Exam, December 13, 2024

The exam is made of three problems. The first one is independent of the two others. The two next ones are not independent. One may obtain the maximal score without answering all the questions and in particular without dealing with the *"more difficult questions"* that one finds at the end of each problem. **These "more difficult questions" may be skipped during the written exam part**. The answers may be written in english or french (choose only one language!). Course notes are allowed during the written exam.

Preliminary question.

Establish the identity

$$\int \mathrm{div}(A\nabla f) f \varrho^2 = -\int A\nabla(f\varrho)\nabla(f\varrho) + \int f^2 A\nabla\varrho\nabla\varrho.$$

Problem 1 - A nonlinear parabolic equation.

The aim of this problem is to prove the existence of a solution $f = f(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$, to the nonlinear parabolic equation

$$\begin{cases} \partial_t f = \mathrm{div}(A_f \nabla f) \ \text{ in } \ (0,\infty)\times\mathbb{R}^d \\ f(0,\cdot) = f_0 \ \text{ on } \ \mathbb{R}^d, \end{cases} \tag{0.1}$$

with $A_f := a(f)I$ and $a : [0,\infty) \to (0,\infty)$ is a continuous function.

Question 1

For some $T, K \in (0,\infty)$, we define the set

$$\mathcal{Z} := \{f \in L^2((0,T)\times\mathbb{R}^d);\ f \geq 0,\ \|f\|_{L^2((0,T)\times\mathbb{R}^d)} \leq K\},$$

and the distance

$$\delta(f,g) := \sum_{n\geq 1} 2^{-n}(\|f-g\|_{L^2((0,T)\times B(0,n))} \wedge 1).$$

Establish that $(\mathcal{Z},\delta)$ is a convex and closed set. We accept that $L^2((0,T)\times\mathbb{R}^d)$ endowed with the topology associated to δ is a locally convex topological vector space.

Question 2

We assume first that

$$0 < a_* \le a(s) \le a^* < \infty, \quad \forall\, s \in \mathbb{R}_+, \quad \text{and} \quad 0 \le f_0 \in L^2(\mathbb{R}^d).$$

We define as usually

$$X_T := L^2(0,T; H^1(\mathbb{R}^d)) \cap H^1(0,T; H^{-1}(\mathbb{R}^d)).$$

Prove that for any $g \in \mathcal{Z}$, there exists a unique variational solution $0 \le h \in X_T$ to the linear problem

$$\begin{cases} \partial_t h = \operatorname{div}(A_g \nabla h) \ \text{ in } \ (0,\infty) \times \mathbb{R}^d \\ h(0,\cdot) = f_0 \ \text{ on } \ \mathbb{R}^d. \end{cases} \tag{0.2}$$

Establish next that there exists at least one solution $f \in \mathcal{Z}$ to the nonlinear parabolic equation (0.1) for any fixed $T > 0$ and a suitable choice of $K > 0$.

Question 3 (more difficult)

We assume next that

$$a(s) := (1+s)^{-1}, \ \forall\, s \in \mathbb{R}_+, \quad \text{and} \quad 0 \le f_0 \in L^\infty(\mathbb{R}^d).$$

Establish that there exists at least one solution $0 \le f \in L^\infty((0,T) \times \mathbb{R}^d)$ to the nonlinear parabolic equation (0.1) for any fixed $T > 0$.

Problem 2 - Kruzkov approach

The aim of this problem is to prove a quantitative version of the strong maximum principle and to deduce in the next problem the Doblin-Harris condition. That will provide an alternative approach to the one used during the course which was based on the Harnack inequality. In the next lines, it is thus forbidden to use the Harnack inequality !

