Exercises about chapter 1

Exercise 0.1. (1) Consider $f \in L^1(\mathbb{R}^d)$ such that $\mathrm{div} f \in L^1(\mathbb{R}^d)$. Show that

$$\int_{\mathbb{R}^d} \mathrm{div} f \, dx = 0.$$

[Hint. That is true for $f \in C^1_c(\mathbb{R}^d)$. For $f \in L^1(\mathbb{R}^d)$ we introduce a mollifier (ρ_ε), a truncation fiunction χ_M and $\rho_\varepsilon * (f\chi_M) \in C^1_c(\mathbb{R}^d)$.]

(2) Deduce that for $f \in H^1(\mathbb{R}^d)$ such that $\Delta f \in L^2(\mathbb{R}^d)$ and $g \in H^1(\mathbb{R}^d)$, there holds

$$\int_{\mathbb{R}^d} g \, \Delta f = - \int_{\mathbb{R}^d} \nabla g \cdot \nabla f.$$

Exercise 0.2. Let (ρ_ε) be a mollifer on the real line, namely $0 \le \rho_\varepsilon \in C^\infty_c(\mathbb{R})$ such that $\|\rho_\varepsilon\|_{L^1} = 1$ and (for instance) $\mathrm{supp}\, \rho_\varepsilon \subset (-\varepsilon, \varepsilon)$. For $f \in L^1_{\mathrm{loc}}(\mathcal{U})$, $\mathcal{U} := (0,T) \times \mathbb{R}^d$, we define $f_\varepsilon := \rho_\varepsilon *_t f$.

(1) For $f \in C([0,T]; L^2(\mathbb{R}^d))$, prove that $f_\varepsilon \in C^1((0,T); L^2(\mathbb{R}^d))$ and $f_\varepsilon \to f$ in $C((0,T); L^2(\mathbb{R}^d))$.

(2) For $f \in L^2(\mathcal{U})$, prove that $f_\varepsilon \in C^1((0,T); L^2(\mathbb{R}^d))$ and $f_\varepsilon \to f$ in $L^2(\mathcal{U})$. *[Hint. Use that for any $\eta > 0$ there exists $g \in C_c(\mathcal{U})$ such that $\|g - f\|_{L^2(\mathcal{U})} < \eta$.]*

(3) For any $f \in X_T$, prove that $f_\varepsilon \in C^1((0,T); H^1(\mathbb{R}^d))$ and $f_\varepsilon \to f$ in X_T.

Exercise 0.3. (1) For $f \in L^2(\mathcal{U})$ prove that $f_\pm, |f| \in L^2(\mathcal{U})$. For $f \in L^2(0,T; H^1(\mathbb{R}^d))$ prove that $f_\pm, |f| \in L^2(0,T; H^1(\mathbb{R}^d))$ and $\nabla f_+ = \nabla f \mathbf{1}_{f>0}$ *[Hint. Consider $\beta_\varepsilon(f)$ with $\beta_\varepsilon(s) := s_+^2(\varepsilon^2 + s^2)^{-1/2}$]*. What about $f \in X_T$?

(2) For $f \in H^1(\Omega)$ prove that $\nabla f = 0$ on $\{f = c\}$ for any $c \in \mathbb{R}$. *[Hint. Consider $\beta_\varepsilon(f)$ and $\gamma_\varepsilon(f)$ with $\beta_\varepsilon(s) := (s+\varepsilon)_+^2(\varepsilon^2 + s^2)^{-1/2}$ and $\gamma_\varepsilon(s) := (s-\varepsilon)_+^2(\varepsilon^2 + s^2)^{-1/2}$].*

Exercise 0.4. Prove that

$$L^2(0,T; H^{-1}(\mathbb{R}^d)) = \{F_0 + \sum_{i=1}^d \partial_{x_i} F_i, \ F_i \in L^2(\mathcal{U}), \ 0 \le i \le d\}.$$

[Hint. Consider the mapping $A : \mathscr{H} := L^2(0,T; H^1) \to \mathscr{E} := (L^2(\mathcal{U}))^{d+1}$, $f \mapsto (f, \nabla f)$, $\mathscr{F} := RA$ and $B := A^{-1} : \mathscr{F} \to L^2(0,T; H^1)$. For a linear form $T \in L^2(0,T; H^{-1}(\mathbb{R}^d)) = \mathscr{H}'$, define the linear form $S : \mathscr{F} \to \mathbb{R}$, $G \in \mathscr{F} \mapsto S(G) := \langle T, BG \rangle$ and prove that there exists $\bar S \in \mathscr{E}'$ and thus $F_i \in L^2(\mathcal{U})$ such that $\bar S_{|\mathscr{F}} = S$ and $\bar S(G) = \sum_i (F_i, G_i)_{L^2(\mathcal{U})}$ for any $G \in \mathscr{E}$. Deduce that $\langle T, f \rangle = S(Af)$ and conclude].

