $\beta : \mathbb{R} \to \mathbb{R}$. More precisely, the following choices are possible :

- De Giorgi method: $\beta(s) := (s-c)_+^2$, $\forall\, c \ge 0$;
- Moser alternative method : $\beta(s) := |s|^p$, $\forall\, p \ne 0, 1$;
- Boccardo-Gallouet method : β such that $\beta(0) = 0$ and $\beta''(s) := \mathbf{1}_{M<|s|<M+1}$, $\forall\, M \ge 0$.

The proofs are based in a fundamental way on the choice of the nonlinear function β, on some suitable interpolation arguments and on the differnentiation or not of the integrability norm in both time and position variables.

3. The Harnack inequality and the Holder estimate

The aim of this section is to present two important properties of general parabolic equations: the Harnack inequality which is somehow a quantitative version of the strong maximum principle and a regularity result which is written in terms of an Holder estimate. We begin by establishing some local versions of the gain of uniform boundedness, in the spirit of the ultracontractivity estimate which is a global version of the same property, and by establishing several other technical but fundamental local properties of subsolutions and supersolutions.

3.1. **The first De Giorgi Lemma.** In this section and the next one, we are concerned with nonnegative subsolutions f to the parabolic equation

$$\partial_t f = \operatorname{div}(A\nabla f) + b\cdot\nabla f \tag{3.1}$$

in a cylinder $\mathbf{Q} \subset \mathbb{R}^{d+1}$, for which we will be able to establish *a priori upper bounds* in a smaller cylinder $\mathbf{q} \subset \mathbf{Q}$. More precisely, we will consider a nonnegative function f satisfying

$$\partial_t f \le \operatorname{div}(A\nabla f) + b\cdot\nabla f \tag{3.2}$$

in a cylinder $\mathbf{Q} \subset \mathbb{R}^{d+1}$, we will not bother with the regularity of f for justifying the computations and for simplicity we will only consider the case $b = 0$ and from time to time the case $A = I$. The generalization to a general parabolic equation starting from the variational formulation is not difficult and it is left as an exercice. When necessary, we introduce the notation $\mathbf{Q} := \mathfrak{C}_r(z_0)$, where for $r > 0$ and $z_0 := (t_0, x_0) \in \mathbb{R}\times\mathbb{R}^d$, we define

$$\mathfrak{C}_r(z_0) := z_0 + \mathfrak{C}_r, \quad \mathfrak{C}_r = \mathfrak{C}_r(0) := (-r^2, 0)\times\mathbb{B}_r. \tag{3.3}$$

We recall again that if f is a solution to (3.1) and β is a convex function then $\beta(f)$ is a subsolution. Similarly, if f is a solution to (3.1) and β is an increasing convex function then $\beta(f)$ is a subsolution.

We start adapting the energy estimate (2.10) in a localized framework. Consider a nonnegative subsolution f to the heat equation (3.1) (with $A = I$ and $b = 0$). Multiplying the equation (3.1) by $f\phi^2$ for $0 \le \phi \in C^1_c(\mathbf{Q})$, $\mathbf{Q} := (T_0, T)\times\mathbb{B}_R$, and integrating in the space and time variables, we obtain

$$\begin{aligned} &\frac{1}{2}\|f\phi(t)\|^2_{L^2} - \frac{1}{2}\|f\phi(s)\|^2_{L^2} \\ &\le -\int_s^t\int \nabla f\cdot\nabla(f\phi^2)\,dxd\tau \\ &= -\int_s^t \|\phi\nabla f\|^2_{L^2}\,d\tau + \int_s^t\int f\phi\nabla f\cdot\nabla\phi\,dxd\tau, \end{aligned}$$

for any $T_0 \le s < t \le T$. Choosing now $0 < r < R$, $\phi(x) = \phi_0(|x|)$, $\phi_0(0) = 1$ and $\phi_0' = -(R-r)^{-1}\mathbf{1}_{[r,R]}$ on $\mathbb{R}_+$, we deduce

$$\begin{aligned} &\int_{\mathbb{B}_r} f(t)^2 dx + \int_s^t\int_{\mathbb{B}_r} |\nabla f|^2\,dxd\tau \\ &\le \int_{\mathbb{B}_R} f(s)^2 dx + \frac{1}{(R-r)^2}\int_s^t\int_{\mathbb{B}_R} f^2\,dxd\tau. \end{aligned}$$

Taking the mean value in $s \in (t_0, t_1)$ with $t_0 < t_1 < t$, we have the first estimate

$$\begin{aligned} &(t_1 - t_0)\int_{\mathbb{B}_r} f(t)^2 dx + (t_1 - t_0)\int_{t_1}^t\int_{\mathbb{B}_r} |\nabla f|^2\,dxds \\ &\le \Big(1 + \frac{t - t_0}{(R-r)^2}\Big)\int_{t_0}^t\int_{\mathbb{B}_R} f^2\,dxds. \end{aligned}$$

Using the Sobolev inequality and the interpolation inequality (2.11), we also get

$$\|f\|^2_{L^p((t_1,T)\times\mathbb{B}_r)} \le C\Big(\frac{1}{t_1 - t_0} + \frac{1}{(R-r)^2}\Big)\|f\|^2_{L^2((t_0,T)\times\mathbb{B}_R)}, \tag{3.4}$$

with $p := 2(1 + 2/d)$.

We express the two above estimates into the following more formalized form. For two cylinders $\mathbf{Q}_i := (a_i, b_i) \times B_i$, we write $\mathbf{Q}_0 \prec \mathbf{Q}_1$ if $a_1 < a_0$, $b_1 \geq b_0$ and $B_0 \subset\subset B_1$.

Lemma 3.1 (Energy estimate)**.** *Let f be a nonnegative subsolution to the parabolic equation* (3.1) *on a cylinder $\mathbf{Q}$ and let $\mathbf{q}$ be another cylinder $\mathbf{q} \prec \mathbf{Q}$. There hold*

$$\|\nabla_x f\|^2_{L^2(\mathbf{q})} \leq C(\mathbf{q}, \mathbf{Q}) \|f\|^2_{L^2(\mathbf{Q})}, \tag{3.5}$$

$$\|f\|^2_{L^p(\mathbf{q})} \leq C(\mathbf{q}, \mathbf{Q}) \|f\|^2_{L^2(\mathbf{Q})}, \quad p := 2(1 + 2/d), \tag{3.6}$$

with $C(\mathbf{q}, \mathbf{Q}) := C((t_1 - t_0)^{-1} + (R - r)^{-2})$ when $\mathbf{Q} := (t_0, T) \times \mathbb{B}_R$, $\mathbf{q} := (t_1, T) \times \mathbb{B}_r$, $0 < r < R$ and $t_0 < t_1 < T$.

We establish now the local gain of uniform boundedness for nonnegative subsolution f to the parabolic equation (1.1) set in a cylinder.

Lemma 3.2 (first De Giorgi lemma)**.** *Let f be a nonnegative subsolution to the parabolic equation* (3.1) *in $\mathfrak{C}_2$. There holds*

$$\|f\|_{L^\infty(\mathfrak{C}_1)} \leq 1/2 \quad \text{if} \quad \|f\|_{L^2(\mathfrak{C}_2)} \leq \delta_{DG},$$

for some constant $\delta_{DG} > 0$ which only depends on the dimension $d \geq 3$.

Remark 3.3. *An alternative and equivalent formulation is that any nonnegative subsolution f to the parabolic equation* (3.1) *satisfies*

$$\|f\|_{L^\infty(\mathfrak{C}_1)} \leq C_{DG} \|f\|_{L^2(\mathfrak{C}_2)}, \tag{3.7}$$

for the constant $C_{DG} := 2/\delta_{DG}$.

Proof of Lemma 3.2. We repeat the Moser iterative argument presented in Section 2.3 and we rather prove (3.7) in the cylinders $\mathfrak{C}_1$ and $\mathfrak{C}_{1/2}$. More precisely, we defined the sequence of (increasing) times, (decreasing) radius and (decreasing) cylinders

$$T_k := -\tfrac{1}{2}(1 + 2^{-k}), \quad r_k := \tfrac{1}{2}(1 + 2^{-k}), \quad \mathcal{U}_k := (T_k, 0) \times \mathbb{B}_{r_k},$$

so that, using (3.4), the local version counterpart of (2.13) is that any nonnegative subsolution g to the heat equation satisfies

$$\|g\|^2_{L^p(\mathcal{U}_{k+1})} \leq C 2^{2k} \|g\|^2_{L^2(\mathcal{U}_k)}, \quad \forall\, k \geq 1.$$

We then conclude the proof in the same manner as for proving (2.14). □

We improve the first De Giorgi lemma by lowering the integrability power at the RHS of estimate (3.7).