We first consider the parabolic equation

$$\partial_t f = \operatorname{div}(A \nabla f) + b \cdot \nabla f \ \text{ in } \ (0,T) \times \mathbb{R}^d, \tag{0.1}$$

with $0 < \nu I \le A \in L^\infty(\mathbb{R}^d)$, $b \in L^\infty_{\text{loc}}(\mathbb{R}^d)$, $d \ge 1$ and $T > 0$. We denote

$$\mathcal{Y} := \{\varphi : (\tau, T) \times \mathbb{R}^d \to \mathbb{R};\ \varphi \in W^{1,\infty},\ D^2_x \varphi \in L^\infty,\ \varphi(t,\cdot) \in C_c(\mathbb{R}^d),\ \forall\, t \in (\tau, T)\}.$$

We say that a function $f = f(t,x)$, $t \ge 0$, $x \in \mathbb{R}^d$, is a **local variational solution** if $f \in X_T := C([0,T]; L^2(\mathbb{R}^d)) \cap L^2(0,T; H^1(\mathbb{R}^d)) \cap H^1(0,T; H^{-1}(\mathbb{R}^d))$ and for any $\varphi \in \mathcal{Y}$ the function $f\varphi \in X_T$ satisfies the naturally associated equation in the variational sense.

We recall that for two cylinders $\mathbf{q} := (\tau_0, \tau_1) \times B_r$ and $\mathbf{q}' := (\tau_0', \tau_1') \times B_{r'}$, we write $\mathbf{q}' \prec \mathbf{q}$ if $\tau_0 < \tau_0' < \tau_1' \le \tau_1$ and $0 < r' < r$. In the sequel, we will consider five cylinders

$$\mathbf{Q} = (0,T) \times B_R, \quad \mathbf{Q}_0 = (T_0, T_2) \times B_{R_0}, \quad \mathbf{Q}_i = (T_i, T) \times B_{R_i}, \ i = 1,2,3,$$

with $0 < T_0 = T_3 < T_2 < T_1 < T$, $0 < R_0 < R$ and $0 < R_1 < R_2 < R_3 < R$, in such a way that

$$\mathbf{Q}_0 \prec \mathbf{Q}, \quad \mathbf{Q}_1 \prec \mathbf{Q}_2 \prec \mathbf{Q}_3 \prec \mathbf{Q}, \quad \mathbf{Q}_0 \cap \mathbf{Q}_2 = \emptyset.$$

Question 4

Consider a local variational solution g to (0.1), $T_0 \in [0,T)$, $\Psi \in \mathcal{Y}$ such that $\Psi(T_0) = 0$ and h the (variational) solution to the heat equation with source term

$$\begin{cases} \partial_t h - \Delta h = g(\partial_t \Psi - \Delta \Psi) \ \text{ in } \ (T_0, T) \times \mathbb{R}^d \\ h(T_0, \cdot) = 0 \ \text{ on } \ \mathbb{R}^d. \end{cases} \tag{0.2}$$

(a) Establish that $G := g\Psi - h$ satisfies

$$\partial_t G - \Delta G = G_0 + \operatorname{div} G_1 \ \text{ in } \ (T_0, T) \times \mathbb{R}^d,$$

with $G_0 := \Psi b \cdot \nabla g - (A \nabla g + \nabla g) \cdot \nabla \Psi$ and $G_1 := (A \nabla g - \nabla g) \Psi$.
(b) For any $t \in (T_0, T)$ and $\varepsilon \in (0,1)$, establish that

$$\frac{1}{2} \int_{\mathbb{R}^d} G(t)_+^2 \le \frac{1}{4} \int_{T_0}^t \int_{\mathbb{R}^d} G_1^2 + \frac{1}{2\varepsilon} \int_{T_0}^t \int_{\mathbb{R}^d} G_0^2 + \frac{\varepsilon}{2} \int_{T_0}^t \int_{\mathbb{R}^d} G_+^2.$$

(c) We fix two cylinders $\mathbf{Q}_2 := (T_2, T) \times B_{R_2} \prec \mathbf{Q}_3 := (T_0, T) \times B_{R_3}$. Integrating in time again and choosing $\varepsilon \in (0,1)$ and Ψ conveniently, deduce that

$$\|(g - h)_+\|_{L^2(\mathbf{Q}_2)} \le C(\mathbf{Q}_2, \mathbf{Q}_3) \|\nabla g\|_{L^2(\mathbf{Q}_3)}.$$

