A C^1-Itô's formula for flows of semimartingale distributions *

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Abstract

We provide an Itô's formula for C^1-functionals of flows of conditional marginal distributions of continuous semimartingales. This is based on the notion of weak Dirichlet process, and extends the C^1-Itô's formula in Gozzi and Russo (2006) to this context. As the first application, we study a class of McKean-Vlasov optimal control problems, and establish a verification theorem which only requires C^1-regularity of its value function, which is equivalently the (viscosity) solution of the associated HJB master equation. It goes together with a novel duality result.

1 Introduction

In its classical formulation, Itô's formula provides a canonical decomposition for C^2-transformations of semimartingales. Since it was introduced, various variations have been proposed. In particular, the C^1-Itô's formula, that was developed in the series of works [18, 19, 20, 14] using the notion of weak Dirichlet process and the stochastic calculus via regularization approach, only requires the transformation to be C^1. In this theory, a C^1-functional of a weak Dirichlet process is again a weak Dirichlet process, which can be (uniquely) decomposed as the sum of a martingale and an orthogonal process.

Recently, motivated by the study of mean-field problems, involving McKean-Vlasov processes, or Mean-Field Games (MFG), an Itô's formula for flows of semimartingale distributions has been introduced, see e.g. [4, 6, 7] or the recent paper [12] and the references therein. It applies to transformations of measure-valued processes, obtained as the flows of (conditional) marginal distributions of semimartingales, and provides a decomposition of C^2-functionals of such measure-valued processes. In particular, it can be used to deduce the master equation for MFGs, or the Hamilton-Jacobi-Bellman (HJB) equation of McKean-Vlasov control problems. However, in practical situations of application, it is usually not easy to check the required C^2-differentiability of the value function defined on the space of probability measures.

In this paper, we provide a C^1-Itô's formula for flows of semimartingale distributions, by using the notion of weak Dirichlet process as in Gozzi and Russo [14]. This requires less regularity on the value function and turns out to be enough in many applications. More precisely, on a filtered

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probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, let X be a $\mathbb{R}^d$-valued continuous semimartingale with decomposition

$$X_t = X_0 + A_t + M_t + \int_0^t \sigma_s^\circ dM_s^\circ,$$

where A is a continuous finite variation process, σ° is progressively measurable, and both M and M° are continuous martingales. Let $\mathcal{G} := \sigma(M_t^\circ, t \geq 0)$ denote the sub-σ-field generated by M°, which is usually referred to as the common noise σ-field in the mean-field literature. Then, one defines a process $m = (m_t)_{t\geq 0}$, taking values in the space $\mathcal{P}(\mathbb{R}^d)$ of probability measures on $\mathbb{R}^d$, by

$$m_t := \mathcal{L}(X_t \big| \mathcal{G}), \quad t \geq 0.$$

Besides, we consider a continuous weak Dirichlet process Y with (unique) decomposition

$$Y_t = Y_0 + A_t^Y + M_t^Y,$$

where M^Y is the martingale part of Y, and A^Y is its orthogonal part (see Section 2 for a precise definition). Under some (essentially related to integrability) technical conditions, we prove that, for a continuous function $F : (t, y, m) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \longrightarrow F(t, y, m) \in \mathbb{R}$ with continuous partial derivative $D_y F$ in y, and continuous intrinsic derivative $D_m F$ (see Section 2 for a precise definition), one has

$$\begin{aligned} F(t, Y_t, m_t) = F(0, Y_0, m_0) &+ \int_0^t D_y F(s, Y_s, m_s)\ dM_s^Y \\ &+ \int_0^t \mathbb{E}\Big[D_m F(s, y, m_s, X_s)\sigma_s^\circ \Big| M^\circ\Big]_{y=Y_s} dM_s^\circ + \Gamma_t, \end{aligned}$$

where Γ is an orthogonal process. When F is a C^2-functional, Γ can be explicitly expressed in terms of the first order time derivative $\partial_t F$, together with the second order derivatives $D_y^2 F$, $D_x D_m F$ and $D_m^2 F$ (see e.g. [8, Section 6]). In particular, we extend the Itô's formula for C^1-transformation of Dirichlet processes of [14] to our context.

Importantly, this formula allows one to identify the martingale part of the process $F(\cdot, Y, m)$, which is enough in many practical situations of application. Let us for instance refer to [14] for an application in optimal control, and to [3] for some applications in mathematical finance.

The second contribution of this paper is to provide a new type of application in the form of a verification argument. Namely, we consider a McKean-Vlasov optimal control problem of the form:

$$\sup_{\nu\in\mathcal{U}} \mathbb{E}\Big[\int_0^T L(t, \rho_t^\nu, \nu_t)dt + g(\rho_T^\nu)\Big],$$

where, given two independent Brownian motions W and W°, X^ν is defined by the controlled McKean-Vlasov SDE:

$$X_t^\nu = X_0 + \int_0^t \sigma(s, X_s^\nu, \rho_s^\nu)dW_s + \int_0^t \sigma_0(s, X_s^\nu, \rho_s^\nu)\big(dW_s^\circ + \nu_s ds\big), \quad \text{with} \quad \rho_t^\nu := \mathcal{L}(X_t^\nu | W^\circ),$$

and where an admissible control process $\nu \in \mathcal{U}$ is a $\mathbb{F}^{W^\circ}$-progressively measurable process taking value in a compact set $\mathrm{U} \subset \mathbb{R}^d$. In the above, W° is the so-called common noise, and $\mathbb{F}^{W^\circ}$ denotes the filtration generated by W°. Notice that one only controls the drift process, and the control depends only on the common noise W°. For this McKean-Vlasov control problem, the value function can be written as $V(t,m)$, where V is the unique (viscosity) solution of the master HJB equation (see e.g. Pham and Wei [17]):

$$\partial_t V(t,m) + \mathbb{L}[V] + H\big(t, m, D_m V(t,m,\cdot)\big) = 0,$$

where $\mathbb{L}[V]$ is a linear operator involving $D_x D_m V$ and $D_m^2 V$ (see Section 3 for an explicit expression), and H is the Hamiltonian given by

$$H(t,m,D_m V(t,m,\cdot)) := \sup_{u\in \mathrm{U}} \Big(L(t,m,u) + um\big(\sigma_0(t,\cdot,m) D_m V(t,m,\cdot)\big)\Big).$$