Lemma 3.4 (Upper bound)**.** *Let f be a nonnegative subsolution to the parabolic equation* (3.1) *in $\mathfrak{C}_R$, $R \leq 1$. For any $r \in (0, R)$ and $q > 0$, there holds*

$$\|f\|_{L^\infty(\mathfrak{C}_r)} \lesssim \|f\|_{L^q(\mathfrak{C}_R)}. \tag{3.8}$$

Proof of Lemma 3.4. Choosing $(t_0, T) \times B_r := \mathfrak{C}_r$ and $(t_1, T) \times B_r := \mathfrak{C}_R$ in (3.4), we find

$$\|f\|^2_{L^p(\mathfrak{C}_r)} \lesssim (R - r)^{-2} \|f\|^2_{L^2(\mathfrak{C}_R)}, \tag{3.9}$$

Repeating the proof of Lemma 3.2 with a suitable choice of (r_k) and tracking the dependence in $R-r$ similarly as we have tracked the dependence in T during the proof of (2.14), we find

$$\|f\|_{L^\infty(\mathfrak{C}_r)} \lesssim (R-r)^{-D}\|f\|_{L^2(\mathfrak{C}_R)},$$

with $D = d/2+1$. That ends the proof estimate (3.4) for $q \ge 2$. For (3.4) when $q \in (0,2)$, we define

$$A(r) := \|f\|_{L^\infty(\mathfrak{C}_r)}$$

and thanks to the Young inequality the above estimate writes

$$\begin{aligned} A(r) &\le C(R-r)^{-D}\|f\|_{L^2(\mathfrak{C}_R)} \\ &\le \frac{1}{2}A(R) + C(R-r)^{-D/q}\|f\|_{L^q(\mathfrak{C}_R)}. \end{aligned}$$

Defining $r_0 := r$ and next $r_n := r_{n-1} + \varepsilon n^{-2}$ with $\varepsilon := (R-r)(\sum n^{-2})^{-1}$, so that $r_n \nearrow R$. Applying the previous estimate with $r = r_n$ and $R = r_{n+1}$, we get

$$A(r_n) \le \frac{1}{2}A(r_{n+1}) + Cn^{2D/q}(R-r)^{-D/q}\|f\|_{L^q(\mathfrak{C}_R)},$$

and summing up

$$A(r) \le \frac{1}{2^n}A(R) + \sum_{k=1}^n C\frac{k^{2D/q}}{2^k}(R-r)^{-D/q}\|f\|_{L^q(\mathfrak{C}_R)}.$$

Passing to the limit, we deduce

$$\|f\|_{L^\infty(\mathfrak{C}_r)} \le \Big(\sum_{k=1}^\infty C\frac{k^{2D/q}}{2^k}\Big)(R-r)^{-D/q}\|f\|_{L^q(\mathfrak{C}_R)},$$

what is the announced estimate. □

3.2. **Some intermediate estimates.** In the two first results, we establish some estimates for a (nonnegative) subsolution g on a cylinder $\mathbf{Q}$. The first step one is a variante of the Poincaré-Wirtinger inequality.

Lemma 3.5 (Parabolic Poincaré-Wirtinger inequality)**.** *Consider a subsolution g to the parabolic equation* (3.1) *on a cylinder $\mathbf{Q}$. For any cylinders $\mathbf{Q}_i := I_i \times B_i \subset \mathbf{Q}$ with $\sup I_0 \le \inf I_1$, any function $0 \le \varphi \in C^1_c(B_0)$ such that $\|\varphi\|_{L^1(\mathbf{Q}_0)} = 1$, we have*

$$\|(g - \langle g\varphi\rangle_{\mathbf{Q}_0})_+\|_{L^1(\mathbf{Q}_1)} \lesssim \|\nabla_x g\|_{L^1(\mathbf{Q})}, \tag{3.10}$$

where we use the notation $\langle u\rangle_Q := \int_Q u dxdt$.

Proof of Lemma 3.5. We compute

$$\begin{aligned} \|(g - \langle\!\langle g\varphi\rangle\!\rangle_{\mathbf{Q}_0})_+\|_{L^1(\mathbf{Q}_1)} \le \int_{\mathbf{Q}_1}\Big(\int_{\mathbf{Q}_0}(g(t,x)-g(t,y))\varphi(y)dsdy\Big)_+ dtdx \\ + \int_{\mathbf{Q}_1}\Big(\int_{\mathbf{Q}_0}(g(t,y)-g(s,y))\varphi(y)dsdy\Big)_+ dtdx =: T_1 + T_2, \end{aligned}$$

and we estimate each term separately. For the first term, we write

$$\begin{aligned} T_1 &\le |I_0|\,\|\varphi\|_\infty \int_{\mathbf{Q}_1}\int_{B_0}|g(t,x)-g(t,y)|dydtdx \\ &\le |I_0|\,\|\varphi\|_\infty \int_{\mathbf{Q}_1}\int_{B_0}\int_0^1 |\nabla_x g(t,\tau x + (1-\tau)y)||x-y|d\tau dydtdx. \end{aligned}$$

We observe that, for any $t \in I_1$,

$$\begin{aligned} &\int_{B_1}\int_{B_0}\int_0^{1/2} |\nabla_x g(t,\tau x+(1-\tau)y)| d\tau dy dx \\ &\le \int_B\int_B\int_0^{1/2} |\nabla_x g(t,\tau x+(1-\tau)y)| d\tau dy dx \\ &\le \int_B\int_B\int_0^{1/2} |\nabla_x g(t,z)| \frac{dz}{(1-\tau)^d} d\tau dx \\ &\le 2^{d-1}|B| \int_B |\nabla_x g(t,z)| dz, \end{aligned}$$

where we have used the change of variables $y \mapsto z := \tau x + (1-\tau)y$. Denoting $B = \mathbb{B}_R$ and performing the same kind of estimate for the integral on $\tau \in [1/2, 1]$, but with the change of variables $x \mapsto z := \tau x + (1-\tau)y$, and summing up the two contributions, we conclude with

$$T_1 \le |I_0| 2^{d+1} |B| R \int_{\mathbf{Q}_1} |\nabla_x g(t,x)| dx dt.$$

For the second term, we compute

$$\begin{aligned} T_2 &= \int_{I_1}\int_{B_1}\Big(\int_{\mathbf{Q}_0}\int_s^t (\partial_\tau g)(\tau,y)\varphi(y) d\tau ds dy\Big)_+ dx dt \\ &\le |B_1| \int_{I_1}\Big(\int_{\mathbf{Q}_0}\int_s^t \nabla g(\tau,y)\cdot\nabla\varphi(y) d\tau ds dy\Big)_+ dt, \end{aligned}$$

by using precisely the fact that $s \le t$ for any $s \in I_0$, $t \in I_1$ and the fact that g is a subsolution on (s,t). We then have

$$\begin{aligned} T_2 &\le |B_1| \int_{I_1}\Big(\int_{\mathbf{Q}_0}\int_s^t \nabla g(\tau,y)\cdot\nabla\varphi(y) d\tau ds dy\Big)_+ dt, \\ &\le |B_1||I_0||I_1|\|\nabla\varphi\|_\infty \int_{I_0\cup I_1}\int_{B_0} |\nabla g|(\tau,y) d\tau dy. \end{aligned}$$

We conclude by gathering the three above estimates. □

In a second step, we establish a variant of the De Giorgi isoperimetric inequality, also named as *intermediate value inequality.*