Question 5

For $0 < T_0 < T_2 < T$, we denote by $\Psi_1 \in \mathcal{Y}$ a function such that

$$\mathbf{1}_{(T_2,T) \times B_{R_1}} \le \Psi_1 \le \mathbf{1}_{(T_0,T) \times B_{R_2}}, \quad \partial_t \Psi_1 \ge \kappa \mathbf{1}_{(T_0,T_2) \times B_{R_2}}, \quad \kappa > 0.$$

For $R := \eta \bar{R}, \ \eta > R_0/R_2$, $\bar{R} := \max(R_0, R_2)$, we define $\Psi(t,x) := \Psi_1(t, x/\eta)$ as well as

$$\mathbf{Q} := (0,T) \times B_R, \quad \mathbf{Q}_0 := (T_0, T_2) \times B_{R_0}, \quad \mathbf{Q}_2 := (T_2, T) \times B_{R_2}.$$

We consider a **nonnegative** local variational solution g to (0.1) such that $g \le M$ on $\mathbf{Q}$ and we define $\mathcal{P}$ and $\mathcal{E}$ the solutions to

$$\begin{cases} (\partial_t - \Delta)\mathcal{P} = S_{\mathcal{P}} := (M - g)(\partial_t \Psi_1)(t, x/\eta) \ \text{ in } \ (T_0, T) \times \mathbb{R}^d \\ (\partial_t - \Delta)\mathcal{E} = S_{\mathcal{E}} := (M - g)\eta^{-2}(\Delta \Psi_1)(t, x/\eta) \ \text{ in } \ (T_0, T) \times \mathbb{R}^d \\ \mathcal{P}(T_0) = \mathcal{E}(T_0) = 0 \ \text{ in } \ \mathbb{R}^d. \end{cases} \tag{0.3}$$

(a) With the notations of Question 4, prove that

$$h - M\Psi = -\mathcal{P} + \mathcal{E}$$

and, denoting by $\gamma = \gamma(s,y,t,x) = \gamma_{t-s}(x-y)$ the heat kernel, prove that

$$u(t,\cdot) = \int_0^t \gamma_{t-s} *_x S_u(s,\cdot) ds, \quad u = \mathcal{P},\, \mathcal{E}.$$

(b) Prove that there exists a constant $C = C(\Psi_1) \geq 0$ such that

$$\|\mathcal{E}\|_{L^\infty((0,T)\times\mathbb{R}^d)} \leq CM\eta^{-2}, \quad \forall\, \eta > 1.$$

(c) Prove that

$$\mathcal{P}(t,x) \geq M\kappa[\min_{\mathbf{Q}_0\times\mathbf{Q}_2} \gamma]\, |\{g=0\}\cap \mathbf{Q}_0| \mathbf{1}_{\mathbf{Q}_2}(t,x).$$

(d) We assume furthermore that $|\{g=0\}\cap\mathbf{Q}_0| \geq \delta$, with $\delta \in (0,1)$. Deduce from the above questions that there exists $\eta > R_0/R_2$ large enough and $\vartheta := \vartheta(\mathbf{Q}_0, \mathbf{Q}, \Psi_1, \delta) \in (0,1)$ such that

$$h \leq \vartheta M \ \text{ on } \ \mathbf{Q}_2.$$

Question 6

With the notations of question 5, we consider a **nonnegative** local variational solution g to the parabolic equation (0.1) such that $g \leq M$ on $\mathbf{Q}$ and $|\{g=0\}\cap\mathbf{Q}_0| \geq \delta$, with $M > 0$ and $\delta \in (0,1)$. For $T_1 \in (T_2, T)$, $R_1 \in (0, R_2)$, we set $\mathbf{Q}_1 := (T_1, T)\times B_{R_1}$.
(a) Establish that $\mathcal{G} := g - \vartheta M$ is a local variational solution to the parabolic equation (0.1) and deduce that

$$\|\mathcal{G}_+\|_{L^\infty(\mathbf{Q}_1)} \lesssim \|\mathcal{G}_+\|_{L^2(\mathbf{Q}_2)}.$$

(b) Together with the estimates established in Questions 4 and 5, deduce that

$$g - \vartheta M \lesssim \|\nabla g\|_{L^2(\mathbf{Q}_3)} \ \text{ on } \ \mathbf{Q}_1,$$

for any cylinder $\mathbf{Q}_3 := (T_3, T)\times B_{R_3}$ with $0 < T_3 < T_2$ and $R_2 < R_3 < R$.
(c) Observe that the same estimate holds when we assume that g is a subsolution (instead of a solution) to the parabolic equation (0.1).