The classical verification theorem states that, given a smooth solution to the HJB equation (or equivalently the value function), the optimizer in the definition of H provides a feedback optimal control. It relies on Itô's formula, assuming that $V \in C^{1,2}$ in the sense that V, $\partial_t V$, $D_m V$, $D_x D_m V$ and $D_m^2 V$ are all well-defined and continuous (see e.g. [15] and [21] for some closely related problems). At the same time, for the above class of optimal control problems, the definition of the Hamiltonian H as well as the associated optimizer only involve the first order derivative $D_m V$. It is then natural to ask whether it is enough to only require C^1-regularity on V (in the sense that V and $D_m V$ are both continuous).
By using our C^1-Itô's formula, we actually establish a verification theorem which only assumes that V is C^1. To the best of our knowledge, this approach is new even for classical optimal control problems. The proof goes together with the proof of a dual formulation which is of own interest.

The rest of this paper is organized as follows. The C^1-Itô's formula for flows of semimartingale distributions is proved in Section 2. Section 3 is dedicated to our verification and duality arguments for a class of McKean-Vlasov optimal control problems.

2 A C^1-Itô's formula for flows of semimartingale distributions

Throughout the section, we fix a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We assume that $\mathcal{F}$ is countably determined.

2.1 Preliminaries

Let $\mathcal{P}(\mathbb{R}^d)$ denote the space of all (Borel) probability measures on $\mathbb{R}^d$, and $\mathcal{P}_2(\mathbb{R}^d)$ denote the set of all probability measures on $\mathbb{R}^d$ with finite second moment, i.e.

$$\mathcal{P}_2(\mathbb{R}^d) = \Big\{ \mu \in \mathcal{P}(\mathbb{R}^d) : \int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty \Big\}.$$

The space $\mathcal{P}_2(\mathbb{R}^d)$ is equipped with the Wasserstein distance

$$\mathcal{W}_2(\mu_1,\mu_2) := \bigg(\inf_{\gamma \in \Gamma(\mu_1,\mu_2)} \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^2 d\gamma(x,y) \bigg)^{\frac{1}{2}},$$

where $\Gamma(\mu_1,\mu_2)$ is the set of all couplings of μ_1 and μ_2, i.e. joint probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ whose marginals are μ_1 and μ_2, respectively.

Definition 2.1. *[7, Definition 5.43] A function $F : \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$ is said to have a linear functional derivative if there exists a function*

$$\frac{\delta F}{\delta m} : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \ni (m,x) \longmapsto \frac{\delta F}{\delta m}(m)(x) \in \mathbb{R},$$

that is continuous for the product topology, such that, for any bounded subset $\mathcal{B} \subset \mathcal{P}_2(\mathbb{R}^d)$, the function $\mathbb{R}^d \ni x \longmapsto [\delta F/\delta m](m)(x)$ is at most of quadratic growth uniformly in $m \in \mathcal{B}$, and such that, for all m and m' in $\mathcal{P}_2(\mathbb{R}^d)$:

$$F(m') - F(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta F}{\delta m}\big(tm' + (1-t)m\big)(x)\,\big[m' - m\big](dx)dt. \tag{1}$$

Assume further that, for any $m \in \mathcal{P}_2(\mathbb{R}^d)$, the function $\mathbb{R}^d \ni x \longmapsto \frac{\delta F}{\delta m}(m)(x)$ is differentiable. Then, one defines the intrinsic derivative

$$D_m F(m,x) := D_x \delta_m F(m,x), \quad \textit{with} \;\; \delta_m F(m,x) := \frac{\delta F}{\delta m}(m)(x), \tag{2}$$

in which D_x is the gradient operator with respect to the x-variable.

In our paper, we will stay in the setting where $D_m F(m,x)$ is jointly continuous in (m,x) and is at most of linear growth in x, uniformly in $m \in \mathcal{B}$, for any bounded subset $\mathcal{B} \subset \mathcal{P}_2(\mathbb{R}^d)$. Then, for any $m \in \mathcal{P}_2(\mathbb{R}^d)$, the function $\mathbb{R}^d \ni x \longmapsto D_m F(m,x)$ is uniquely defined m-almost everywhere on $\mathbb{R}^d$.

We shall make use of the notion of weak Dirichlet process and stochastic calculus by regularization, for which we now define the notions of quadratic variation and of orthogonal process (see e.g. [14, Definition 3.4]). Recall that a sequence of stochastic processes $\big\{(X_t^n)_{t\geq 0}, n \geq 1\big\}$ is said to converge to the process $(X_t)_{t\geq 0}$ in the u.c.p topology (uniform convergence on compacts in probability) if, for all $\varepsilon > 0$ and $t \geq 0$,

$$\lim_{n\to\infty} \mathbb{P}\Big[\sup_{s\leq t} |X_s^n - X_s| > \varepsilon\Big] = 0.$$

Definition 2.2. (i) *Given two càdlàg processes X and Y, the quadratic covariation $[X,Y]$ is defined by*

$$[X,Y]_s = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^s (X_{r+\varepsilon} - X_r)(Y_{r+\varepsilon} - Y_r)dr, \;\; s \geq 0,$$

if the limit exists in the sense of u.c.p.