Lemma 3.6 (Intermediate value inequality)**.** *Consider a subsolution f to the parabolic equation* (3.1) *on $\mathbf{Q} := I \times B$ such that $f \le 1$ on Q and some cylinders $(\mathbf{Q}_i)_{i=0,1,2}$, with $\mathbf{Q}_i := I_i \times B_i \subset \mathbf{Q}_2 \prec \mathbf{Q}$ for $i = 0,1$ and $\sup I_0 \le \inf I_1$. For any $\delta > 0$, $\delta_1 > 0$ and $C_0 > 0$, there exist $\vartheta \in (0,1)$ and $\eta > 0$ such that*

$$\|\nabla f_+\|_{L^2(\mathbf{Q}_2)} \le C_0, \quad |\{f \ge 1-\vartheta\} \cap \mathbf{Q}_1| \ge \delta_1|\mathbf{Q}_1|, \quad |\{f \le 0\} \cap \mathbf{Q}_0| \ge \delta|\mathbf{Q}_0|$$

imply

$$|\{0 < f < 1-\vartheta\} \cap \mathbf{Q}_2| \ge \eta.$$

Proof of Lemma 3.5. We have $B_0 = \mathbb{B}_{r_0}$ for some $r_0 > 0$. For any $\varepsilon > 0$, we choose $\psi \in C^1_c(\mathbb{R}^d)$ such that $\mathbf{1}_{\mathbb{B}_{(1-\varepsilon)r_0}} \le \psi \le \mathbf{1}_{B_0}$ and we set $\varphi := \psi / \langle\psi\rangle_{\mathbf{Q}_0}$. We observe that

$$\langle f_+\varphi\rangle_{\mathbf{Q}_0} := \frac{1}{\langle\psi\rangle_{\mathbf{Q}_0}} \int_{\mathbf{Q}_0} f_+\psi \le \frac{1}{(1-\varepsilon)^d}\frac{1}{|\mathbf{Q}_0|}\int_{\mathbf{Q}_0} f_+,$$

with

$$\frac{1}{|\mathbf{Q}_0|}\int_{\mathbf{Q}_0} f_+ \quad \leq \quad \frac{|\{f>0\}\cap \mathbf{Q}_0|}{|\mathbf{Q}_0|} = 1 - \frac{|\{f\leq 0\}\cap \mathbf{Q}_0|}{|\mathbf{Q}_0|} \leq 1-\delta.$$

Choosing $\varepsilon > 0$ small enough, we get

$$\langle f_+ \varphi\rangle_{\mathbf{Q}_0} \leq 1 - \delta/2,$$

from what we deduce that

$$\begin{aligned} ⨍_{\mathbf{Q}_1} (f_+ - \langle f_+\varphi\rangle_{\mathbf{Q}_0})_+ \quad &\geq \quad \frac{1}{|Q_1|}\int_{\mathbf{Q}_1} \mathbf{1}_{f\geq 1-\vartheta}(f-(1-\frac{\delta}{2})) \\ &\geq \quad (\frac{\delta}{2}-\vartheta)\frac{1}{|\mathbf{Q}_1|}|\{f\geq 1-\vartheta\}\cap Q_1| \geq (\frac{\delta}{2}-\vartheta)\delta_1, \end{aligned}$$

for any $\vartheta \in (0,\delta/2)$. Applied to the subsolution $g := f_+$, the parabolic Poincaré-Wirtinger inequality (3.10) implies

$$⨍_{\mathbf{Q}_1} (f_+ - \langle f_+\varphi\rangle_{\mathbf{Q}_0})_+ \quad \leq \quad C_1 \int_{\mathbf{Q}_2} |\nabla f_+|.$$

On the one hand, we have

$$C_1 \int_{\mathbf{Q}_2} |\nabla f_+| \mathbf{1}_{0<f<1-\vartheta} \leq C_1 \|\nabla f_+\|_{L^2(\mathbf{Q}_2)} |\{0<f<1-\vartheta\}\cap \mathbf{Q}_2|^{1/2},$$

by using the Cauchy-Schwarz inequality. On the other hand, we have

$$\begin{aligned} C_1 \int_{\mathbf{Q}_2} |\nabla f_+| \mathbf{1}_{f>1-\vartheta} \quad &\leq \quad C_1 \|\nabla (f-(1-\vartheta))_+\|_{L^1(\mathbf{Q}_2)} \\ &\leq \quad C_1 |\mathbf{Q}_2|^{1/2} \|\nabla (f-(1-\vartheta))_+\|_{L^2(\mathbf{Q}_2)} \\ &\leq \quad C_2 \|(f-(1-\vartheta))_+\|_{L^2(\mathbf{Q})} \\ &\leq \quad C_2 |\mathbf{Q}|\vartheta, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the second line, the energy estimate (3.5) on the subsolution $(f-(1-\vartheta))_+$ in the third line and the upper bound condition on f in the last one.
Altogether, we have

$$C_1 \|\nabla f_+\|_{L^2} |\{0<f<1-\vartheta\}\cap \mathbf{Q}_2|^{1/2} \quad \geq \quad (\frac{\delta}{2}-\vartheta)\delta_1 - C_2|Q|\vartheta \geq \frac{\delta\delta_1}{8},$$

by choosing $\vartheta > 0$ such that $\vartheta \leq \delta/4$ and $\vartheta \leq \delta\delta_1/(4C_2|Q|)$. We conclude by setting $\eta := \delta\delta_1/(8C_0C_1)$. □

In our last intermediate result, gathering the first De Giorgi Lemma 3.2 and the intermediate value Lemma 3.6, we establish a lowering of the maximum lemma, also named as *measure-to-pointwise estimate* or second De Giorgi Lemma. It is worth emphasizing here that it is fundamental that this estimate is uniform with respect to the size of the cylinders and that it is true because the estimates are invariant by scaling. For a cylinder $\mathfrak{C}_r(z_0)$ as defined in (3.3), we set

$$\mathfrak{C}_r^-(z_0) := z_0 + (-2r^2, -r^2)\times \mathbb{B}_r, \quad \mathfrak{C}_r^+(z_0) := z_0 + (0, r^2)\times \mathbb{B}_r,$$

and we use the shorthand $\mathfrak{C}_r^\pm := \mathfrak{C}_r^\pm(0)$.

Lemma 3.7 (Measure-to-pointwise estimate). *For any $\delta \in (0,1)$, there exists a constant $\lambda \in (0,1)$ such that for any $r > 0$ and any subsolution g to the parabolic equation* (3.1) *in $\mathfrak{C}_{4r}$ satisfying $g \le 1$ on $\mathfrak{C}_{4r}$, we have*

$$|\{g \le 0\} \cap \mathfrak{C}_r^-| \ge \delta |\mathfrak{C}_r^-| \quad \textit{implies} \quad g < 1 - \lambda \quad \textit{on} \quad \mathfrak{C}_r. \tag{3.11}$$

Proof of Lemma 3.7. Step 1. We assume $r = 1$. We set $C_0 := C(\mathfrak{C}_2, \mathfrak{C}_4)|\mathfrak{C}_4|$, with $C(\mathfrak{C}_2, \mathfrak{C}_4)$ defined in the Energy estimate lemma 3.1, $\delta_1 := \delta_{DG}^2/|\mathfrak{C}_2|$, with δ_{DG} defined in the first De Giorgi lemma 3.2, and we denote by $\vartheta, \eta \in (0,1)$ the constant defined in the *intermediate value inequality* as stated in lemma 3.6 associated to δ, δ_1 and C_0. We define the sequence

$$g_k := \vartheta^{-k}[g - (1 - \vartheta^k)] = \vartheta g_{k-1} + (1 - \vartheta),$$

so that

$$g_{k+1} \le 1 \quad \text{and} \quad \{g_{k+1} \ge 0\} = \{g_k \ge 1 - \vartheta\}. \tag{3.12}$$

In particular, from the first above estimate and Lemma 3.1, we know that

$$\|\nabla g_k\|_{L^2(\mathfrak{C}_2)} \le C(\mathfrak{C}_2, \mathfrak{C}_4)\|g_k\|_{L^2(\mathfrak{C}_4)} \le C(\mathfrak{C}_2, \mathfrak{C}_4)|\mathfrak{C}_4|^{1/2} =: C_0.$$

From the very definition of (g_k) and the hypothesis, we have

$$|\{g_k \le 0\} \cap \mathfrak{C}_1^-| \ge |\{g \le 0\} \cap \mathfrak{C}_1^-| \ge \delta |\mathfrak{C}_1^-|.$$