Exercise 0.5. Consider a sequence (f_n) such that $f_n \to f$ in $L^2(\mathcal{U})$ and, for some $k > d/2$,

$$f_n \in \mathcal{Z} := \{g \in L^2(\mathcal{U}); \, g \ge 0, \ \|g(t)\|_{L^1} \le A(t), \ \|g(t)\|_{L^2_k} \le B(t)\}.$$

(1) Prove that $f \ge 0$. *[Hint. Prove that for $g \in L^1(\mathcal{U})$, we have $g \ge 0$ if and only if $\langle g, \varphi \rangle \ge 0$ for any $\varphi \in L^\infty(\mathcal{U})$.]*

(2) Prove that $\|f\|_{L^1(\mathbb{R}^d)} \le A$ a.e. on $(0,T)$. *[Hint. Prove that $\|f_n(t,\cdot)\|_{L^1} \to \|f(t,\cdot)\|_{L^1}$ in $L^1(0,T)$ by using the Cauchy-Schwartz inequality and conclude by using the reverse sense of the dominated convergence Lebesgue theorem.]*

(3) Prove that $\|f\|_{L^2_k(\mathbb{R}^d)} \le B$ a.e. on $(0,T)$. *[Hint. For any $k' \in [0,k)$, prove that $f_n \langle x \rangle^{k'} \to f \langle x \rangle^{k'}$ strongly $L^2(\mathcal{U})$ and that $\|f_n\|_{L^2_{k'}} \to \|f_n\|_{L^2_{k'}}$ a.e. on $(0,T)$. Next deduce that $\|f(t,\cdot)\|_{L^2_{k'}} \le B$ a.e. on $(0,T)$ for any $k' \in (0,k)$ and conclude.]*

Exercise 0.6. Consider a sequence (f_n) such that $f_n \rightharpoonup f$ in $L^2(\mathcal{U})$ and $f_n \in \mathcal{Z}$, for some $k > d/2$.

(1) Prove that $f \ge 0$. *[Hint. Prove that for $g \in L^1(\mathcal{U})$, we have $g \ge 0$ if and only if $\langle g, \varphi \rangle \ge 0$ for any $\varphi \in \mathcal{D}(\mathcal{U})$.]*

(2) Prove that $\|f\|_{L^1(\mathbb{R}^d))} \le A$ a.e. on $(0,T)$. *[Hint. Prove that $f_n \rightharpoonup f$ in $L^2(0,T;L^2_{k'})$ for any $k' \in [0,k)$ and there exists a sequence (g_n) such that g_n is a convex combination of $f_1,\dots,f_n$ and $g_n \to f$ in $L^2(0,T;L^2_{k'})$. Conclude with the help of Exercise 5.]*

(3) - Prove that $f_n^2 \rightharpoonup g$ weakly and $g \ge f^2$. *[Hint. Consider the family $\mathscr{A}$ of real affine functions such that $\ell \in \mathscr{A}$ iff $\ell(s) \le s^2$ for any $s \in \mathbb{R}$ and observe that $\ell(f_n) \rightharpoonup \ell(f)$ weakly.]*

- We define $G_n(t) := \|f_n(t,\cdot)\|^2_{L^2_k}$. Prove that, up to the extraction of a subsequence, $G_n \rightharpoonup G$ weakly and $G(t) \ge \langle g(t,\cdot)\rangle$ a.e. on $(0,T)$. *[Hint. Take $\psi(t)\chi_R(x)$ as a test function].*

- Conclude that $\|f\|_{L^2_k(\mathbb{R}^d)} \le B$ a.e. on $(0,T)$.

(4) - For $0 \le F \in L^2(\mathcal{U})$ such that

$$\int F\varphi \le C\|\varphi\|_{L^1(0,T;L^2)}, \quad \forall\, \varphi,$$

establish that $\|F\|_{L^\infty(0,T;L^2)} \le C$ by proving first

$$\int (F \wedge n)^2 \psi \le C\|\psi\|_{L^1(0,T)}, \quad \forall\, \psi, \forall\, n.$$

- Establish that $\|f(t,\cdot)\|_{L^2_k} \le C = B(T)$ a.e. on $(0,T)$.