(ii) *Let A be a $\mathbb{F}$-adapted càdlàg process, we say that A is an orthogonal process if $[A,N] = 0$ for every continuous $\mathbb{F}$-local martingale N.*

(iii) *Let X be a $\mathbb{F}$-adapted càdlàg process, it is called a weak Dirichlet process if it has the decomposition*

$$X_t = X_0 + A_t + M_t, \quad t \geq 0,$$

where M is a local martingale, and A is an orthogonal process w.r.t. the filtration $\mathbb{F}$.

Remark 2.3. (i) *When X and Y are càdlàg semimartingales, $[X,Y]$ coincides with the usual bracket (see e.g. [19, Proposition 1.1]).*

(ii) *In the definition of the orthogonal process, it is equivalent to consider all bounded continuous martingales N in place of all continuous local martingales.*

(iii) *For a continuous weak Dirichlet process, its decomposition as the sum of an orthogonal process and a local martingale is unique.*

(iv) *We will consider later a sub-filtration $\mathbb{G}$ of $\mathbb{F}$. Nevertheless, throughout the paper, the notion of orthogonal and weak Dirichlet process are all w.r.t. the filtration $\mathbb{F}$.*

2.2 Main Result

From now on, we fix a continuous semimartingale $(X_t)_{t\geq 0}$ on the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with decomposition

$$X_t = X_0 + A_t + M_t^X, \quad \text{with} \;\; M_t^X = M_t + \int_0^t \sigma_s^\circ dM_s^\circ, \; t \geq 0, \tag{3}$$

where $(A_t)_{t\geq 0}$ is continuous with finite variation, $(M_t)_{t\geq 0}$ and $(M_t^\circ)_{t\geq 0}$ are continuous martingales, with $A_0 = M_0 = M_0^\circ = 0$. Let us define $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ as the (raw) filtration generated by M°, i.e.

$$\mathcal{G}_t := \sigma(M_s^\circ, \; 0 \leq s \leq t), \quad t \geq 0,$$

and define

$$\mathcal{G} := \sigma(M_s^\circ, \; s \geq 0).$$

Assumption 2.1. (i) *The process σ° is $\mathbb{F}$-progressively measurable, and there exists an increasing sequence of stopping times $(\tau_n)_{n\geq 1}$ w.r.t. $\mathbb{G}$ such that $\tau_n \longrightarrow \infty$, a.s. as $n \longrightarrow \infty$, and*

$$\mathbb{E}\Big[[M]_{\tau_n\wedge t} + \big|A\big|^2_{\tau_n\wedge t} + \int_0^{\tau_n\wedge t} \big|\sigma_s^\circ\big|^2 d[M^\circ]_s\Big] < +\infty, \; \textit{for all } t \geq 0 \textit{ and } n \geq 1, \tag{4}$$

where $(|A|_t)_{t\geq 0}$ denotes the total variation of A.

(ii) *The martingale M is orthogonal to N (i.e. $[M, N] = 0$), for all (càdlàg) $\mathbb{G}$-martingale N.*

(iii) *(H)-hypothesis condition:*

$$\mathbb{E}\big[\mathbf{1}_D\big|\mathcal{G}_t\big] = \mathbb{E}\big[\mathbf{1}_D\big|\mathcal{G}\big], \; \textit{a.s., for all } D \in \mathcal{F}_t, \; t \geq 0.$$

We next introduce the $\mathcal{P}(\mathbb{R}^d)$-valued process $m = (m_t)_{t\geq 0}$ associated to the $\mathcal{G}$-conditional law of X:

$$m_t := \mathcal{L}\left(X_t \,|\mathcal{G}_t\right) = \mathcal{L}(X_t \,|\mathcal{G}), \quad t \geq 0.$$

Assumption 2.1 ensures that $m = (m_t)_{t\geq 0}$ is continuous under $\mathcal{W}_2$, as shown in the following lemma.

Lemma 2.4. *Let Assumption 2.1 hold true, and $(\tau_n)_{n\geq 1}$ be the sequence of $\mathbb{G}$-stopping times therein, then for all bounded $\mathcal{F}_t$-measurable random variable ξ,*

$$\mathbb{E}\big[\xi\mathbf{1}_{\{t\leq\tau_n\}}\big|\mathcal{G}_{\tau_n\wedge t}\big] \;=\; \mathbb{E}\big[\xi\mathbf{1}_{\{t\leq\tau_n\}}\big|\mathcal{G}_t\big] \;=\; \mathbb{E}\big[\xi\mathbf{1}_{\{t\leq\tau_n\}}\big|\mathcal{G}\big], \; \textit{a.s.}$$

Consequently,

$$m_t = \mathcal{L}(X_{\tau_n\wedge t}|\mathcal{G}_t) = \mathcal{L}(X_{\tau_n\wedge t}|\mathcal{G}), \; \textit{a.s., on } \{t \leq \tau_n\}, \; \textit{for all } t \geq 0 \textit{ and } n \geq 1.$$

In particular, one can choose $m = (m_t)_{t\geq 0}$ to be a continuous $\mathcal{P}_2(\mathbb{R}^d)$-valued process, under the Wasserstein distance $\mathcal{W}_2$ on $\mathcal{P}_2(\mathbb{R}^d)$.