We next assume that for some $k_0 \ge 0$ and any $k \in \{0, \cdots, k_0\}$, we have

$$\int_{\mathfrak{C}_2} (g_{k+1})_+^2 \, dxdt > \delta_{DG}^2. \tag{3.13}$$

Under this condition, for any $k \in \{0, \cdots, k_0\}$, we have thus

$$|\{g_k \ge 1 - \vartheta\} \cap \mathfrak{C}_2| = |\{g_{k+1} \ge 0\} \cap \mathfrak{C}_2| \ge \int_{\mathfrak{C}_2} (g_{k+1})_+^2 \, dxdt > \delta_{DG}^2,$$

where we have used (3.12) and the upper bound hypothesis. Applying Lemma 3.6, we know that independently of k

$$|\{0 < g_k < 1 - \lambda\} \cap \mathfrak{C}_2| \ge \eta.$$

Using (3.12) again and repeatedly the above lower bound, we have

$$\begin{aligned} |\mathfrak{C}_2| &\ge |\{g_{k+1} \le 0\} \cap \mathfrak{C}_2| \\ &\ge |\{g_k \le 0\} \cap \mathfrak{C}_2| + |\{0 < g_k < 1 - \vartheta\} \cap \mathfrak{C}_2| \\ &\ge k\eta, \end{aligned}$$

which provides a finite bound on $k_0 \le |\mathfrak{C}_2|\eta^{-1}$. For the first $k = k_0 - 1$ such that (3.13) fails, we have $\|(g_{k_0})_+\|_{L^2(\mathfrak{C}_2)} \le \delta_{DG}$, and thus $g_{k_0} \le 1/2$ in $\mathfrak{C}_1$ from the first De Giorgi Lemma 3.2. Rescaling back to g gives the result with $\lambda := \vartheta^{k_0}/3$.

Step 2. We now consider the general case $r > 0$, which proof comes from a mere scaling argument. Defining $g_r(t,x) := g(r^2 t, rx)$ and $A_r(x) := A(rx)$ for $(t,x) \in \mathfrak{C}_4$, we observe that A_r satisfies the same ellipticity and boundedness conditions as A and that g_r is a (weak) subsolution to the parabolic equation (3.1) associated to A_r on the cylinder $\mathfrak{C}_4$. The condition in (3.11) translates into $|\{g_r \le 0\} \cap \mathfrak{C}_1^-| \ge \delta |\mathfrak{C}_1^-|$ and the first step implies $g_r \le 1 - \lambda$ on $\mathfrak{C}_1$, or equivalently, $g \le 1 - \lambda$ on $\mathfrak{C}_r$. □

3.3. **The Harnack inequality.** We start recalling the classical Vitali Lemma and a variant of the classical Lebesgue Theorem.

Lemma 3.8 (Vitali)**.** *Consider a family $\mathcal{F}$ of cylinders of size bounded by a same constant. There exists an at most countable family $\mathcal{D} \subset \mathcal{F}$ of disjoints cylinders such that*

$$\bigcup_{\tilde{\mathbf{q}} \in \mathcal{F}} \tilde{\mathbf{q}} \subset \bigcup_{\tilde{\mathbf{q}} \in \mathcal{D}} \tilde{\mathbf{q}}_5,$$

where here, for $\tilde{\mathbf{q}} := z_0 + (-\frac{1}{2}r^2, \frac{1}{2}r^2) \times \mathbb{B}_r$, $z_0 \in \mathbb{R}^{d+1}$, $r > 0$ and for $a > 0$, we define $\tilde{\mathbf{q}}_a := z_0 + (-\frac{1}{2}(ar)^2, \frac{1}{2}(ar)^2) \times \mathbb{B}_{ar}$.

Proof of Lemma 3.8. We denote by $R > 0$ a common upper bound of the size of the cylinders. For any $n \ge 0$, let us denote by $\mathcal{F}_n$ the subfamily of cylinders of size $r \in]2^{-n-1}R, 2^{-n}R]$. We define recursively a maximal and finite subfamily $\mathcal{D}_n \subset \mathcal{F}_n$ such that the cylinders in $\mathcal{D}_n$ are disjoints and they are disjoints of those of $\mathcal{D}_0, \dots, \mathcal{D}_{n-1}$. In such a way, any cylinder $q' \in \mathcal{F}_n$ intersect at least one cylinder $q \in \mathcal{D}_n$, which implies $[t_0 - r^2, t_0] \cap [t'_0 - (r')^2, t'_0] \neq \emptyset$, $B_r(x_0) \cap B_{r'}(x'_0) \neq \emptyset$ and thus using $r' \le 2r$ and the triangular inequality, we get $q' \subset q_5$. The family $\mathcal{D} := \cup \mathcal{D}_n$ is suitable. ☐

Theorem 3.9 (Lebesgue)**.** *For any locally integrable function f on an open set $\mathcal{U} \subset \mathbb{R}^{d+1}$, there exists $\mathcal{N} \subset \mathcal{U}$ mesurable and negligible such that*

$$\forall\, z \in \mathcal{U} \backslash \mathcal{N}, \quad f(z) = \lim_{r \to 0} \fint_{\mathfrak{C}_r(z)} f(t,y)\,dtdy.$$

We are now in position to state the Harnack inequality between the supremum and the infimum of a solution to a parabolic equation.

Theorem 3.10 (Harnack inequalities)**.** *There exists $q > 0$ such that any nonnegative supersolution f to the parabolic equation in $\mathfrak{C}_8$ satisfies the weak Harnack inequality*

$$\|f\|_{L^q(\mathfrak{C}_1^-)} \lesssim \inf_{\mathfrak{C}_1} f \tag{3.14}$$

and any nonnegative solution f to the parabolic equation in $\mathfrak{C}_{16r}(z_0)$, $r > 0$, satisfies the Harnack inequality

$$\sup_{\mathfrak{C}_r^-(z_0)} f \le C \inf_{\mathfrak{C}_r(z_0)} f, \tag{3.15}$$

for a constant $C \ge 1$ independent of $r > 0$ and $z_0 \in \mathbb{R}^{d+1}$.

Proof of Theorem 3.10. Step 1. We first establish the weak Harnack inequality (3.14). We fix $\delta := 10^{-D-1}$, $D := d+2$, we denote by $\lambda > 0$ the associated constant given by Lemma 3.7 and we set $c := 1/\lambda$. For any $\mathfrak{C}_r^+(z) \subset \mathfrak{C}_1 \cup \mathfrak{C}_1^- \subset \mathfrak{C}_2$, we may apply Lemma 3.7 to the subsolution $g := 1 - f/c$ and we get that

$$|\{f \ge c\} \cap \mathfrak{C}_r(z)| > \delta |\mathfrak{C}_r(z)| \ \text{ implies } \ f > 1 \text{ on } \mathfrak{C}_r^+(z), \tag{3.16}$$

what is the fundamental information we will use. In particular, that tells us

$$\inf_{\mathfrak{C}_1} f \le 1 \ \text{ implies } \ |\{f > c\} \cap \mathfrak{C}_1^-| \le \ \delta |\mathfrak{C}_1^-|. \tag{3.17}$$

We now prove recursively that for any $k \ge 1$, there holds

$$\inf_{\mathfrak{C}_1} f \le 1 \ \text{ implies } \ |\{f > c^k\} \cap \mathfrak{C}_1^-| \le \delta_k |\mathfrak{C}_1^-|, \tag{3.18}$$

with $\delta_k := 10^{-D-k}$. The case $k=1$ is nothing but (3.17). We assume that (3.18) holds up to order $k \ge 1$ and we wish to prove that it holds at order $k+1$. For that purpose, we define

$$A_k := \{f > c^k\} \cap \mathfrak{C}_1^-$$

and we define $\mathcal{F}$ as the family of cylinders $\mathbf{q} = \mathfrak{C}_r(z)$ such that

$$z \in A_{k+1}, \ |A_{k+1} \cap \widetilde{\mathbf{q}}_{10}| \le \delta |\widetilde{\mathbf{q}}_{10}|, \ |A_{k+1} \cap \widetilde{\mathbf{q}}| > \delta |\widetilde{\mathbf{q}}|, \tag{3.19}$$

where we use the notation $\widetilde{\mathbf{q}} = \widetilde{\mathbf{q}}_1$ and $\widetilde{\mathbf{q}}_a := \widetilde{\mathfrak{C}}_{ar}(z)$ as defined in Lemma 3.8. From now on in Steps 2, 3 and 4, we assume

$$\inf_{\mathfrak{C}_1} f \le 1. \tag{3.20}$$

Step 2. A covering argument. We claim that $(\widetilde{\mathbf{q}}_2)_{\mathbf{q}\in\mathcal{F}}$ is covering A_{k+1}. More precisely, recalling that from Theorem 3.9 there exists exists a negligible set $\mathcal{N} \subset \mathfrak{C}_1^-$ such that

$$\forall\, z \in \mathfrak{C}_1^- \backslash \mathcal{N}, \quad \mathbf{1}_{A_{k+1}}(z) = \lim_{r\to 0} \frac{|A_{k+1} \cap \tilde{\mathfrak{C}}_r(z)|}{|\tilde{\mathfrak{C}}_r(z)|},$$

we claim that

$$A_{k+1} \backslash \mathcal{N} \subset \bigcup_{\mathbf{q}\in\mathcal{F}} (\widetilde{\mathbf{q}}_2 \cap A_{k+1}). \tag{3.21}$$

