Exercises about chapter 2

Exercise 0.1. (Representation formulas) We are concerned with the heat equation

$$\partial_t f = \frac{1}{2}\Delta f + g \tag{0.1}$$

with either an initial datum f_0 or a source term $g \neq 0$.

(1) Show that γ_t provides a fundamental solution to the heat equation (0.1) and that $\gamma_{t+s} = \gamma_t * \gamma_s$ for any $t, s > 0$.

(2) Show that $f_t := \gamma_t * f_0$ provides a solution to the heat equation (0.1) for any initial datum $f_0 \in L^q$, $q \in [1, \infty]$.

(3) Show that

$$\|\nabla_x \gamma_t\|_{L^r} = \frac{C_{d,r}}{t^{\frac{d}{2}(1-\frac{1}{r})+\frac{1}{2}}} \tag{0.2}$$

and recover the estimate $\|\nabla f_t\|_{L^2} \lesssim t^{-1}\|f_0\|_{L^2}$.

(4) We denote $\mathcal{U} := (0,T) \times \mathbb{R}^d$. For $g : \mathcal{U} \to \mathbb{R}$ (smooth and rapidly decaying) show that

$$f := \gamma *_{t,x} g = \int_0^t \gamma_{t-s} *_x g(s,\cdot) ds \tag{0.3}$$

provides a solution to the heat equation with source term g and vanishing initial datum.

(5) For $g \in L^1(\mathcal{U})$ establish that the solution f to the heat equation with source term given by (0.3) satisfies $f \in L^p(\mathcal{U})$ for any $1 < p < 1 + 2/d$. More generally and more precisely, establish that

$$\|f\|_{L^p(\mathcal{U})} \lesssim C T^{1-(1+\frac{d}{2})(\frac{1}{q}-\frac{1}{p}-1)} \|g\|_{L^q(\mathcal{U})}, \quad C := \frac{C_{r,d}}{(1-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})r)^{1/r}},$$

under the condition $1 \le q < p$, $(1+\frac{d}{2})(\frac{1}{q}-\frac{1}{p}) < 1$ and where $C_{r,d}$ and r are defined in (0.2).

Exercise 0.2. (Fourier transform) We consider the heat equation with source term

$$\partial_t f = \Delta f + \mathrm{div} G,$$

with $f, G \in L^2(\mathbb{R}^{d+1})$. Prove that there exists $p > 2$ such that $f \in L^p(\mathbb{R}^{d+1})$.

Exercise 0.3. (Variant proofs of Nash inequality using the Sobolev inequality)

1. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d \ge 3$. (Hint. Write the interpolation estimate

$$\|f\|_{L^2} \le \|f\|_{L^1}^{\theta} \|f\|_{L^{2^*}}^{1-\theta}$$

and then use the Sobolev inequality associated to the Lebesgue exponent $p = 2$).

2. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d = 2$. (Hint. Prove the interpolation estimate

$$\|f\|_{L^2} \le \|f\|_{L^1}^{1/4} \|f^{3/2}\|_{L^2}^{1/2},$$

then use the Sobolev inequality associated to the Lebesgue exponent $p = 1$ and $p^* := 2$ and finally the Cauchy-Schwartz inequality in order to bound the second term).

3. Give another proof of the Nash inequality by using the Sobolev inequality in dimension $d = 1$. (Hint. Prove the interpolation estimate

$$\|f\|_{L^2} \le \|f\|_{L^1}^{1/2} \|f^{3/2}\|_{L^\infty}^{1/3},$$

then use the Sobolev inequality associated to the Lebesgue exponent $p = 1$ and $p^* := \infty$ and finally the Cauchy-Schwartz inequality in order to bound the second term).

Exercise 0.4. (A variant proof of Nash inequality using the Poincaré-Wirtinger inequality) 1) Prove the Poincaré-Wirtinger inequality

$$\|f - f_r\|_{L^2} \le C\, r \|\nabla f\|_{L^2}, \quad f_r(x) := \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy,$$

for any $r > 0$ and some constant $C = C(d) > 0$.

2) Recover the Nash inequality in any dimension $d \ge 1$. *(Hint. Write that $\|f\|^2_{L^2} = (f, f - f_r) + (f, f_r)$ and deduce that $\|f\|^2_{L^2} \le C_1\, r\, \|f\|_{L^2}\, \|\nabla f\|_{L^2} + C_2\, r^{-d}\, \|f\|^2_{L^1}$, for any $r > 0$).*

Exercise 0.5. (Nash using the representation formula) Let us consider a solution f to the usual parabolic equation.