Proof. (i) Let us fix $n \geq 1$ and $t \geq 0$. Since $\mathcal{G}$, $\mathcal{G}_t$ and $\mathcal{G}_{\tau_n\wedge t}$ are all countably generated, let us take respectively a regular conditional probability $(\mathbb{P}_\omega)_{\omega\in\Omega}$ of $\mathbb{P}$ w.r.t. $\mathcal{G}$, a regular conditional probability $(\mathbb{P}^t_\omega)_{\omega\in\Omega}$ of $\mathbb{P}$ w.r.t. $\mathcal{G}_t$, and a regular conditional probability $(\mathbb{P}^{n,t}_\omega)_{\omega\in\Omega}$ of $\mathbb{P}$ w.r.t. $\mathcal{G}_{\tau_n\wedge t}$. Under the (H)-hypothesis condition, one has, for all bounded $\mathcal{F}_t$-measurable random variables ξ,

$$\mathbb{E}\big[\xi\mathbf{1}_{\{t\leq\tau_n\}}\big|\mathcal{G}_{\tau_n\wedge t}\big] \;=\; \mathbb{E}\big[\xi\mathbf{1}_{\{t\leq\tau_n\}}\big|\mathcal{G}_t\big] \;=\; \mathbb{E}\big[\xi\mathbf{1}_{\{t\leq\tau_n\}}\big|\mathcal{G}\big], \text{ a.s.}$$

Since $\mathcal{F}$ is assumed to be countably determined and $\mathcal{F}_t \subset \mathcal{F}$, this implies that

$$\mathbb{P}^{n,t}_\omega[B] = \mathbb{P}^t_\omega[B] = \mathbb{P}_\omega[B], \text{ for all } B \in \mathcal{F}_t, \text{ for } \mathbb{P}\text{-a.e. } \omega \in \{t \le \tau_n\}.$$

As $\{t \le \tau_n\} \in \mathcal{G}_{\tau_n \wedge t}$, one can then deduce that

$$m_t(\omega) = \mathbb{P}_\omega \circ X_t^{-1} = \mathbb{P}^t_\omega \circ X_t^{-1} = \mathbb{P}^{n,t}_\omega \circ X_t^{-1} = \mathbb{P}^{n,t}_\omega \circ X^{-1}_{\tau_n(\omega)\wedge t}, \text{ for } \mathbb{P}\text{-a.e. } \omega \in \{t \le \tau_n\}.$$

(ii) Given the above, one can assume that the integrability condition in (4) holds for (X, A, M, M°) in place of $(X_{\tau_n \wedge \cdot}, A_{\tau_n \wedge \cdot}, M_{\tau_n \wedge \cdot}, M^\circ_{\tau_n \wedge \cdot})$, up to using a standard localizing technique. Then,

$$\mathbb{E}^{\mathbb{P}}\left[\sup_{0\le s\le t} X_s^2\right] \le 2X_0^2 + 16\mathbb{E}^{\mathbb{P}}\left[|M^X_t|^2\right] + 4\mathbb{E}^{\mathbb{P}}\left[\left|A\right|_t^2\right] < \infty, \text{ for all } t > 0,$$

in which we also used Doob's inequality. This implies that $\mathbb{E}^{\mathbb{P}_\omega}\left[\sup_{0\le s\le t} X_s^2\right] < \infty$, for $\mathbb{P}$- a.e. ω. Define $m_s(\omega) := \mathbb{P}_\omega \circ X_s^{-1}$ for $0 \le s \le t$, then

$$\lim_{\varepsilon\to 0} \mathcal{W}_2^2(m_s(\omega), m_{s+\varepsilon}(\omega)) \le \lim_{\varepsilon \to 0} \mathbb{E}^{\mathbb{P}_\omega}\left[(X_{s+\varepsilon} - X_s)^2\right] = 0, \text{ for } \mathbb{P}\text{- a.e. } \omega.$$

□

Remark 2.5. (i) Let W and W° be two independent Brownian motions, and $\sigma = (\sigma_s)_{s\ge 0}$ and $\sigma^\circ = (\sigma^\circ_s)_{s\ge 0}$ be progressively measurable such that

$$\mathbb{E}\Big[\int_0^t \big(|\sigma_s|^2 + |\sigma^\circ_s|^2\big) ds\Big] < \infty, \quad \text{for all } t \ge 0.$$

Let us define M and M° by

$$M_t := \int_0^t \sigma_s dW_s, \quad M^\circ_t := W^\circ_t, \quad t \ge 0,$$

together with a continuous process A with square integrable total variation. Then, Assumption 2.1 holds true.

(ii) Since σ° is assumed to be adapted to the filtration $\mathbb{F}$ (rather than the sub-filtration $\mathbb{G}$), the form of processes X as in (3) covers the usual McKean-Vlasov SDEs with common noise:

$$X_t = X_0 + \int_0^t b(X_s, m_s) ds + \int_0^t \sigma(X_s, m_s) dW_s + \int_0^t \sigma_0(X_s, m_s) dW^\circ_s,$$

for two independent Brownian motions W and W° and $m_s := \mathcal{L}(X_s|W^\circ)$. In our general formulation, the process M° in (3) plays the role of the common noise.