We fix $z \in A_{k+1}\backslash\mathcal{N}$. We denote $R := 10^{1/D-1-k/D}$ and we consider the property

$$\forall\, r \in [R10^{-j}, R10^{1-j}), \quad |A_{k+1} \cap \widetilde{\mathbf{q}}_{10}| \le \delta |\widetilde{\mathbf{q}}_{10}|. \tag{3.22}$$

For any $r \ge R$, we have

$$|A_{k+1} \cap \widetilde{\mathbf{q}}_{10}| \le |A_k \cap \mathfrak{C}_1^-| \le \delta_k |\mathfrak{C}_1^-| = \delta_k (10r)^{-D} |\widetilde{\mathbf{q}}_{10}| \le \delta |\widetilde{\mathbf{q}}_{10}|,$$

where we have successively used the inclusions $A_{k+1} \cap \widetilde{\mathbf{q}}_{10} \subset A_{k+1} \subset A_k \cap \mathfrak{C}_1^-$, the induction property (3.18) and the scaling property $\widetilde{\mathbf{q}}_a = a^D |\mathfrak{C}^-|$, so that the property (3.22) is true for $j=0$. We claim that the property (3.22) cannot be true for any $j \ge 1$. Otherwise, there would exist a sequence $r_j \to 0$ as $j \to \infty$ such that

$$\mathbf{1}_{A_{k+1}}(z) = \lim_{j\to\infty} \frac{|A_{k+1} \cap \tilde{\mathfrak{C}}_{10r_j}(z)|}{|\tilde{\mathfrak{C}}_{10r_j}(z)|} \le \delta < 1,$$

so that $\mathbf{1}_{A_{k+1}}(z) = 0$ and $z \notin A_{k+1}$, what is a contradiction. Choosing $j \ge 1$ the first integer such that (3.22) is not true, we may pick up $r \in [R10^{1-j}, R10^{2-j})$ such that (3.19) holds. We have established (3.21).

Step 3. Proof of (3.18) *at order* $k+1$. Because of Step 2 and of Vitali covering lemma, we have

$$A_{k+1} \backslash \mathcal{N} \subset \bigcup_{\widetilde{\mathbf{q}}\in\mathcal{D}} (\widetilde{\mathbf{q}}_{10} \cap A_{k+1})$$

with $\mathcal{D} \subset \mathcal{F}$ countable and $(\widetilde{\mathbf{q}}_2)_{\mathbf{q}\in\mathcal{D}}$ disjoint sets. By definition $\widetilde{\mathbf{q}}^+ \subset \widetilde{\mathbf{q}}_2$, so that $(\widetilde{\mathbf{q}}^+)_{\mathbf{q}\in\mathcal{D}}$ is also a family of disjoint sets. Because of (3.16) applied to the nonnegative supersolution f/c^k, we have $\widetilde{\mathbf{q}}^+ \subset A_k$ if $\mathbf{q} \in \mathcal{F}$. We compute

$$\begin{aligned} |A_{k+1}| &\le \Big|\bigcup_{\mathbf{q}\in\mathcal{D}} (\widetilde{\mathbf{q}}_{10} \cap A_{k+1})\Big| \le \sum_{\mathbf{q}\in\mathcal{D}} \big|\widetilde{\mathbf{q}}_{10} \cap A_{k+1}\big| \\ &\le \sum_{\mathbf{q}\in\mathcal{D}} \delta\big|\widetilde{\mathbf{q}}_{10}\big| = \sum_{\mathbf{q}\in\mathcal{D}} 10^D \delta \big|\widetilde{\mathbf{q}}^+\big| \\ &= 10^{-1}\Big|\bigcup_{\mathbf{q}\in\mathcal{D}} \widetilde{\mathbf{q}}^+\Big| \le 10^{-1}\big|A_k\big| \\ &\le 10^{-1}\delta_k|\mathfrak{C}_1^-|, \end{aligned}$$

what is nothing but the property (3.18) at order $k+1$. By induction, the property (3.18) is thus true for any $k \ge 1$.

Step 4. Conclusion of (3.14). We write

$$\begin{aligned} \int_{\mathfrak{C}_1^-} f^q &= \int_{\mathfrak{C}_1^-} f^q \mathbf{1}_{\{f \le c\}} + \sum_{k\ge1} \int_{\mathfrak{C}_1^-} f^q \mathbf{1}_{\{c^k < f \le c^{k+1}\}} \\ &\le c^q |\mathfrak{C}_1^-| + \sum_{k\ge1} c^{(k+1)q} |\{f > c^k\} \cap \mathfrak{C}_1^-| \\ &\le c^q|\mathfrak{C}_1^-| + \sum_{k\ge1} c^q 10^{-D} (c^q 10^{-1})^k |\mathfrak{C}_1^-| < \infty, \end{aligned}$$

provided that $c^q 10^{-1} < 1$, what is possible by choosing $q > 0$ small enough. In other words, we have $\|f\|_{L^q(\mathfrak{C}_1)} \le C$, for a constant $C > 0$ depending on f only by the condition (3.20). Applying this estimate to f/ε with $\varepsilon := \inf f$ if $\inf f > 0$ and with $\varepsilon > 0$ arbitrary small if $\inf f = 0$, we finish the proof of (3.14).

Step 5. Proof of (3.15). For a nonnegative solution f to the parabolic equation, we have

$$\|f\|_{L^\infty(\mathfrak{C}^-_{1/2})} \lesssim \|f\|_{L^q(\mathfrak{C}^-_1)} \lesssim \inf_{\mathfrak{C}_1} f \lesssim \inf_{\mathfrak{C}_{1/2}} f,$$

where we have combining the upper bound for nonnegative subsolution provided by Lemma 3.4 and the weak Harnack inequality (3.14) for nonnegative supersolution. We generalize to any $r > 0$ thanks to the scaling invariance of the estimate. □

3.4. **The Holder continuity.** We now establish a Holder continuity result and thus drastically improve the L^∞ estimate provided by the first De Giorgi Lemma.

Theorem 3.11. *Let $f \in X_T$ be be a variational solution to the parabolic equation* (1.1). *There exists $\alpha \in (0,1)$ such that for any $t_0 \in (0,T)$ there holds*

$$f \in C^\alpha((t_0,T)\times\mathbb{R}^d). \tag{3.23}$$

Proof of Theorem 3.11. Step 1. For $h : O \to \mathbb{R}$, we define

$$\mathrm{osc}_O h := \sup_O h - \inf_O h.$$

Assume first f defined in $\mathfrak{C}_1$. We write

$$g := \frac{2}{\mathrm{osc}_{\mathfrak{C}_1} f}\Big(f - \frac12(\sup_{\mathfrak{C}_1} f + \inf_{\mathfrak{C}_1} f)\Big)$$

so that $-1 \le g \le 1$ on $\mathfrak{C}_1$. We have either

$$|\{g \le 0\} \cap \mathfrak{C}^-_{1/4}| \ge |\mathfrak{C}^-_{1/4}|/2 \quad \text{or} \quad |\{g \ge 0\} \cap \mathfrak{C}^-_{1/4}| \ge |\mathfrak{C}^-_{1/4}|/2.$$

In the first case, we apply Lemma 3.7 to g and we deduce $g \le 1 - \lambda$ on $\mathfrak{C}_{1/4}$. In the second case, we apply Lemma 3.7 to $-g$ and we deduce $g \ge -1 + \lambda$ on $\mathfrak{C}_{1/4}$. In both cases, we conclude with $\mathrm{osc}_{\mathfrak{C}_{1/4}} g \le 2 - \lambda$. Hence, we have

$$\mathrm{osc}_{\mathfrak{C}_{1/4}} f \le \vartheta \mathrm{osc}_{\mathfrak{C}_1} f, \quad \vartheta := 1 - \lambda/2.$$