Question 7

For a **nonnegative** local variational solution f to the parabolic equation (0.1) and two parameters $\varepsilon, \lambda > 0$, we define define $\beta(s) := (\log s)_-$ and $g := \beta(f/\lambda + \varepsilon)$.
(a) Establish that g is a subsolution to the parabolic equation (0.1) and more precisely that g satisfies

$$\partial_t g - \operatorname{div}(A\nabla g) - b\cdot\nabla g \leq -\nu|\nabla g|^2.$$

(b) Deduce that

$$\int_{\mathbf{Q}_3} |\nabla g|^2 \lesssim \int_{\mathbf{Q}} g + 1.$$

Question 8 (more difficult) (Quantitative strong maximum principle)

We consider a **nonnegative** local variational solution f to the parabolic equation (0.1). We define the cylinders $\mathbf{Q}$ and $\mathbf{Q}_0$ as in Question 5 and we assume that

$$|\{f \geq \lambda\} \cap \mathbf{Q}_0| \geq \delta \quad \text{for some } \delta, \lambda > 0.$$

(a) We define g as in Question 4 for some $\varepsilon > 0$. Prove that $|\{g = 0\} \cap \mathbf{Q}_0| \geq \delta$ and $g \leq \beta(\varepsilon)$.
(b) We define $\mathbf{Q}_1$ and ϑ as in Question 3. Using Questions 6 and 7, deduce that

$$g - \vartheta\beta(\varepsilon) \lesssim \sqrt{\beta(\varepsilon) + 1} \quad \text{on} \quad \mathbf{Q}_1.$$

(c) Choosing $\varepsilon \in (0, 1)$ small enough, deduce that there exists $\vartheta_1 \in (\vartheta, 1)$ such that

$$g \leq \vartheta_1 \beta(\varepsilon) \quad \text{on} \quad \mathbf{Q}_1,$$

and next that there exists $\ell > 0$ such that

$$f \geq \ell\lambda \quad \text{on} \quad \mathbf{Q}_1.$$

Problem 3 - The Doblin-Harris condition.

From now-on, we consider the Fokker-Planck equation

$$\begin{cases} \partial_t f = \operatorname{div}(A\nabla f) + \operatorname{div}(xf) & \text{in} \;\; (0, \infty) \times \mathbb{R}^d \\ f(0, \cdot) = f_0 & \text{on} \;\; \mathbb{R}^d \end{cases} \tag{0.4}$$

on the function $f = f(t, x)$, $t \geq 0$, $x \in \mathbb{R}^d$, with $0 < \nu I \leq A \in L^\infty(\mathbb{R}^d)$.

Question 9

For $f_0 \in H := L^2_\varrho$, $\varrho := e^{\alpha|x|^2}$, $\alpha > 0$ conveniently chosen, recall why there exists a unique variational solution

$$f \in \mathcal{X}_T := C([0, T]; H) \cap L^2(0, T; V) \cap H^1(0, T; V'), \; \forall T > 0, \quad V := H^1_\varrho,$$

to the Fokker-Planck equation (0.4) and that $f(t, \cdot) \geq 0$ for any $t \geq 0$ if $f_0 \geq 0$.

Question 10

We set $\omega := \langle x \rangle$ and we consider $0 \leq f_0 \in L^2_\varrho$ such that $\operatorname{supp} f_0 \subset B_{R_*}$, $R_* > 0$.
(a) Establish that the associated solution satisfies

$$\int f_t = \int f_0,$$

and that under the additional assumption $A \in W^{1,\infty}(\mathbb{R}^d)$ it satisfies

$$\int f_t \omega \le C\omega(R_*) \int f_0, \quad \forall\, t \ge 0, \tag{0.5}$$

for some $C \ge 1$. Until the end of this question and in the next two questions we will always assume that (0.5) holds true.

