CHAPTER 5 - MARKOV SEMIGROUP

Contents

1. Markov semigroup

In this chapter, we are interested in Markov semigroups which is a class of semigroups which enjoy both a positivity and a "conservativity" property. The importance of Markov semigroups comes from its deep relation with Markov processes in stochastic theory as well as from the fact that a quite satisfactory description of the longtime behaviour of such a semigroups can be performed.

We start with the notion of positivity. It can be formulated in the abstract framework of Banach lattices $(X, \|\cdot\|, \geq)$ which are Banach spaces endowed with compatible order relation or equivalently with an appropriate positive cone X_+. To be more concrete, we just observe that the following three examples are Banach lattices when endowed with their usual order relation:

• $X := C_0(E)$, the space of continuous functions which tend to 0 at infinity (when E is not a compact set) endowed with the uniform norm $\|\cdot\|$;

• $X := L^p(E) = L^p(E, \mathcal{E}, \mu)$, the Lebesgue space of functions associated to the Borel σ-algebra $\mathcal{E}$, a positive σ-finite measure μ and an exponent $p \in [1, \infty]$;

• $X := M^1(E) = (C_0(E))'$, the space of Radon measures defined as the dual space of $C_0(E)$.

Here E denotes a σ-locally compact metric space (typically $E \subset \mathbb{R}^d$) and in the last example the positivity can be defined by duality: $\mu \geq 0$ if $\langle \mu, \varphi \rangle \geq 0$ for any $0 \leq \varphi \in C_0(E)$.

Lemma 1.1. *Consider X a Banach lattice (one of the above examples), a bounded linear operator A on X and its dual operator A^* on X'. The following equivalence holds:*
(1) A is positive, namely $Af \geq 0$ for any $f \in X$, $f \geq 0$;
(2) A^* is positive, namely $A^*\varphi \geq 0$ for any $\varphi \in X'$, $\varphi \geq 0$.

The (elementary) proof is left as an exercise. We emphasize that $\langle f, \varphi \rangle \geq 0$ for any $\varphi \in X'_+$ (resp. for any $f \in X_+$) implies $f \in X_+$ (resp. $\varphi \in X'_+$).

There are two "equivalent" (or "dual") ways to formulate the notion of Markov semigroup.

Definition 1.2. *On a Banach lattice $Y \supset C_0(E)$ we say that (P_t) ia a (constants conservative) Markov semigroup if*
(1) (P_t) is a continuous semigroup in Y;
(2) (P_t) is positive, namely $P_t \geq 0$ for any $t \geq 0$;
(3) (P_t) is conservative, namely $\mathbf{1} \in Y$ and $P_t\mathbf{1} = \mathbf{1}$ for any $t \geq 0$.

Definition 1.3. *On a Banach lattice $X \subset M^1(E)$ we say that (S_t) ia a (mass conservative) Markov semigroup if*
(1) (S_t) is a (strongly or weakly $*$ continuous) continuous semigroup in X;
(2) (S_t) is positive, namely $S_t \ge 0$ for any $t \ge 0$;
(3) (S_t) is conservative, namely $\langle S_t f\rangle = \langle f\rangle$, $\forall\, t \ge 0$, $\forall\, f \in X$, where $\langle g\rangle := \langle g, \mathbf{1}\rangle$.

The two notions are dual. In particular, if (P_t) is a (constants conservative) Markov semigroup on $Y \supset C_0(E)$, the dual semigroup (S_t) defined by $S_t := P_t^*$ on $X := Y'$ is a (mass conservative) Markov semigroup. In the sequel we will only consider (mass conservative) Markov semigroups defined on $X \subset L^1(E)$.

Markov semigroup and semigroup of contractions for the L^1 are closely linked.

Proposition 1.4. *A Markov semigroup is a semigroup of contractions for the L^1 norm. In the other way round, a mass conservative semigroup of contractions for the L^1 norm is postive, and thus it is a Markov semigroup.*

Proof of Proposition 1.4. We fix $f \in X$ and $t \ge 0$. We write

$$\begin{aligned} |S_t f| &= |S_t f_+ - S_t f_-| \\ &\le |S_t f_+| + |S_t f_-| \\ &= S_t f_+ + S_T f_- \\ &= S_t |f|, \end{aligned}$$

where we have used the positivity property in the third line. We deduce

$$\int |S_t f| \le \int S_t |f| = \int |f|,$$

because of the mass conservation. The reciprocal part is left to the reader. □

We may also characterize a Markov semigroup in terms of its generator.