(5) For $0 \le F \in L^1(\mathcal{U})$ such that

$$\int F\psi \le C\|\psi\|_{L^1(0,T))}, \quad \forall\, \psi,$$

establish that $\|F\|_{L^\infty(0,T;L^1)} \le C$. *[Hint. Consider $\psi := \mathbf{1}_{F \ge C+\varepsilon}$, $\varepsilon > 0$].* Recover (2).

Exercise 0.7. Consider a parabolic equation where the operator $\mathcal{L}$ incloses a kernel term

$$\mathcal{L}f := \Delta f + b\cdot\nabla f + cf + \mathcal{K}f, \quad (\mathcal{K}f)(x) := \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with coefficients satisfying

$$b, c \in L^\infty(\mathbb{R}^d), \quad k \in L^2(\mathbb{R}^d \times \mathbb{R}^d),$$

and establish the existence of a variational solution in the usual X_T space.
[Hint. Observe that $\mathcal{K} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$.]

Exercise 0.8. Consider the parabolic equation with coefficients $b \in L^\infty + L^d$ and $c \in L^1_{\text{loc}}$, $c_+ \in L^\infty + L^{d/2}$ with $d \ge 3$. Establish the existence of a variational solution in the space X_T associated to $H := L^2$ and $V := \{g \in H^1;\ \sqrt{c_-}g \in L^2\}$.
[Hint. Observe that $f(|b|\mathbf{1}_{|b|>M} + \sqrt{c_+}\mathbf{1}_{c_+>M}) \to 0$ in L^2 when $M \to \infty$ and that $2/d + 2/2^* = 1$, where 2^* denotes the Sobolev exponent.]

Exercise 0.9. **(L^p estimates).** For $b, c \in L^\infty(\mathbb{R}^d)$, $(\operatorname{div} b)_- \in L^\infty(\mathbb{R}^d)$, $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, we consider the linear parabolic equation

$$\partial_t f = \Lambda f := \Delta f + b\cdot\nabla f + cf, \quad f(0) = f_0. \tag{0.1}$$

We introduce the usual notations $H := L^2$, $V := H^1$ and X_T the associated space for some given $T > 0$.

1) Consider a convex function $\beta \in C^2(\mathbb{R})$ such that $\beta(0) = \beta'(0) = 0$ and $\beta'' \in L^\infty$. Prove that any variational solution $f \in X_T$ to the above linear parabolic equation satisfies

$$\int_{\mathbb{R}^d} \beta(f_t)\,dx \le \int_{\mathbb{R}^d} \beta(f_0)\,dx + \int_0^t \int_{\mathbb{R}^d} \{c\,f\,\beta'(f) - (\operatorname{div} b)\,\beta(f)\}\,dxds,$$

for any $t \ge 0$.

2) Assuming moreover that $\beta \ge 0$ and there exists a constant $K \in (0,\infty)$ such that $0 \le s\,\beta'(s) \le K\beta(s)$ for any $s \in \mathbb{R}$, deduce that for some constant $C := C(b,c,K)$, there holds

$$\int_{\mathbb{R}^d} \beta(f_t)\,dx \le e^{Ct} \int_{\mathbb{R}^d} \beta(f_0)\,dx, \quad \forall\, t \ge 0.$$

3) Prove that for any $p \in [1,2]$, for some constant $C := C(b,c)$ and for any $f_0 \in L^2 \cap L^p$, there holds

$$\|f(t)\|_{L^p} \le e^{Ct} \|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

[Hint. For $p \in (1,2]$, define $\beta \simeq s^p$ on $\mathbb{R}_+$ and extend it to $\mathbb{R}$ by symmetry. More precisely, define $\beta''_\alpha(s) = 2\theta \mathbf{1}_{s\le\alpha} + p(p-1)s^{p-2}\mathbf{1}_{s>\alpha}$, with $2\theta = p(p-1)\alpha^{p-2}$, and then the primitives which vanish at the origin, which are thus defined by $\beta'_\alpha(s) = 2\theta s \mathbf{1}_{s\le\alpha} + (ps^{p-1} + p(p-2)\alpha^{p-1})\mathbf{1}_{s>\alpha}$, $\beta_\alpha(s) = \theta s^2 \mathbf{1}_{s\le\alpha} + (s^p + p(p-2)\alpha^{p-1}s + A\alpha^p)\mathbf{1}_{s>\alpha}$, $A := p(p-1)/2 - 1 - p(p-2)$. Observe that $s\beta'_\alpha(s) \le 2\beta_\alpha(s)$ because $s\beta''_\alpha(s) \le \beta'_\alpha(s)$ and $\beta_\alpha(s) \le \beta(s)$ because $\beta''_\alpha(s) \le \beta''(s)$. Pass to the limit $p \to 1$ in order to deal with the case $p = 1$.]