(b) By writing

$$\int_{B_R} f_t \;=\; \int f_t - \int_{B_R^c} f_t$$

for any $t, R > 0$, deduce that one may choose $R_0 > R_*$ large enough so that

$$\int_{B_{R_0}} f_t \ge \frac{1}{2} \int_{B_{R_*}} f_0, \quad \forall\, t \ge 0.$$

(c) By writing

$$\int_{B_{R_0}} f_t \le \int_{B_{R_0}} f_t \mathbf{1}_{f_t > \lambda} + \int_{B_{R_0}} f_t \mathbf{1}_{f_t \le \lambda},$$

for any $t, \lambda > 0$, deduce that one may choose $\lambda_0 > 0$ small enough so that

$$\|f_t\|_{L^\infty} |\{f_t > \lambda_0\} \cap B_{R_0}| \ge \frac{1}{4} \int_{B_{R_*}} f_0, \quad \forall\, t \ge 0.$$

(d) Establish that

$$\|f_t\|_{L^\infty} \le C \frac{e^{(d/2)t}}{t^{d/2}} \int_{B_{R_*}} f_0, \quad \forall\, t > 0,$$

for some $C > 0$ and deduce that

$$\frac{t^{d/2}}{4Ce^{(d/2)t}} \le |\{f_t \ge \lambda_0\} \cap B_{R_0}|, \quad \forall\, t > 0.$$

Question 11

We fix $T > 0$ and $R_* > 0$. Deduce from Questions 8 and 10, that there exist $K, r_1 > 0$ such that the solution f associated to an initial condition $0 \le f_0 \in L^2_\varrho$ satisfies

$$f_T \ge K \mathbf{1}_{B_{r_1}} \int_{B_{R_*}} f_0. \tag{0.6}$$

Question 12 (more difficult)

Prove that there exists a unique steady state

$$0 \le f_\infty \in X, \quad \operatorname{div}(A\nabla f_\infty) + \operatorname{div}(x f_\infty) = 0, \quad \langle f_\infty \rangle := \int f_\infty = 1$$

and there exist two constants $\lambda_1 > 0$ and $C \ge 0$ such that any variational solution f associated to the Fokker-Planck equation (0.4) satisfies

$$\|f(t) - f_\infty \langle f_0 \rangle\|_X \le C e^{-\lambda_1 t} \|f_0 - f_\infty \langle f_0 \rangle\|_X, \quad \forall\, t \ge 0,$$

for a convenient choice of X *(Hint.* $X := L^1_\omega$*)*.

Question 13 (more difficult)

We do not make the assumption $A \in W^{1,\infty}(\mathbb{R}^d)$ anymore. We define now $\omega := \langle x \rangle^k$, $k := d/2 + 1$. We observe that $L^2_\omega = L^2_k \subset L^1_1$.

(a) Prove that for any solution f to the Fokker-Planck equation (0.4), there holds

$$\frac{d}{dt} \int f^2 \omega^2 \le -c_1 \|f\omega\|^2_{H^1} + C_1 \int_{B_R} f^2$$

for some constants $R, c_1, C_1 > 0$.

(b) Prove that for any $\varepsilon, R > 0$ there exists a constant $C = C(\varepsilon, R)$ such that

$$\int_{B_{R_*}} g^2 \le \varepsilon \|g\omega\|^2_{H^1} + C\|g\|^2_{L^1}$$

for any $g\omega \in H^1(\mathbb{R}^d)$.

(c) Deduce that there exists a constant $C > 0$ such that any solution f to the Fokker-Planck equation (0.4) satisfies

$$\sup_{t \ge 0} \|f\omega\|_{L^2} \le C \|f_0 \omega\|_{L^2}.$$

(c) We fix $T, R_* > 0$. Adapting Questions 10 and 11, establish that for any $Q > 0$, there exists $K, r_1 > 0$ such that for any initial condition $0 \le f_0 \in L^2_\varrho$ satisfying $\|f_0\|_{L^2_k} \le Q \|f_0\|_{L^1}$, the associated solution f to the Fokker-Planck equation (0.4) satisfies (0.6).

(d) Recover the conclusions of Question 12 with a convenient choice of X *(Hint.* $X := L^2_\varrho$*)*.