Step 2. We come to the general case and we assume f defined in $\mathcal{U}$. Take $y_0 \in \mathcal{U}$ and $d_0 := \min(d(y_0, \mathcal{U}^c), 1)$. We define

$$\widetilde{f}(y) := f(y_0 + \frac{d_0}{4} y) \quad \text{on } \mathfrak{C}_1$$

and recursively

$$\widetilde{f}_1 = \widetilde{f}, \quad \widetilde{f}_k(y) = \widetilde{f}_{k-1}(y/4), \ k \ge 2.$$

Applying the first Step to $\widetilde{f}_k$ gives

$$\mathrm{osc}_{\mathfrak{C}_{1/4}} \widetilde{f}_k \le \vartheta \, \mathrm{osc}_{\mathfrak{C}_1} \widetilde{f}_k,$$

with $\vartheta := 1 - \lambda/2 \in (0,1)$, and thus

$$\mathrm{osc}_{\mathfrak{C}_{1/4^k}} \widetilde{f} \le \vartheta^k \, \mathrm{osc}_{\mathfrak{C}_2} \widetilde{f} \le 2\vartheta^k \|f\|_{L^\infty(\mathcal{U})}.$$

$$\sup_{|y_0 - y| \le 4^{-n}} |f(y) - f(y_0)| \le 2\vartheta^k \|f\|_{L^\infty(\mathcal{U})}.$$

In other words, for any y such that $4^{-k-1} \le |y| \le 4^{-k}$, we have

$$|\widetilde{f}(y) - \widetilde{f}(0)| \le \mathrm{osc}_{\mathfrak{C}_{1/4^k}} \widetilde{f} \le \vartheta^{k+1} [2\vartheta \|f\|_{L^\infty(\mathcal{U})}] \le |y|^\alpha [2\vartheta \|f\|_{L^\infty(\mathcal{U})}],$$

by choosing $\alpha := -\log \theta / \log 4$.
we have

$$\sup_{4^{-k-1} \le |y| \le 4^{-k}} |\widetilde{f}(y) - \widetilde{f}(0)| \le (4^\alpha \vartheta)^k |y - z|^\alpha \|f\|_{L^\infty(\mathcal{U})}$$

$$\sup_{4^{-k-1} \le |y-z| \le 4^{-k}} |\widetilde{f}(y) - \widetilde{f}(z)| \le (4^\alpha \vartheta)^k |y - z|^\alpha \|f\|_{L^\infty(\mathcal{U})}$$

We have established that $\widetilde{f}$ is α-Holder near 0, and thus also f on $\mathcal{U}$. □

4. The fundamental solution

In this section, we are interest in the fundamental solution to the parabolic equation (1.1), namely to the solution $\Gamma = \Gamma(t, x; x_0)$ to

$$\frac{\partial \Gamma}{\partial t} = \mathrm{div}(A \nabla \Gamma) \quad \text{in } (0, \infty) \times \mathbb{R}^d, \qquad \Gamma(0, \cdot) = \delta_{x_0} \quad \text{in } \mathbb{R}^d. \tag{4.1}$$

We will successively exhibit pointwise upper and lower bounds and next the existence, regularity and uniqueness of such a solution.

4.1. **The upper bound.**

Theorem 4.1. *There exists $C, k > 0$ such that the fundamental solution Γ satisfies*

$$\Gamma(t,\cdot) \le C\gamma_{kt}, \tag{4.2}$$

where γ stands for the kernel of the heat equation.

Proof of Theorem 4.1. We assume $x_0 = 0$. We give a proof for the heat equation which follows Nash argument and which can be adapted to a general parabolic equation with smooth and bounded coefficients. Without regularity assumption on the coefficients, the L^1 norm is not so easy to estimate and one may should rather follow Moser proof in a similar way as here.

We consider a smooth, positive and fast decaying solution f to the heat equation with initial datum f_0, and for a given $\alpha \in \mathbb{R}^d$, we define $g := f\,e^{\psi}$, $\psi(x) := \alpha \cdot x$. The equation satisfied by g is

$$\begin{aligned} \partial_t g &= \frac{1}{2} e^{\psi} \Delta(g\, e^{-\psi}) = \frac{1}{2}\Delta g - \nabla\psi\cdot\nabla g + \frac{1}{2}|\nabla\psi|^2 g \\ &= \frac{1}{2}\Delta g - \alpha\cdot\nabla g + \frac{1}{2}|\alpha|^2 g. \end{aligned}$$

For the L^1 norm, we have

$$\frac{d}{dt}\|g\|_{L^1} = \frac{1}{2}|\alpha|^2\,\|g\|_{L^1},$$

and then $\|g(t,.)\|_{L^1} = e^{|\alpha|^2 t/2}\,\|g_0\|_{L^1}$ for any $t \ge 0$. For the L^2 norm and thanks to the Nash inequality (2.3), we have

$$\begin{aligned} \frac{d}{dt}\|g\|^2_{L^2} &= -\|\nabla g\|^2_{L^2} + |\alpha|^2\,\|g\|^2_{L^2} \\ &\le -K_0\, e^{-2|\alpha|^2 t/d}\,\|g\|_{L^2}^{2(1+2/d)} + |\alpha|^2\,\|g\|^2_{L^2}, \end{aligned}$$

with $K_0 := C_N\,\|g_0\|_{L^1}^{-4/d}$. We see that the function $u(t) := e^{-|\alpha|^2 t}\,\|g(t)\|^2_{L^2}$ satisfies the differential inequality

$$u' \le -K_0\, u^{1+2/d},$$

from what, exactly as in the Section 2.1, we deduce

$$\|g(t)\|^2_{L^2}\, e^{-|\alpha|^2 t} \le \frac{\|g_0\|^2_{L^1}}{(2/d\, C_N\, t)^{d/2}}, \quad \forall\, t > 0.$$

Denoting by $T(t)$ the semigroup associated to the parabolic equation satisfies by g, the above estimate writes

$$\|T(t)g_0\|_{L^2} \le \frac{C\, e^{|\alpha|^2 t/2}}{t^{d/4}}\,\|g_0\|_{L^1}, \quad \forall\, t > 0.$$

Because the equation associated to the dual operator is

$$\partial_t h = \frac{1}{2}\Delta h + \alpha\cdot\nabla h + \frac{1}{2}|\alpha|^2 h, \quad h(0) = h_0,$$

the same estimate holds on $T^*(t)h_0 = h(t)$, and we thus deduce

$$\|T(t)g_0\|_{L^\infty} \le \frac{C\, e^{|\alpha|^2 t/2}}{t^{d/4}}\,\|g_0\|_{L^2}, \quad \forall\, t > 0.$$

Using the trick $T(t) = T(t/2)T(t/2)$, both estimates together give an accurate time depend estimate on the mapping $T(t) : L^1 \to L^\infty$ for any $t > 0$. More precisely and in other words, we have proved that the heat semigroup S satisfies

$$\|(S(t)f_0)\, e^{\psi}\|_{L^\infty} \le \frac{C}{t^{d/2}}\, e^{|\alpha|^2 t/2}\, \|f_0\, e^{\psi}\|_{L^1}, \quad \forall\, t > 0. \tag{4.3}$$

Denoting $\Gamma(t, y; x) := (S(t)\delta_x)(y)$ the fundamental solution associated to the heat equation when starting from the Dirac function in $x \in \mathbb{R}^d$, the above estimate rewrites as

$$\Gamma(t, y; x) \le \frac{C}{t^{d/2}}\, e^{\alpha\cdot(x-y) - |\alpha|^2 t/2}, \quad \forall\, t > 0, \forall\, x, y, \alpha \in \mathbb{R}^d.$$

Choosing $\alpha := (x - y)/t$, we end with

$$\Gamma(t, y; x) \le \frac{C}{t^{d/2}}\, e^{-\frac{|x-y|^2}{2t}}, \quad \forall\, t > 0, \forall\, x, y \in \mathbb{R}^d,$$

what is the announced estimate with $k = 1$. □

4.2. **The lower bound.** We first establish with a simple lower bound the first moment $\|\Gamma\|_{\dot L^1}$, where we define

$$\|f\|_{\dot L^1_1} := \|xf\|_{L^1}.$$

We start with a classical interpolation estimate.