Theorem 1.5. *Let $S = S_{\mathcal{L}}$ be a strongly continuous semigroup on a Banach space $X \subset L^1$. There is equivalence between*
(a) $S_{\mathcal{L}}$ is a Markov semigroup;
(b) $\mathcal{L}^* 1 = 0$ and $\mathcal{L}$ satisfies Kato's inequality

$$(\operatorname{sign} f)\mathcal{L} f \le \mathcal{L}|f|, \quad \forall\, f \in D(\mathcal{L}).$$

Partial proof of Theorem 1.5. Step 1. We prove $(a) \Rightarrow (b)$. On the one hand, for any $f \in D(\mathcal{L})$ and any $0 \le \psi \in D(\mathcal{L}^*)$, we have

$$\begin{aligned} \langle \psi, (\operatorname{sign} f)\mathcal{L} f\rangle &= \lim_{t\to 0} \frac{1}{t} \langle \psi, (\operatorname{sign} f)(S(t) f - f)\rangle \\ &\le \lim_{t\to 0} \frac{1}{t} \langle \psi, |S(t) f| - |f|\rangle \\ &\le \lim_{t\to 0} \frac{1}{t} \langle \psi, S(t)|f| - |f|\rangle \\ &= \lim_{t\to 0} \frac{1}{t} \langle S^*(t)\psi - \psi, |f|\rangle \\ &= \langle \mathcal{L}^*\psi, |f|\rangle, \end{aligned}$$

where we have used the inequality $(\operatorname{sign} f) g \le |g|$ in the second line and the positivity assumption in the third line. That inequality is the weak formulation of Kato's inequality. On the other hand and similarly, for any $f \in D(\mathcal{L})$, we have

$$\begin{aligned} \langle \mathcal{L}^* \mathbf{1}, f\rangle &= \langle \mathbf{1}, \mathcal{L} f\rangle \\ &= \lim_{t\to 0} \frac{1}{t} \langle \mathbf{1}, S(t) f - f\rangle = 0, \end{aligned}$$

by just using the mass conservation property. The reciprocal part is left to the reader. □

2. Asymptotic of Markov semigroups

2.1. **Strong positivity condition and Doeblin Theorem.** We consider the case of a strong positivity condition.

Theorem 2.1 (Doeblin)**.** *Consider a Markov semigroup S_t such that*

$$S_T f \ge \alpha\, \nu\, \langle f \rangle, \quad \forall\, f \in X_+,$$

for some constants $T > 0$ and $\alpha \in (0,1)$ and some probability measure ν. There holds

$$\|S_t f\|_{L^1} \le C\, e^{at} \|f\|_{L^1}, \quad \forall\, t \ge 0,\ \forall\, f \in X,\ \langle f \rangle = 0,$$

for some constants $C \ge 1$ and $a < 0$.

Proof of Theorem 2.1. We fix $f \in X$ such that $\langle f \rangle = 0$ and we define $\eta := \alpha\nu\langle f_+\rangle = \alpha\nu\langle f_-\rangle$. We write

$$\begin{aligned} |S_T f| &= |S_T f_+ - \eta - S_T f_- + \eta| \\ &\le |S_T f_+ - \eta| + |S_T f_- - \eta| \\ &= S_T f_+ - \eta + S_T f_- - \eta, \end{aligned}$$

where in the last equality we have used the Doeblin condition. Integrating, we deduce

$$\begin{aligned} \int |S_T f| &\le \int S_T f_+ - \alpha\langle\nu\rangle\langle f_+\rangle + \int S_T f_- - \alpha\langle\nu\rangle\langle f_-\rangle \\ &\le \int f_+ - \alpha\, \langle f_+\rangle + \int f_- - \alpha\, \langle f_-\rangle \\ &\le (1-\alpha) \int |f|. \end{aligned}$$

By induction, we obtain $a := [\log(1-\alpha)]/T$ and $C := \exp[|a|T]$. □

2.2. **Geometric stability under Harris and Lyapunov conditions.** We consider now a semigroup S with generator $\mathcal{L}$ and we assume that