(1) Repeating the first above argument, establish that

$$\nu \int_0^T \varphi^2 \int |\nabla f|^2 \le \int_0^T \varphi'_+ \varphi \int f^2,$$

for any $0 \le \varphi \in \mathcal{D}(0, T)$.

(2) Establish that $f\varphi$ satisfies

$$\partial_t (f\varphi) - \Delta(f\varphi) = \mathrm{div}((A - I)\nabla f\varphi) + f\varphi'.$$

Using Exercice 1, prove that

$$\|f\varphi\|_{L^p} \le C \|(A - I)\nabla f\varphi\|_{L^2} + C\|f\varphi'\|_{L^2},$$

for some exponent $p = p(d) > 2$ and a constant $C = C(d) > 0$.

(3) Recover the ultracontractivity estimate by combining (1) and (2).

Exercise 0.6. (Interpolation inequality) (1) For any $1 \le p, q \le \infty$, $\theta \in (0, 1)$ and $f \in L^p \cap L^q$, prove that $f \in L^r$ and

$$\|f\|_{L^r} \le \|f\|^\theta_{L^p} \|f\|^{1-\theta}_{L^q}, \quad \text{with} \quad \frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}.$$

(2) For any $1 \le p_i, q_i \le \infty$, $\theta \in (0, 1)$ and $f \in L^{p_1} L^{p_2} \cap L^{q_1} L^{q_2}$, prove that $f \in L^{r_1} L^{r_2}$ and

$$\|f\|_{L^{r_1} L^{r_2}} \le \|f\|^\theta_{L^{p_1} L^{p_2}} \|f\|^{1-\theta}_{L^{q_1} L^{q_2}}, \quad \text{with} \quad \frac{1}{r_i} = \frac{\theta}{p_i} + \frac{1-\theta}{q_i}.$$

Hint. For (1), write $f^r = f^{r\theta} f^{r(1-\theta)}$ and use the Holder inequality with $s := p/(\theta r)$ and $t := q/((1-\theta)r)$. Verify that s and t are conjugate exponents.

For (2), start applying the interpolation inequality from (1) to $\|f\|_{L^{r_2}}$ and then the Holder inequality with $s := p_1/(\theta r_1)$ and $t := q_1/((1-\theta)r_1)$.

Exercise 0.7.

(1) Establish the ultracontractivity estimate for a variational solution to the heat equation starting from its very definition.

(2) Establish the ultracontractivity estimate for a solution to the heat equation set on a bounded domain with a Dirichlet or a Neumann boundary condition.

(3) Establish the ultracontractivity estimate for a solution to the parabolic equation

$$\frac{\partial f}{\partial t} = \mathrm{div}(A\nabla f) \quad \text{in } (0, \infty) \times \mathbb{R}^d \tag{0.4}$$

for a measurable, bounded and strictly elliptic matrix A by using Nash's method.

(4) Establish the ultracontractivity estimate for a solution to the parabolic equation

$$\frac{\partial f}{\partial t} = \operatorname{div}(A\nabla f + af) + b \cdot \nabla f \quad \text{in } (0,\infty) \times \mathbb{R}^d, \tag{0.5}$$

for a bounded strictly elliptic matrix A and bounded vector fields a, b by using Moser's method. What about Nash's method?

Exercise 0.8. **(Moser estimate)** Consider $q \in (0,\infty)$ and two cylinders $\mathbf{q} \subset\subset \mathbf{Q}$. Establish that there exists $p > q$ such that for any nonnegative solution f to the parabolic equation (0.4), there holds

$$\|f\|_{L^p(\mathbf{q})} \lesssim \|f\|_{L^q(\mathbf{Q})},$$

by just considering the equation satisfied by f^q. Generalize the argument for dealing with supersolution and subsolution depending of the value of q.

Exercise 0.9. Consider a variational solution f to the parabolic equation (0.5) in the cylinder $\mathfrak{C}_r$, $r > 0$, and define $f_r(t,x) := f(r^2 t, rx)$ on $\mathfrak{C}_1$. Show that f_r is the variational solution to the parabolic equation (0.5) with coefficients $A_r(t,x) = A(r^2 t, rx)$, $a_r(t,x) = ra(r^2 t, rx)$ and $b_r(t,x) = rb(r^2 t, rx)$. *(Hint. Prove the result for smooth coefficients and then use an regularization argument or make all the job on the variational formulation.)*