4) Prove that for any $p \in [2,\infty]$ and for some constant $C := C(a,c,p)$ there holds

$$\|f(t)\|_{L^p} \le e^{Ct} \|f_0\|_{L^p}, \quad \forall\, t \ge 0.$$

[Hint. Define $\beta''_R(s) = p(p-1)s^{p-2}\mathbf{1}_{s\le R} + 2\theta \mathbf{1}_{s>R}$, with $2\theta = p(p-1)R^{p-2}$, and then the primitives which vanish in the origin and which are thus defined by $\beta'_R(s) = ps^{p-1}\mathbf{1}_{s\le R} + (pR^{p-1} + 2\theta(s-R))\mathbf{1}_{s>R}$, $\beta_R(s) = s^p \mathbf{1}_{s\le R} + (R^p + pR^{p-1}(s-R) + \theta(s-R)^2)\mathbf{1}_{s>R}$. Observe that $s\beta'_R(s) \le p\beta_R(s)$ because $s\beta''_R(s) \le (p-1)\beta'_R(s)$ and $\beta_R(s) \le \beta(s)$ because $\beta''_R(s) \le \beta''(s)$. Pass to the limit $p \to \infty$ in order to deal with the case $p = \infty$.]

5) Prove that for any $f_0 \in L^p(\mathbb{R}^d)$, $1 \le p \le \infty$, there exists at least one weak (in the sense of distributions) solution to the linear parabolic equation (0.1). *[Hint: Consider $f_{0,n} \in L^1 \cap L^\infty$ such that $f_{0,n} \to f_0$ in L^p, $1 \le p < \infty$, and prove that the associate variational solution $f_n \in X_T$ is a Cauchy sequence in $C([0,T];L^p)$. Conclude the proof by passing to the limit $p \to \infty$.]* Prove that $f \ge 0$ if furthermore $f_0 \ge 0$.

Exercise 0.10. (McKean-Vlasov equation) Consider the linear parabolic equation

$$\partial_t f = \mathcal{L}_g f := \Delta f + \operatorname{div}(a_g f), \quad f(0) = f_0, \tag{0.2}$$

with

$$a_g := a * g, \quad a \in L^\infty(\mathbb{R}^d)^d, \tag{0.3}$$

associated to the nonlinear McKean-Vlasov equation. We prove the existence and uniqueness of the solution to this equation by using directly the J.-L. Lions theorem in the flat L^2 and associated Sobolev spaces.

1) Defining $F := f\langle x\rangle^{2k}$, establish that F is a solution to the linear parabolic equation

$$\partial_t F = \mathcal{M}_g F := \Delta F + \operatorname{div}(a_g F) + b \cdot \nabla F + c_g F, \tag{0.4}$$

with b and c_g to be determined. *[Hint. $b := -4kx/\langle x\rangle^2$, $c_g := \langle x\rangle^{-2k}(8|\nabla\langle x\rangle^k|^2 - \Delta\langle x\rangle^{2k}) + \frac{1}{2}a_g \cdot b$.]*

2) Establish that for any $F_0 \in L^2$ and $g \in L^\infty(0,T;L^1)$, there exists a unique variational solution $F \in X_T$ to the parabolic equation (0.4).

3) Establish that for $f_0 \in L^2_k$ and $g \in L^\infty(0,T;L^1)$, there exists a unique variational solution $f \in Y_T$ to the parabolic equation (0.2) with $Y_T = C([0,T];H) \cap L^2(0,T;V) \cap H^1(0,T;V')$, $H := L^2_k$, $V := H^1_k$.

Exercise 0.11. (McKean-Vlasov equation again) We consider the same linear parabolic equation as in Exercise 10 and the associated nonlinear McKean-Vlasov equation. We extend the existence of solutions to a larger class of initial data.

1) Prove that for $f_0 \in L^2_k$, $k > d/2$, and $g \in L^1(\mathcal{U})$, the solution $f \in X_T$ to the linear parabolic equation satisfies

$$\|f(t,\cdot)\|_{L^1} \le \|f_0\|_{L^1}, \quad \forall\, t \ge 0. \tag{0.5}$$

[Hint. Define $f^\pm$ the solutions associated to the initial data $f_{0\pm} \ge 0$. Prove that $f = f^+ - f^-$ and conclude.]