(H1) there exists some weight function $m : \mathbb{R}^d \to [1,\infty)$ satisfying $m(x) \to \infty$ as $x \to \infty$ and there exist some constants $\alpha > 0, b > 0$ such that

$$\mathcal{L}^* m \le -\alpha\, m + b;$$

(H2) for any $R > 0$ there exist a constant $T > 0$ and a positive and not zero measure ν such that

$$S_T f \ge \nu \int_{B_R} f, \quad \forall\, f \in X_+.$$

Proposition 2.2 (Doeblin)**.** *Consider a Markov semigroup S on $X := L^1(m)$ which satisfies (H1) and (H2). There holds*

$$\|S_t f\|_{L^1(m)} \le C\, e^{at} \|f\|_{L^1(m)}, \quad \forall\, t \ge 0,\ \forall\, f \in X,\ \langle f \rangle = 0,$$

for some constants $C \ge 1$ and $a < 0$.

We start with a variant of the key argument in the above Doeblin's Proposition.

Lemma 2.3 (Doeblin's variant)**.** *Under assumption (H2), if $f \in L^1(m)$, with $m(x) \to \infty$ as $|x| \to \infty$, satisfies*

$$\|f\|_{L^1} \ge \frac{4}{m(R)} \|f\|_{L^1(m)} \quad \text{and} \quad \langle f \rangle = 0, \tag{2.1}$$

we then have

$$\|S_T f\|_{L^1} \le \big(1 - \frac{\langle \nu \rangle}{2}\big) \|f\|_{L^1}.$$

Proof of Lemma 2.3. From the hypothesis (2.1), we have

$$\begin{aligned}\int_{B_R} f_\pm &= \int f_\pm - \int_{B_R^c} f_\pm \\ &\geq \frac{1}{2}\int |f| - \frac{1}{m(R)}\int |f| m \geq \frac{1}{4}\int |f|.\end{aligned}$$

Together with (H2), we get

$$S_T f_\pm \geq \frac{\nu}{4}\int |f| =: \eta.$$

We deduce

$$|S_T f| \leq |S_T f_+ - \eta| + |S_T f_- - \eta| = S_T f_+ - \eta + S_T f_- - \eta = S_T |f| - 2\eta,$$

and next

$$\int |S_T f| \leq \int S_T |f| - 2\int \eta = \int |f| - \langle \nu \rangle \frac{1}{2}\int |f|,$$

which is nothing but the announced estimate. □

Proof of Theorem 2.2. We split the proof in several steps.

Step 1. We fix $f_0 \in L^1(m)$, $\langle f_0 \rangle = 0$ and we denote $f_t := S_t f_0$. From (H1), we have

$$\frac{d}{dt}\|f_t\|_{L^1(m)} \leq -\alpha \|f_t\|_{L^1(m)} + b\|f_t\|_{L^1},$$

from what we deduce

$$\|f_t\|_{L^1(m)} \leq e^{-\alpha t}\|f_0\|_{L^1(m)} + \big(1 - e^{-\alpha t}\big)\frac{b}{\alpha}\|f_0\|_{L^1} \quad \forall\, t \geq 0.$$

In other words, we have proved

$$\|S_T f_0\|_{L^1(m)} \leq \gamma \|f_0\|_{L^1(m)} + K\|f_0\|_{L^1}, \tag{2.2}$$

with $\gamma \in (0,1)$ and $K > 0$. We fix $R > 0$ large enough such that $K/A \leq (1-\gamma)/2$ with $A := m(R)/4$. We also recall that

$$\|S_T f_0\|_{L^1} \leq \|f_0\|_{L^1}. \tag{2.3}$$

We define

$$\|f\|_\beta := \|f\|_{L^1} + \beta \|f\|_{L^1(m)}, \quad \beta > 0,$$

and we observe that the following altenative holds

$$\|f_0\|_{L^1(m)} \leq A\|f_0\|_{L^1} \tag{2.4}$$

or

$$\|f_0\|_{L^1(m)} > A\|f_0\|_{L^1}. \tag{2.5}$$

Step 2. We observe that under condition (2.4), there holds

$$\|S_T f_0\|_{L^1} \leq \gamma_1 \|f_0\|_{L^1}, \quad \gamma_1 \in (0,1), \tag{2.6}$$

and more precisely $\gamma_1 := 1 - \langle \nu \rangle/2$, which is nothing but the conclusion of Lemma 2.3.