For sake of more generality, we now also consider a $\mathbb{R}^d$-valued continuous weak Dirichlet process $(Y_t)_{t\ge 0}$ (see Definition 2.2), on the same filtered space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, with finite quadratic variation (i.e. $[Y, Y]_t < \infty$ for all $t \ge 0$), whose unique decomposition is given by

$$Y_t = Y_0 + A^Y_t + M^Y_t, \ t \ge 0, \tag{5}$$

for an orthogonal process A^Y and a local martingale M^Y, such that $A^Y_0 = M^Y_0 = 0$. Notice that there is no condition on the joint law or dynamics of X and Y, it is just required that X is a semimartingale and Y is a weak Dirichlet process w.r.t. the same filtration $\mathbb{F}$.

For a function $F : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$, we denote by $D_m F : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ its partial derivative in the sense that $(m, x) \longmapsto D_m F(t, y, m, x)$ is the derivative of $m \longmapsto F(t, y, m)$ as defined in (2). Then, we say that

$$F \in C^{0,1,1}(\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)) \tag{6}$$

if F and its partial derivatives $D_y F$ and $D_m F$ are well-defined and all (jointly) continuous.

In the following, for a random variable ξ, we use the notation

$$\mathbb{E}^\circ\big[\xi\big] := \mathbb{E}\big[\xi \big| \mathcal{G}\big]$$

whenever the right-hand side is well-defined, and

$$\mathbb{E}^\circ\big[D_m F(s, \cdot, m_s, X_s)\sigma_s^\circ\big](Y_t) := \mathbb{E}\Big[D_m F(s, y, m_s, X_s)\sigma_s^\circ \Big| \mathcal{G}\Big]_{y=Y_t}, \text{ for all } s, t \geq 0.$$

We shall require the following local boundedness assumption on $D_m F$.

Assumption 2.2. *With the same sequence $(\tau_n)_{n\geq 1}$ of stopping times as in Assumption 2.1, for all $n \geq 1$, $T > 0$ and compact subsets $K \subset \mathbb{R}^d$, there exists a constant $C > 0$ satisfying*

$$\mathbb{E}^\circ\Big[\Big(D_m F(r, y, m_s^{n,\lambda,t}, X_s^{n,\eta,t})\Big)^2\Big] \leq C, \text{ a.s., for all } r \in [0, 2T], s \in [0, t],\ t \in [0, T],\ \lambda, \eta \in [0, 1],\ y \in K,$$

where $m_s^{n,\lambda,t} := m_{\tau_n \wedge s} + \lambda(m_{\tau_n \wedge t} - m_{\tau_n \wedge s})$ and $X_s^{n,\eta,t} := X_{\tau_n \wedge s} + \eta(X_{\tau_n \wedge t} - X_{\tau_n \wedge s})$.

Remark 2.6. *If there exists a constant $C > 0$ such that*

$$\big|D_m F(r, y, m, x)\big| \;\leq\; C(1 + |x|), \text{ for all } (r, y, m, x) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d,$$

then one can check that Assumption 2.2 holds true whenever Assumption 2.1 does.

We can now state the main result of this section.

Theorem 2.3. *Let $F \in C^{0,1,1}(\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$, and Assumptions 2.1 and 2.2 hold true. Then,*

$$\begin{aligned} F(t, Y_t, m_t) = F(0, Y_0, m_0) &+ \int_0^t D_y F(s, Y_s, m_s)\, dM_s^Y \\ &+ \int_0^t \mathbb{E}^\circ\big[D_m F(s, \cdot, m_s, X_s)\sigma_s^\circ\big](Y_s) dM_s^\circ + \Gamma_t, \quad t \geq 0, \end{aligned} \tag{7}$$

where $(\Gamma_t)_{0\leq t\leq T}$ is an orthogonal process.

Before to provide the proof of this result, let us make some comments.

Remark 2.7. (i) *The above theorem proves that $(\mathbb{F}(t, Y_t, m_t))_{t\geq 0}$ is a weak Dirichlet process. Moreover, it is continuous so that the decomposition in (7) is unique.*

(ii) *The above result extends the classical C^1-Itô's formula such as in Gozzi and Russo [14]. Our new feature is that F depends on $(m_t)_{t\geq 0}$, the conditional marginal distribution of the semimartingale X. The main reason to consider a semimartingale X rather than a general weak Dirichlet process is that, technically, we will use the integral w.r.t. the finite variation part A of X to handle to conditional expectation terms in the proof (see in particular the proof of Lemma 2.11). It is an open question to us whether this formula still holds true if the process A in (3) is only assumed to be orthogonal, so that X is only a weak Dirichlet process.*

Proof of Theorem 2.3. (i) Let $(\tau_n^Y)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times such that $\tau_n^Y \longrightarrow \infty$ as $n \longrightarrow \infty$, and Y is uniformly bounded on $[0, \tau_n^Y]$, for each $n \geq 1$. Then, given the sequence $(\tau_n)_{n\geq 1}$ of localizing stopping times given in Assumption 2.1, we define

$$X_t^n := X_{\tau_n \wedge t}, \quad Y_t^n := Y_{\tau_n^Y \wedge t}, \quad \text{and} \quad m_t^n := \mathcal{L}(X_t^n | \mathcal{G}_t), \quad t \geq 0, \quad n \geq 1.$$

It is enough to prove that (7) holds for (X^n, Y^n, m^n), and then use Lemma 2.4 and let $n \longrightarrow \infty$. For simplicity, we also assume that $F(0, Y_0, m_0) = 0$ and $d = 1$. Upon replacing the processes (X, Y, m) in (7) by the localized process (X^n, Y^n, m^n), one can assume w.l.o.g. that

$$\begin{gathered} F(0, Y_0, m_0) = 0,\ d = 1, \\ Y \text{ is bounded}, \\ \text{Assumption 2.1 and 2.2 hold with } \tau_n \equiv \infty \text{ a.s. for all } n \geq 1 \end{gathered} \tag{8}$$

which we do in the following.