Lemma 4.2. *We have*

$$\|f\|_{L^1} \lesssim \|f\|_{L^2}^{\frac{2}{d+2}} \|f\|_{\dot L^1_1}^{\frac{d}{d+2}}.$$

Proof of Lemma 4.2. We write

$$\begin{aligned} \|f\|_{L^1} &= \int_{B_R} |f| + \int_{B_R^c} |f| \\ &\le |B_R|^{1/2} \|f\|_{L^2} + \frac{1}{R} \|f\|_{\dot L^1} \\ &= c_d R^{d/2} \|f\|_{L^2} + \frac{1}{R} \|f\|_{\dot L^1}, \end{aligned}$$

and we conclude by choosing $R := (\|f\|_{\dot L^1_1}/\|f\|_{L^2})^{\frac{2}{d+2}}$. □

Together with the ultracontractivity property, we immediately obtain a rough lower bound.

Lemma 4.3. *Any solution f to the parabolic equation* (1.1) *satisfies*

$$\|f(t, \cdot)\|_{\dot L^1_1} \gtrsim t^{1/2}, \quad \forall\, t \ge 0.$$

Proof of Lemma 4.3. Gathering the ultracontractivity estimate $\|f\|_{L^2} \lesssim t^{-d/4} \|f\|_{L^1}$ and the just above interpolation inequality, we get

$$t^{\frac{d}{4}\frac{2}{d+2}} \lesssim \|f\|_{\dot L^1_1}^{\frac{d}{d+2}}$$

from what we immediately conclude. □

[The few last pages are still a draft and have to be checked again]

Theorem 4.4. *Assume $f_0 \in L^1_1$ with $\|f_0\|_{L^1} = 1$, $\|f_0\|_{L^1_1} \le M$. There exist $c, K > 0$ only depending of M such that*

$$f(t,\cdot) \ge c\gamma_{Kt}, \quad \forall\, t \ge 1. \tag{4.4}$$

For the fundamental solution, we have

$$\Gamma(t,\cdot) \ge c\gamma_{Kt}, \quad \forall\, t > 0. \tag{4.5}$$

Proof of Theorem 4.4. Step 1. We claim that there exists $C, c > 0$ such that for any $t > 0$ and $x, y \in \mathbb{R}^d$, $|x-y|^2/t > 4$, there holds

$$f(2t,y) \ge Ce^{-c|x-y|^2/t} f(t,x).$$

We define

$$k := \lfloor |x-y|^2/t \rfloor + 1, \quad r := t/|x-y|$$

and the chain

$$z_i := (t_i, x_i) = (t + t\frac{i}{k}, x + (y-x)\frac{i}{k}), \quad \forall\, i \in \{1, \dots, k\}.$$

We observe that $\mathfrak{C}_r(z_i) \cap \mathfrak{C}_r(z_{i+1}) \ni \{(t_{i+1} - r^2, x_{i+1} - r(y-x))\} \neq \emptyset$, due to the fact that $t/r^2 = |y-x|/r = |y-x|^2/t < k$, and we deduce

$$\inf_{\mathfrak{C}_r(z_{i+1})} f \ge \gamma \sup_{\mathfrak{C}_r(z_{i+1})} f \ge \gamma \inf_{\mathfrak{C}_r(z_i)} f,$$

for a constant $\gamma \in (0,1)$ independent of f and i given by the Harnack inequality provided that $\mathfrak{C}_{2r}(z_0) \subset \mathbb{R}_+ \times \mathbb{R}^d$, or equivalently, $|x-y|^2/t > 4$. Iterating, we get

$$f(2t,y) \ge \inf_{\mathfrak{C}_r(z_k)} f \ge \gamma^k \inf_{\mathfrak{C}_r(z_1)} f \ge \gamma^{k+1} \sup_{\mathfrak{C}_r(z_0)} f \ge \gamma^{k+1} f(t,x),$$

and the result is thus proved for the constants $c := -\ln\gamma$ and $C := \gamma^2$.

Step 2. We claim that there exists $c = c(d, M) \ge 4$ such that

$$\int_{B(0,R)} f(t,x)dx \ge \frac{1}{2}, \quad R := \sqrt{ct},$$

when $f_0 = \delta_0$. We may indeed write

$$\begin{aligned} \int_{B(0,R)} f_t(x)dx &= \int_{\mathbb{R}^d} f(x)dx - \int_{B(0,R)^c} f_t(x)dx \\ &\ge 1 - \frac{1}{R}\int_{\mathbb{R}^d} f_t(x)|x|dx \\ &\ge 1 - Cd(2\pi)^{d/2}\frac{t}{R^2} \ge 1/2, \end{aligned}$$

for $c > 0$ large enough. As a consequence, for any $t > 0$, there exists $x_t \in B(0, \sqrt{tc})$ such that

$$f(t,x_t)C_2 t^{d/2} = \|f(t,\cdot)\|_{L^\infty(B(0,\sqrt{tc})} C_2 t^{d/2} \ge \|f(t,\cdot)\|_{L^1(B(0,\sqrt{tc})} \ge \frac{1}{2}.$$

Step 3. For any $y \in B^c(0, 2\sqrt{ct})$, we have $|y - x_t| \ge \sqrt{ct}$, so that

$$\frac{|y-x_t|^2}{t} \ge c \ge 4,$$

and

$$\frac{|y-x_t|^2}{t} \le 2\frac{|y|^2}{t} + 2\frac{|x_t|^2}{t} \le 2\frac{|y|^2}{t} + 2c.$$

Thanks to steps 1 and 2, we deduce

$$f(2t,y) \ge C e^{-\frac{c}{t}|y-x_t|^2} f(t,x_t) \ge C e^{-\frac{2c}{t}|y|^2 - 2c^2} (2C_2 t^{d/2})^{-1},$$

so that (4.5) is proved on $B^c(0, 2\sqrt{ct})$.

Step 4. For any $x \in B(0, 2\sqrt{ct})$, we take $y \in B(0, 4\sqrt{ct}) \backslash B(0, 3\sqrt{ct})$, so that

$$\frac{|y-x|^2}{t} \ge c \ge 4$$

and

$$\frac{|y-x|^2}{t} \le 2\frac{|x|^2}{t} + 2\frac{|y|^2}{t} \le 40c.$$

Thanks to steps 1 and 3, we deduce

$$f(2t,x) \ge C e^{-\frac{c}{t}|x-y|^2} f(t,y) \ge C e^{-40c^2} K t^{-d/2} e^{-C_1 32 c^2},$$

so that (4.5) is also proved on $B(0, 2\sqrt{ct})$. □

4.3. **Existence, regularity and uniqueness.** In this section, we use the shorthand $\|f\|_{\dot L^1_1} := \| |x| f\|_{L^1(\mathbb{R}^d)}$ and $(\tau_{x_0} f)(x) := f(x - x_0)$.

Theorem 4.5. *We assume $A \in L^\infty$, $A \ge \nu I$, $\nu > 0$. For any $x_0 \in \mathbb{R}^d$, there exists a unique fundamental solution $\Gamma = \Gamma(t,x;x_0)$ to the parabolic equation (4.1), that is a function Γ on $(0,\infty) \times \mathbb{R}^d$ such that*

$$\Gamma \ge 0, \quad \|\Gamma(t,\cdot)\|_{L^1} = 1, \quad \|\tau_{x_0}\Gamma(t,\cdot)\|_{\dot L^1} \lesssim \sqrt{t}, \quad \forall\, t > 0; \tag{4.6}$$

$$\Gamma(t_0 + \cdot, \cdot) \in X_T \quad \forall\, T, t_0 > 0; \tag{4.7}$$

which is a weak solution to (4.1) in the following sense

$$\int_{\mathbb{R}^d} (\varphi\Gamma)(t_0,x)\,dx + \int_0^T \int_{\mathbb{R}^d} \{\Gamma \partial_t \varphi - \nabla\varphi \cdot A \nabla \Gamma\} = 0, \tag{4.8}$$

for any $\varphi(t_0 + \cdot, \cdot) \in X_T$, $t_0, T > 0$, and

$$\int_{\mathbb{R}^d} \Gamma(t_0,x)\varphi(x)\,dx \to \varphi(x_0), \quad \text{as} \quad t_0 \to 0, \tag{4.9}$$

for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$. The fundamental solution Γ also satisfies the upper bound (4.2), the lower bound (4.4) and the Holder continuity (3.23).