Step 3. We claim that under condition (2.4), there holds

$$\|S_T f_0\|_\beta \leq \gamma_2 \|f_0\|_\beta, \quad \gamma_2 := \max\Big(\frac{\gamma_1 + 1}{2}, \gamma\Big) \tag{2.7}$$

for $\beta > 0$ small enough. Indeed, using (2.2) and (2.7), we compute

$$\begin{aligned}\|S_T f_0\|_\beta &= \|S_T f_0\|_{L^1} + \beta \|S_T f_0\|_{L^1(m)} \\ &\leq (\gamma_1 + K\beta)\|f_0\|_{L^1} + \gamma\beta\|f_0\|_{L^1(m)},\end{aligned}$$

and we take $\beta > 0$ such that $\gamma_1 + K\beta \leq \gamma_2$.

Step 4. We claim that under condition (2.5), there holds

$$\|S_T f_0\|_{L^1(m)} \leq \gamma_3 \|f_0\|_{L^1(m)}, \quad \gamma_3 := \frac{\gamma + 1}{2}. \tag{2.8}$$

Indeed we compute

$$\|S_T f_0\|_{L^1(m)} \le \gamma \|f_0\|_{L^1(m)} + \frac{K}{A}\|f_0\|_{L^1(m)} = \gamma_3 \|f_0\|_{L^1(m)}$$

Step 5. We claim that under condition (2.5), there holds

$$\|S_T f_0\|_\beta \le \gamma_4 \|f_0\|_\beta, \quad \gamma_4 := \frac{\gamma_3 + 1/\beta}{1+1/\beta}. \tag{2.9}$$

Indeed, using (2.3) and (2.8), we compute

$$\begin{aligned} \|S_T f_0\|_\beta &= \|S_T f_0\|_{L^1} + \beta \|S_T f_0\|_{L^1(m)} \\ &\le \|f_0\|_{L^1} + \gamma_3 \beta \|f_0\|_{L^1(m)}, \\ &\le (1-\varepsilon)\|f_0\|_{L^1} + (\varepsilon + \gamma_3\beta)\|f_0\|_{L^1(m)}, \end{aligned}$$

and we choose $\varepsilon \in (0,1)$ such that $1-\varepsilon = \varepsilon/\beta + \gamma_3$.

Step 6. By gathering (2.7) and (2.9), we see that we have

$$\|S_T f_0\|_\beta \le \gamma_5 \|f_0\|_\beta, \quad \gamma_5 := \max(\gamma_2, \gamma_4) \in (0,1),$$

for some well choosen $\beta > 0$. By iteration, we get

$$\|S_{nT} f_0\|_\beta \le \gamma_5^n \|f_0\|_\beta,$$

and we then conclude in a standard way. □

3. An example: the renewal equation

We will discuss now the renewal equation for which we apply some of the results of the preceding sections in order to get some insight about its qualitative behavior in the large time asymptotic. We are thus interesting by the renewal equation

$$\begin{cases} \partial_t f + \partial_x f + af = 0 \\ f(t,0) = \rho_{f(t)}, \quad f(0,x) = f_0(x), \end{cases} \tag{3.1}$$

where $f = f(t,x)$, $t \ge 0$, $x \ge 0$, and

$$\rho_g := \int_0^\infty g(y)\, a(y)\, dy.$$

Here f typically represents a population of cells (particles) which are aging (getting holder), die (disappear) with rate $a \ge 0$, born again (reappear) with age $x = 0$ and has distribution f_0 at initial time. At least at a formal level, any solution of (3.1) satisfies

$$\frac{d}{dt}\int_0^\infty f\, dx = \int_0^\infty (-\partial_x f - af)\, dx = \big[-f \big]_0^\infty - \int_0^\infty af\, dx = 0,$$

so that the mass is conserved. Similarly, we have

$$\frac{d}{dt}\int_0^\infty |f|\, dx = \int_0^\infty (-\partial_x |f| - a|f|)\, dx = \big[-|f| \big]_0^\infty - \int_0^\infty a|f|\, dx \le 0,$$

so that the sign of the solution is preserved by observing that $g_- = (|g| + g)/2$ and using the above two informations. That seems to indiquate that if (3.1) defines a semigroup, this one is a L^1 Markov semigroup.