Let us define the process $(\Gamma_t)_{t\geq 0}$ by

$$\Gamma_t \ := \ F(t, Y_t, m_t) - \int_0^t D_y F(s, Y_s, m_s) dM_s^Y - \int_0^t \mathbb{E}^\circ\big[D_m F(s, \cdot, m_s, X_s)\sigma_s^\circ\big](Y_s) dM_s^\circ.$$

To prove the theorem, it is enough to show that Γ is an orthogonal process, i.e. $[\Gamma, N] = 0$ for any bounded continuous martingale N. That is, for N given:

$$I_t^\varepsilon := \frac{1}{\varepsilon}\int_0^t \big[F(s+\varepsilon, Y_{s+\varepsilon}, m_{s+\varepsilon}) - F(s, Y_s, m_s)\big](N_{s+\varepsilon} - N_s) ds \ \longrightarrow \ I_t,\ t \geq 0,\ \text{u.c.p.},$$

as $\varepsilon \longrightarrow 0$, where

$$I_t := \int_0^t D_y F(s, Y_s, m_s)\ d[M^Y, N]_s + \int_0^t \mathbb{E}^\circ\Big[D_m F(s, \cdot, m_s, X_s)\sigma_s^\circ\Big](Y_s)\ d[M^\circ, N]_s,\ t \geq 0.$$

To this end, we use the decomposition

$$I_t^\varepsilon = I_t^{1,\varepsilon} + I_t^{2,\varepsilon} + I_t^{3,\varepsilon},\ t \geq 0,$$

with

$$I_t^{1,\varepsilon} \ := \ \int_0^t \big[F(s+\varepsilon, Y_{s+\varepsilon}, m_{s+\varepsilon}) - F(s+\varepsilon, Y_s, m_{s+\varepsilon})\big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds, \tag{9}$$

$$I_t^{2,\varepsilon} \ := \ \int_0^t \big[F(s+\varepsilon, Y_s, m_{s+\varepsilon}) - F(s+\varepsilon, Y_s, m_s)\big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds, \tag{10}$$

and

$$I_t^{3,\varepsilon} \ := \ \int_0^t \big[F(s+\varepsilon, Y_s, m_s) - F(s, Y_s, m_s)\big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds. \tag{11}$$

We shall prove in Lemmas 2.8 and 2.9 below that

$$I_t^{1,\varepsilon} \longrightarrow \int_0^t D_y F(s, Y_s, m_s) d[M^Y, N]_s,\ \text{and}\ I_t^{3,\varepsilon} \longrightarrow 0,\ t \geq 0,\ \text{u.c.p.},$$

and, in Lemma 2.12, that

$$I_t^{2,\varepsilon} \ \longrightarrow \ \int_0^t \mathbb{E}^\circ\big[D_m F(s, \cdot, m_s, X_s)\sigma_s^\circ\big](Y_s) d[M^\circ, N]_s,\ t \geq 0, \text{u.c.p.}$$

This will provide the required result. □

Lemma 2.8. *Let the conditions of Theorem 2.3 and Condition (8) hold. Let $(I^{1,\varepsilon})_{\varepsilon>0}$ be defined as in (9). Then,*

$$I_t^{1,\varepsilon} \longrightarrow \int_0^t D_y F(s, Y_s, m_s) d[M^Y, N]_s, \; t \geq 0, \; \textit{u.c.p., as } \varepsilon \longrightarrow 0.$$

Proof. Let us define

$$I_t^{11,\varepsilon} := \int_0^t D_y F(s, Y_s, m_s)(Y_{s+\varepsilon} - Y_s) \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds, \; t \geq 0,$$

and

$$I_t^{12,\varepsilon} := \int_0^t \Delta_s^\varepsilon (Y_{s+\varepsilon} - Y_s) \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds, \; t \geq 0,$$

with

$$\Delta_s^\varepsilon := \int_0^1 \left(D_y F(s+\varepsilon, Y_s + \lambda(Y_{s+\varepsilon} - Y_s), m_{s+\varepsilon}) - D_y F(s, Y_s, m_s) \right) d\lambda,$$

so that $I_t^{1,\varepsilon} = I_t^{11,\varepsilon} + I_t^{12,\varepsilon}$ for all $t \geq 0$.

First, by similar arguments as in [14, Proposition 3.10], one easily obtains that

$$I_t^{11,\varepsilon} \xrightarrow[\varepsilon \to 0]{\text{u.c.p.}} \int_0^t D_y F(s, Y_s, m_s) d\left[M^Y, N\right]_s.$$

Further, by (uniform) continuity of $(s, y) \longmapsto D_y F(s, y, m_s)$ on compact sets, there exists random variables $(\delta(D_y F, \varepsilon))_{\varepsilon>0}$ such that

$$\sup_{0 \leq s \leq t} \left| \Delta_s^\varepsilon \right| \; \leq \; \delta(D_y F, \varepsilon) \; \longrightarrow \; 0, \; \text{a.s., as } \varepsilon \longrightarrow 0.$$

Recall that Y has finite quadratic variation and that N is square integrable, so that

$$\left(\int_0^t \frac{(Y_{s+\varepsilon} - Y_s)^2}{\varepsilon} ds \right) \left(\int_0^t \frac{(N_{s+\varepsilon} - N_s)^2}{\varepsilon} ds \right) \xrightarrow[\varepsilon \to 0]{\text{u.c.p.}} [Y]_t [N]_t < \infty.$$