2) When $\mathrm{div} a \in L^\infty$, recover (0.5) by using a convenient family of renormalizing functions.

3) Prove the existence and uniqueness of a solution to the nonlinear McKean-Vlasov equation for any $f_0 \in L^2_k$, $k > d/2$.

4) Prove the existence of a weak solution to the nonlinear McKean-Vlasov equation (0.2) for any initial datum $f_0 \in L^1 \cap L^2_k$, $k > 0$.

Exercise 0.12. Consider the Fokker-Planck equation

$$\partial_t f = \Delta f + \mathrm{div}(xf) \ \text{ in } \ (0,T) \times \mathbb{R}^d, \quad f(0,\cdot) = f_0 \ \text{ in } \ \mathbb{R}^d. \tag{0.6}$$

Under convenient assumptions on f_0 show the existence and uniqueness of a variational solution to the Fokker-Planck equation (0.6) by using Lions' theorem.

[Hint. Choose $H = L^2_\omega$ with a convenient weight $\omega : \mathbb{R}^d \to (0,\infty)$. We may accept and use the functional inequality

$$\frac{1}{4} \int_{\mathbb{R}^d} h^2 M dx \le \int |\nabla h|^2 M dx + \frac{d}{2} \int h^2 M dx,$$

for any $h \in W^{1,\infty}(\mathbb{R}^d)$ and M is the standard gaussian function. For the proof of that last inequality one may write $h := g M^{-1.2}$ and compute $\|M^{1/2} \nabla h\|^2_{L^2}$ as a function of g and ∇g.]

Exercise 0.13. (The viscosity method) Consider the transport equation

$$\partial_t f = b \cdot \nabla f + cf \ \text{ in } \ (0,T) \times \mathbb{R}^d, \quad f(0,\cdot) = f_0 \in L^2(\mathbb{R}^d), \tag{0.7}$$

and its small viscosity regularized version

$$\partial_t f = \varepsilon \Delta f + b \cdot \nabla f + cf \ \text{ in } \ (0,T) \times \mathbb{R}^d, \quad f(0,\cdot) = f_0 \in L^2(\mathbb{R}^d), \tag{0.8}$$

with $\varepsilon > 0$.

(1) Under convenient assumptions on a, b show the existence of a weak solution to the transport equation (0.7) by using Lions' variant of the Lax-Milgram theorem.

(2) Establish the same result by proving first the existence of a solution to the parabolic equation (0.8) with $\varepsilon > 0$ and next passing to the limit $\varepsilon \to 0$.

[Hint for (1). Define $H := L^2(\mathcal{U})$,

$$\mathcal{E}(f,\varphi) := \int_{\mathcal{U}} f(\mathrm{div}(b\varphi) - c\varphi + \lambda\varphi - \partial_t \varphi)$$

for $\lambda \in \mathbb{R}$ such that $\mathrm{div} b/2 - c + \lambda \ge 1$ and use J.-L. Lions' variant of the Lax-Milgram theorem.]

Exercise 0.14. (Semigroup). For any $f_0 \in L^2(\mathbb{R}^d)$, consider (for instance) the parabolic equation

$$\partial_t f = \Delta f + b \cdot \nabla f + cf, \quad f(0) = f_0$$

with time independent coefficients b, $c \in L^\infty(\mathbb{R}^d)$ and denote by $f \in X_T$ the unique variational solution. Defining S by $S(t) f_0 := f(t)$ for any $t \ge 0$, establish that S is a continuous semigroup of bounded operators in $H := L^2(\mathbb{R}^d)$, and more precisely

(i) $f_0 \mapsto S(t) f_0$ is a linear and continuous mapping in H for any $t \ge 0$;

(ii) $t \mapsto S(t) f_0 \in C(\mathbb{R}_+; H)$ for any $f_0 \in H$;

(iii) $S(0) = I$ and $S(t+s) = S(t)S(s)$ for any $t, s \ge 0$.

The only property which has to be proved is the semigroup property in (iii) which is a consequence of the fact that the operator $\mathcal{L}$ does not depend of time and of the uniqueness of solutions.

[Hint. The only real thing to prove is the semigroup property $S_{t_1+t_2} = S_{t_2} S_{t_1}$ for any $t_i \ge 0$. For that purpose, defining $\tilde f(t) := f(t+t_1)$, establish that $\tilde f$ is a variational solution to the equation associated to the initial datum $\tilde f(0)$. Conclude thanks to the uniqueness result.]