Preliminarily, we consider the (simpler) transport equation with boundary condition

$$\begin{cases} \partial_t f + \partial_x f + af = 0 \\ f(t,0) = \rho(t), \quad f(0,x) = f_0(x), \end{cases} \tag{3.2}$$

with f_0 and ρ are given data. We observe that when f is smooth (C^1) and satisfies (3.2), we have

$$\frac{d}{ds}[f(t+s,x+s)e^{A(x+s)}] = 0, \quad A(x) := \int_0^x a(y)\, dy,$$

from what we deduce

$$f(t,x)e^{A(x)} = f(t-s,x-s)e^{A(x-s)},$$

when both terms are well defined. Choosing either $s = t$ or $s = x$, we get

$$f(t,x) = f_0(x-t)\,e^{A(x-t)-A(x)}\,\mathbf{1}_{x>t} \;+ \rho(t-x)\,e^{-A(x)}\,\mathbf{1}_{x<t}. \tag{3.3}$$

In the other way round, we may check that for any smooth functions a, f_0, ρ, the above formula gives a classical solution to (3.2) at least in the region $\{(t,x) \in \mathbb{R}_+^2,\ x \neq t\}$, and thus a weak solution to (3.2) in the sense

$$\int_0^\infty\int_0^\infty f\,(-\partial_t\varphi - \partial_x\varphi + a\varphi)\,dxdt - \int_0^\infty f_0(x)\varphi(0,x)\,dx - \int_0^\infty \rho(t)\varphi(t,0)\,dt = 0, \tag{3.4}$$

for any $\varphi \in C^1_c(\mathbb{R}^2_+)$. It is worth noticing that this last equation is also the weak formulation of the evolution equation with source term

$$\partial_t f + \partial_x f + af = \rho(t)\delta_0, \quad f(0,x) = f_0(x),$$

defined on the all line (that is for any $x \in \mathbb{R}$).

At least at a formal level, for any solution f to (3.2), we may compute

$$\frac{d}{dt}\int_0^\infty |f|\,dx = \big[\,-|f|\,\big]_0^\infty - \int_0^\infty a|f|\,dx \le |\rho(t)|,$$

so that

$$\sup_{[0,T]} \|f(t)\|_{L^1} \le \|f_0\|_{L^1} + \int_0^T |\rho(t)|\,dt. \tag{3.5}$$

Lemma 3.1. *Assume $a \in L^\infty$. For any $f_0 \in L^1(\mathbb{R}_+)$ and $\alpha \in L^1(0,T)$ there exists a unique weak solution $f \in C([0,T];L^1(\mathbb{R}_+))$ associated to equation (3.2).*

Proof Lemma 3.1. Step 1. Existence. When $a \in C_b(\mathbb{R}_+)$ and $f_0, \rho \in C^1_c(\mathbb{R}_+)$ the solution is explicitly given thanks to the characteristics formula (3.3). In the general case, we consider three sequences (a_ε), $(f_{0,\varepsilon})$ and (ρ_ε) of $C_b(\mathbb{R}_+)$ and $C^1_c(\mathbb{R}_+)$ which converge appropriately, namely $a_\varepsilon \to a$ a.e. and (a_ε) bounded in L^∞, $f_{0,\varepsilon} \to f_0$ in $L^1(\mathbb{R}_+)$ and $\rho_\varepsilon \to \rho$ in $L^1(0,T)$, and we see immediately from (3.5) that the functions (f_ε) and f defined thanks to the characteristics formula (3.3) satisfy $f_\varepsilon \to f$ in $C([0,T];L^1)$. As a consequence, we may pass to the limit in (3.2) and we deduce that f is a weak solution to equation (3.2).