It follows that

$$|I_t^{12,\varepsilon}| \; \leq \; \delta\left(D_y F, \varepsilon\right) \sqrt{\int_0^t \frac{(Y_{s+\varepsilon} - Y_s)^2}{\varepsilon} ds \int_0^t \frac{(N_{s+\varepsilon} - N_s)^2}{\varepsilon} ds} \; \longrightarrow \; 0, \; t \geq 0, \; \text{u.c.p.}$$

as $\varepsilon \longrightarrow 0$. □

Lemma 2.9. *Let the conditions of Theorem 2.3 and Condition (8) hold. Let $(I^{3,\varepsilon})_{\varepsilon>0}$ be defined as in (11). Then, $I_t^{3,\varepsilon} \longrightarrow 0$, $t \geq 0$, u.c.p. as $\varepsilon \to 0$.*

Proof. By the integration by parts formula, one can rewrite $I_t^{3,\varepsilon}$ as

$$I_t^{3,\varepsilon} := \int_0^{t+\varepsilon} \eta_r^\varepsilon dN_r, \; \text{with } \eta_r^\varepsilon := \frac{1}{\varepsilon} \int_{(r-\varepsilon)_+}^{r \wedge t} \big(F(s+\varepsilon, Y_s, m_s) - F(s, Y_s, m_s) \big) ds.$$

One observes that $\eta_r^\varepsilon \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. Moreover, $|\eta^\varepsilon|$ is bounded by the (locally bounded) continuous adapted process $(\bar{\eta}_r)_{r \geq 0}$ defined as:

$$\bar{\eta}_r := 2 \max_{s \leq r+1} \max_{r' \leq r} \left| F(s, Y_{r'}, m_{r'}) \right|.$$

Then, one can apply e.g. [16, Theorem I.4.31] to deduce that

$$I_t^{3,\varepsilon} \longrightarrow 0, \; t \geq 0, \; \text{u.c.p.}$$

□

To prove Lemma 2.12 below, we need the following two intermediate lemmas.

Lemma 2.10. *Let the conditions of Theorem 2.3 and Condition (8) hold. Let H be a $\mathbb{F}$-progressively measurable process such that*

$$\overline{M}_t := \int_0^t H_s dM_s,\ t \geq 0,\ \textit{is a martingale.}$$

Then,

$$\mathbb{E}^\circ\big[\overline{M}_t\big] = 0,\ \textit{a.s., for all } t \geq 0.$$

Moreover,

$$\mathbb{E}^\circ\Big[(M_t - M_s)^2\Big] = \mathbb{E}^\circ\Big[[M]_t - [M]_s\Big],\ \textit{a.s., for all } t \geq s \geq 0.$$

Proof. (i) Recall that $\mathbb{G}$ is the filtration generated by M°. Then, for all $\phi \in C_b(C^d[0,T];\mathbb{R}^d)$, there exists a bounded $\mathbb{G}$-martingale $\Phi = (\Phi_r)_{r\geq 0}$ such that $\Phi_r = \mathbb{E}\big[\phi(M^\circ)\big|\mathcal{G}_r\big]$, for all $r \geq 0$. By Assumption 2.1.(ii), one knows that $[\overline{M}, \Phi]_r = 0$ for all $r \geq 0$. Up to a localization argument, one obtains that

$$\mathbb{E}\big[\overline{M}_t\phi(M^\circ)\big] \;=\; \mathbb{E}\left[\overline{M}_0\Phi_0 + \int_0^t \overline{M}_r d\Phi_r + \int_0^t \Phi_r d\overline{M}_r + [\overline{M}, \Phi]_t\right] \;=\; 0.$$

Using the (H)-hypothesis condition in Assumption 2.1, it follows that

$$\mathbb{E}^\circ[\overline{M}_t] \;:=\; \mathbb{E}\big[\overline{M}_t\big|\mathcal{G}\big] = \mathbb{E}\big[\overline{M}_t\big|\mathcal{G}_t\big] = 0.$$

(ii) Given the square integrability conditions on M of Condition (8), both processes

$$\int_0^t M_r dM_r = \frac{1}{2}M_t^2 - \frac{1}{2}[M]_t, \quad \int_0^t M_s \mathbf{1}_{\{r\geq s\}} dM_r, \quad t \geq 0,$$

are true martingales. Thus

$$\mathbb{E}^\circ\Big[\int_s^t (M_r - M_s) dM_r\Big] = 0.$$

Then, it follows that

$$\mathbb{E}^\circ\Big[(M_t - M_s)^2\Big] \;=\; \mathbb{E}^\circ\Big[\int_s^t 2(M_r - M_s)dM_r + [M]_t - [M]_s\Big] \;=\; \mathbb{E}^\circ\Big[[M]_t - [M]_s\Big].$$

□

Lemma 2.11. *Let the conditions of Theorem 2.3 and Condition (8) hold. Then,*

$$\lim_{\varepsilon\to 0} \mathbb{E}^\circ\left[\frac{1}{\varepsilon}\int_0^t \left(|A|_{s+\varepsilon} - |A|_s\right)^2 ds\right] = 0,\ t \geq 0,\ \textit{u.c.p.},$$

$$\lim_{\varepsilon\to 0} \mathbb{E}^\circ\left[\frac{1}{\varepsilon}\int_0^t \left(M_{s+\varepsilon} - M_s\right)^2 ds\right] = \mathbb{E}^\circ\big[[M]_t\big],\ t \geq 0,\ \textit{u.c.p.}$$

and

$$\lim_{\varepsilon\to 0} \mathbb{E}^\circ\left[\frac{1}{\varepsilon}\int_0^t \left(\int_s^{s+\varepsilon} \sigma_r^\circ dM_r^\circ\right)^2 ds\right] \;=\; \int_0^t \mathbb{E}^\circ\big[(\sigma_s^\circ)^2\big] d[M^\circ]_s,\ t \geq 0,\ \textit{u.c.p.}$$