Remark 4.6. *In fact, we may also establish that*

$$\|\nabla_x \Gamma\|_{L^q((0,T)\times B_R)} < \infty, \quad \forall\, T, R > 0, \tag{4.10}$$

for some $q > 1$, and that Γ satisfies the weak formulation

$$\int_0^T \int_{\mathbb{R}^d} \{f \partial_t \varphi - \nabla\varphi \cdot A \nabla f\} = \varphi(0,x_0), \tag{4.11}$$

for any $\varphi \in \mathcal{D}([0,\infty) \times \mathbb{R}^d)$.

Proof of Theorem 4.5. Step 1. Existence. For simplicity, we only consider the case when $x_0 = 0$. Let (ρ_ε) be a mollifier, and more precisely $\rho_\varepsilon \ge 0$, $\|\rho_\varepsilon\|_{L^1} = 1$, $\operatorname{supp}\rho_\varepsilon \subset B(0,\varepsilon)$, and thus $\rho_\varepsilon \rightharpoonup \delta_0$. We consider (f_ε) the variational solution to the parabolic equation (1.1) associated to the initial datum $f_\varepsilon(0,\cdot) = \rho_\varepsilon$ which exists and is nonnegative because of the analysis made in the first chapter. Because of the previous section, we have

$$\|f_\varepsilon(t,\cdot)\|_{L^1} = 1, \quad \|f_\varepsilon(t,\cdot)\|_{L^p} \le \frac{C}{t^{d/2(1-1/p)}}, \tag{4.12}$$

for any $p \in (1,\infty]$. Using the second estimate with $p = 2$ and the energy estimate starting from $t_0 > 0$, se also have

$$\|\nabla_x f_\varepsilon\|_{L^2((t_0,T)\times\mathbb{R}^d)} \le C t_0^{-d/4},$$

for any $T > t_0$. From (4.3), we have

$$f_\varepsilon(t,x)e^{\alpha\cdot x} \le \frac{C}{t^{d/2}}\, e^{|\alpha|^2 t/2} \|\rho_\varepsilon e^{|\alpha|\varepsilon}\|_{L^1},$$

or equivalently

$$f_\varepsilon \le \frac{C}{t^{d/2}}\, e^{|\alpha|^2 t/2 + |\alpha|\varepsilon - \alpha\cdot x},$$

for any $t,\varepsilon > 0$, $x,\alpha \in \mathbb{R}^d$. Choosing again $\alpha := x/t$, we obtain

$$f_\varepsilon \le \frac{C}{t^{d/2}}\, e^{-\frac{1}{2t}(|x|^2 - 2\varepsilon|x|)} \le \frac{C}{t^{d/2}}\, e^{-\frac{1}{4t}|x|^2} \mathbf{1}_{|x|\ge 4\varepsilon}$$

Together with (4.12), we thus deduce

$$\begin{aligned}\int |x||f_\varepsilon|dx &= \int_{B^c_{4\varepsilon}} |x||f_\varepsilon|dx + \int_{B_{4\varepsilon}} |x||f_\varepsilon|dx \\ &\lesssim \int_{B^c_{4\varepsilon}} |x| e^{-\frac{1}{4t}|x|^2} \frac{dx}{t^{d/2}} + 4\varepsilon \int_{B_{4\varepsilon}} |f_\varepsilon|dx \\ &\lesssim \sqrt{t}\int_{\mathbb{R}^d} |y| e^{-\frac{1}{4}|y|^2} dy + \varepsilon,\end{aligned}$$

or in other words

$$\|f_\varepsilon(t,\cdot)\|_{\dot L^1_1} \le C\sqrt{t} + \varepsilon, \quad \forall\, t > 0,\ \varepsilon \in (0,1).$$

With these pieces of information, there exists a function Γ satisfying the estimates (4.6)-(4.7) and a subsequence $(f_{\varepsilon'})$ such that $f_{\varepsilon'} \rightharpoonup \Gamma$ weakly in $\mathcal{D}'(\mathcal{U})$. We may then pass to the limit in the weak formulation of the equations satisfied by $f_{\varepsilon'}$ and we obtain that Γ satisfies (4.8). The convergence (4.9) is a straightforward consequence of (4.6).

Step 2. Uniqueness. We know from (4.7)-(4.8) that Γ is a variational solution on $(t_0,T)\times\mathbb{R}^d$, and we may thus write

$$\int_{\mathbb{R}^d} \psi_T \Gamma_T + \int_{t_0}^T \int_{\mathbb{R}^d} (-\Gamma\partial_t\psi + \nabla\psi\cdot A\nabla\Gamma) = \int_{\mathbb{R}^d} \psi_{t_0}\Gamma_{t_0},$$

for any $\psi \in W^{1,\infty}([0,T]\times\mathbb{R}^d)$. For $\phi \in L^1 \cap L^\infty$, we define the solution $\varphi \in X_T$ to the backward problem

$$-\partial_t\varphi = \operatorname{div}(A^T\nabla\varphi) \ \text{ in } \ (0,T)\times\mathbb{R}^d, \qquad \varphi(T,\cdot) = \phi \ \text{ in } \ \mathbb{R}^d. \tag{4.13}$$

We define $\varphi^\varepsilon = \varphi *_x \rho_\varepsilon$ for a mollifier (ρ_ε). Observing that

$$\varphi^\varepsilon,\, \nabla\varphi^\varepsilon, \partial_t\varphi^\varepsilon = (-\mathrm{div}(A^T\nabla\varphi)) * \rho_\varepsilon \in L^\infty(\mathcal{U}),$$

we may take $\psi = \varphi^\varepsilon$ in the above variational formulation and we get

$$\begin{aligned}\Big[\int_{\mathbb{R}^d} \varphi^\varepsilon\Gamma\Big]_{t_0}^T &= \int_{t_0}^T\int_{\mathbb{R}^d} (\{\Gamma(-\mathrm{div}(A^T\nabla\varphi)) * \rho_\varepsilon - \nabla\varphi^\varepsilon \cdot A\nabla\Gamma) \\ &= \int_{t_0}^T\int_{\mathbb{R}^d} (\nabla\Gamma^\varepsilon \cdot A^T\nabla\varphi + \nabla\varphi^\varepsilon \cdot A\nabla\Gamma),\end{aligned}$$

with $\Gamma^\varepsilon := \Gamma * \check\rho_\varepsilon$, $\check\rho_\varepsilon(x) := \rho_\varepsilon(-x)$. Using that $\nabla\Gamma^\varepsilon \to \nabla\Gamma$ and $\nabla\varphi^\varepsilon \to \nabla\varphi$ in $L^2((t_0,T)\times\mathbb{R}^d)$, as well as $\varphi^\varepsilon_s \to \varphi_s$ in $L^2(\mathbb{R}^d)$ for $s = t_0, T$, we may pass to the limit $\varepsilon \to 0$ in the previous equation and we conclude that

$$\int_{\mathbb{R}^d} \phi\Gamma_T = \int_{\mathbb{R}^d} \varphi_{t_0}\Gamma_{t_0}, \quad \forall\, t_0 > 0.$$

From (4.6) and the De Giorgi-Nash regularity estimate $\varphi \in C_b([0,T/2]\times\mathbb{R}^d)$ provided by Theorem 3.11, we may pass to the limit $t_0 \to 0$ and we obtain

$$\int_{\mathbb{R}^d} \phi\Gamma_T = \lim_{t\to 0}\int_{\mathbb{R}^d} \varphi_t f_t = \varphi(t_0, 0).$$

Let us consider now another fundamental solution Γ' to the parabolic equation (4.1) associated to the same initial datum δ_0. For the same reasons, the function Γ' satisfies the same equation as above and thus the difference $\Upsilon := \Gamma - \Gamma'$ satisfies

$$\int_{\mathbb{R}^d} \phi\Upsilon_T = 0.$$

Because $\phi \in L^1 \cap L^\infty$ is arbitrary, we deduce that $\Upsilon_T = 0$ for any $T > 0$, and that concludes the uniqueness of the fundamental solution. $\square$