Step 2. Uniqueness. Consider two weak solutions f_1 and f_2 to equation (3.2). The difference $f := f_2 - f_1$ satisfies

$$\int_0^\infty\int_0^\infty f\,(-\partial_t\varphi - \partial_x\varphi + a\varphi)\,dxdt = 0, \tag{3.6}$$

for any $\varphi \in C^1_c(\mathbb{R}^2_+)$ and thus also for any $\varphi \in C_c(\mathbb{R}^2_+) \cap W^{1,\infty}(\mathbb{R}^2_+)$. Introducing the semigroup

$$(S_t g)(x) := g(x-t)\,e^{A(x-t)-A(x)}\,\mathbf{1}_{x>t},$$

associated to equation (3.2) with no boundary term, its dual is

$$(S^*_t\psi)(x) := \psi(x+t)\,e^{A(x)-A(x+t)}, \quad \forall\,\psi \in L^\infty(\mathbb{R}_+),$$

and (S^*_t) is well-defined as a semigroup in $C_c \cap W^{1,\infty}(\mathbb{R}_+)$. Now, for $\psi \in C^1_c(\mathbb{R}^2_+)$, we define

$$\begin{aligned}\varphi(t,x) &:= \int_t^T (S^*_{s-t}\psi(s,\cdot))(x)\,ds \\ &= \int_t^T \psi(s,x+s-t)\,e^{A(x)-A(x+s-t)}\,ds \in C_c(\mathbb{R}^2_+) \cap W^{1,\infty}(\mathbb{R}^2_+),\end{aligned}$$

and we compute

$$\partial_x\varphi(t,x) = \int_t^T [\partial_x\psi(s,x+s-t) + \psi(s,x+s-t)(a(x) - a(x+s-t))]\,e^{A(x)-A(x+s-t)}\,ds,$$

from what we deduce

$$\begin{aligned}\partial_t\varphi(t,x) &= -\psi(t,x) + \int_t^T [-\partial_x\psi(s,x+s-t) + \psi(s,x+s-t)a(x+s-t)]\,e^{A(x)-A(x+s-t)}\,ds \\ &= -\psi(t,x) - \partial_x\varphi(t,x) + a(x)\varphi(t,x).\end{aligned}$$

Using then this test function φ in (3.6), we get

$$\int_0^\infty \int_0^\infty f\,\psi\,dxdt = 0, \quad \forall\,\psi \in C^1_c(\mathbb{R}^2_+),$$

and finally $f_1 = f_2$. □

We are now in position to come back to the renewal equation (3.1).

Lemma 3.2. *Assume $a \in L^\infty$. For any $f_0 \in L^1(\mathbb{R}_+)$, there exists a unique global weak solution $f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+))$ associated to equation (3.1). We may then associate to the renewal evolution a Markov semigroup.*

Proof Lemma 3.2. We define $\mathcal{E}_T := C([0,T]; L^1(\mathbb{R}_+))$ and for any $g \in \mathcal{E}_T$, we define $f := \Phi(g) \in \mathcal{E}_T$ the unique solution to equation (3.2) associated to f_0 and $\rho(t) := \rho_{g(t)} \in C([0,T])$. For two given functions $g_1, g_2 \in \mathcal{E}_T$ and the two associated images $f_i := \Phi(g_i)$, we observe that $f := f_2 - f_1$ is a weak solution to equation (3.2) associated to $f(0) = 0$ and $\rho(t) := \rho_{g_2(t)-g_1(t)}$. The estimate (3.5) reads here

$$\begin{aligned}\sup_{[0,T]} \|(f_2 - f_1)(t)\|_{L^1} &\le \int_0^T |\rho_{g_2(t)-g_1(t)})|\,dt \le \int_0^T \int_0^\infty a(y)|(g_2 - g_1)(t,y)\,dydt \\ &\le T\,\|a\|_{L^\infty} \sup_{[0,T]} \|(g_2 - g_1)(t)\|_{L^1}.\end{aligned}$$

Taking first T small enough such that $T\,\|a\|_{L^\infty} < 1$, we get the existence and uniqueness of a fixed point $f = \Phi(f) \in \mathcal{E}_T$, which is nothing but a weak solution to the renewal equation (3.1). Iterating the argument, we get the desired global weak solution $f \in C(\mathbb{R}_+; L^1(\mathbb{R}_+))$.

We may apply the results of the first section in the semigroup chapter 3 in order to get the existence of a semigroup S_t associated to the evolution problem (3.1). This semigroup is clearly positive. That can be seen by construction for instance. Indeed, if $g \in \mathcal{E}_{T,+} := \{\, g \in \mathcal{E}_T,\, g \ge 0\}$, then $f = \Phi(g) \in \mathcal{E}_{T,+}$ from the representation formula (3.3), and the fixed point argument can be made in that set. Next, from (3.4), we classical deduce (see chapter 2) that

$$\int_0^\infty f\,\varphi_R\,dx = \int_0^\infty f_0\,\varphi_R\,dx + \int_0^t \int_0^\infty (\partial_x \varphi_R + a\varphi_R)\,dxds + \int_0^t \rho(s)\,ds$$

for $\varphi_R(x) := \varphi(x/R)$, $\varphi \in C^1_c(\mathbb{R}_+)$, $\mathbf{1}_{[0,1]} \le \varphi \le \mathbf{1}_{[0,2]}$. We get the mass conservation by passing to the limit as $R \to \infty$. □

Lemma 3.3. *Assume furthermore* $\liminf a \ge a_0 > 0$. *There exists a unique stationary solution $F \in W^{1,\infty}(\mathbb{R}_+)$ to the stationary problem*

$$\partial_x F + aF = 0, \quad F(0) = \rho_F, \quad F \ge 0, \quad \langle F \rangle = 1.$$

Proof Lemma 3.3. From the first equation we have $F(x) = Ce^{-A(x)}$, so that the boundary condition is immediately fulfilled and the normalized condition is fulfilled by choosing $C := \langle e^{-A(x)} \rangle^{-1}$. It is worth noticing that the additional assumption implies $\langle e^{-A(x)} \rangle < \infty$ so that $C > 0$ and the same is true for F. □

Lemma 3.4. *We still assume $a \in L^\infty$ and* $\liminf a \ge a_0 > 0$. *There exist $C \ge 1$ and $\alpha < 0$ such that for any $f_0 \in L^1(\mathbb{R}_+)$ the associated global solutionf to the renewal equation* (3.1) *satisfies*

$$\|f(t) - \langle f_0 \rangle F\|_{L^1} \le C\,e^{\alpha t}\,\|f_0 - \langle f_0 \rangle F\|_{L^1}, \quad \forall\, t \ge 0.$$

Proof Lemma 3.4. We check Harris condition. We observe that $a \ge a_0/2\,\mathbf{1}_{x \ge x_0}$ for some $x_0 > 0$. We then set $T := 2x_0 > 0$ and we take $0 \le f_0 \in L^1(\mathbb{R}_+)$. From (3.3), we have

$$f(T,x) \ge \rho_{f(T-x,\cdot)}\,e^{-A(x)}\,\mathbf{1}_{x<T/2}. \tag{3.7}$$

with

$$\begin{aligned}\rho_{f(T-x,\cdot)} &= \int_0^\infty a(y) f(T-x,y)\,dy \\ &\ge \frac{a_0}{2} \int_{x_0}^\infty f(T-x,y)\,dy,\end{aligned}$$

Using the representation formula (3.3) again, we have

$$\begin{aligned} f(T-x,y) &\geq f_0(y+x-T)\, e^{-(A(y)-A(y-(x-T)))}\, \mathbf{1}_{y>T-x} \\ &\geq f_0(y+x-T)\, e^{-(x-T)\|a\|_\infty}\, \mathbf{1}_{y>T-x}, \end{aligned}$$

so that

$$\begin{aligned} \rho_{f(T-x,\cdot)} &\geq \frac{a_0}{2} \int_{x_0}^\infty f_0(y+x-T)\, \mathbf{1}_{y>T-x}\, dy\, e^{-(x-T)\|a\|_\infty} \\ &\geq \frac{a_0}{2} \int_0^\infty f_0(z)\, \mathbf{1}_{z>x_0+x-T}\, dz\, e^{-(x-T)\|a\|_\infty}. \end{aligned}$$

Together with (3.7), we obtain

$$\begin{aligned} f(T,x) &\geq \frac{a_0}{2} \int_0^\infty f_0(z)\, \mathbf{1}_{z>x_0+x-T}\, dz\, e^{-(x-T)\|a\|_\infty}\, e^{-A(x)}\, \mathbf{1}_{x<T/2} \\ &= \nu(x) \int_0^\infty f_0(z)\, dz, \quad \nu(x) := \frac{a_0}{2}\, e^{-(x-T)\|a\|_\infty}\, e^{-A(x)}\, \mathbf{1}_{x<T/2}, \end{aligned}$$

which is precisely a Harris type lower bound. We conclude thanks to Theorem 2.1. □