Proof. (i) Notice that the total variation process $(|A|_t)_{t\geq 0}$ of A is a continuous non-decreasing process. By direct computations, it follows that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\mathbb{E}^\circ\left[\frac{1}{\varepsilon}\int_0^t \left(|A|_{s+\varepsilon}-|A|_s\right)^2 ds\right]\\
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big[\int_t^{t+\varepsilon}\mathbb{E}^\circ\big[|A|_s^2\big]ds-\int_0^{\varepsilon}\mathbb{E}^\circ\big[|A|_s^2\big]ds-2\int_0^t\mathbb{E}^\circ\big[|A|_s\big(|A|_{s+\varepsilon}-|A|_s\big)\big]ds\Big]\\
&= \mathbb{E}^\circ\big[|A|_t^2\big]-\lim_{\varepsilon\to 0}\mathbb{E}^\circ\left[2\int_0^t\frac{1}{\varepsilon}\int_s^{s+\varepsilon}|A|_s\,d\,|A|_r\,ds\right]\\
&= \mathbb{E}^\circ\big[|A|_t^2\big]-\lim_{\varepsilon\to 0}\mathbb{E}^\circ\left[2\int_0^{t+\varepsilon}\frac{1}{\varepsilon}\int_{(r-\varepsilon)_+}^{r\wedge t}|A|_s\,dsd\,|A|_r\right]\\
&= \mathbb{E}^\circ\big[|A|_t^2\big]-\mathbb{E}^\circ\left[\int_0^t 2\,|A|_r\,d\,|A|_r\right] \;=\; 0,\ \text{a.s.}
\end{aligned}$$

In the above, we used the square integrability condition on $|A|_t$ in Condition (8) to deduce that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\mathbb{E}^\circ\left[\int_0^{t+\varepsilon}\frac{1}{\varepsilon}\int_{(r-\varepsilon)_+}^{r\wedge t}|A|_s\,dsd\,|A|_r\right] &= \mathbb{E}^\circ\left[\lim_{\varepsilon\to 0}\int_0^{t+\varepsilon}\frac{1}{\varepsilon}\int_{(r-\varepsilon)_+}^{r\wedge t}|A|_s\,dsd\,|A|_r\right]\\
&= \mathbb{E}^\circ\left[\int_0^t |A|_r\,d\,|A|_r\right],\ \text{a.s.}
\end{aligned}$$

(ii) Next, by Lemma 2.10, it follows that

$$\lim_{\varepsilon\to 0}\mathbb{E}^\circ\left[\frac{1}{\varepsilon}\int_0^t\left(M_{s+\varepsilon}-M_s\right)^2 ds\right]=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t\mathbb{E}^\circ\big[[M]_{s+\varepsilon}-[M]_s\big]ds=\mathbb{E}^\circ\big[[M]_t\big],\ \text{a.s.}$$

(iii) Let us set

$$\overline{M}_t^\circ:=\int_0^t\sigma_s^\circ dM_s^\circ,\ t\geq 0.$$

Then,

$$\begin{aligned}
&\mathbb{E}^\circ\Big[\frac{1}{\varepsilon}\int_0^t\big(\overline{M}_{s+\varepsilon}^\circ-\overline{M}_s^\circ\big)^2ds\Big]\\
&= \mathbb{E}^\circ\Big[\int_0^t\frac{1}{\varepsilon}\Big(\int_s^{s+\varepsilon}2(\overline{M}_r^\circ-\overline{M}_s^\circ)d\overline{M}_r^\circ+\int_s^{s+\varepsilon}d[\overline{M}^\circ]_r\Big)ds\Big]\\
&= 2\int_0^{t+\varepsilon}\frac{1}{\varepsilon}\int_{(r-\varepsilon)_+}^{r\wedge t}\mathbb{E}^\circ\big[(\overline{M}_r^\circ-\overline{M}_s^\circ)\sigma_r^\circ\big]ds\;dM_r^\circ\;+\;\mathbb{E}^\circ\Big[\int_0^{t+\varepsilon}\frac{r\wedge t-(r-\varepsilon)_+}{\varepsilon}d[\overline{M}^\circ]_r\Big]. \qquad (12)
\end{aligned}$$

Let us define

$$H_r^\varepsilon:=\frac{1}{\varepsilon}\int_{(r-\varepsilon)_+}^{r\wedge t}\mathbb{E}^\circ\big[(\overline{M}_r^\circ-\overline{M}_s^\circ)\sigma_r^\circ\big]ds.$$

Under the square integrability conditions in (8), it is easy to check that

$$\big|H_r^\varepsilon\big|\;\leq\;\sqrt{\mathbb{E}^\circ\big[|\sigma_r^\circ|^2\big]}\;\frac{1}{\varepsilon}\int_{(r-\varepsilon)_+}^{r\wedge t}\sqrt{\mathbb{E}^\circ\big[\big(\overline{M}_r^\circ-\overline{M}_s^\circ\big)^2\big]}ds\;\longrightarrow\;0,\ \text{as }\varepsilon\longrightarrow 0. \qquad (13)$$

Moreover, one has

$$\big|H_r^\varepsilon\big|\;\leq\;H_r\;:=\;2\sqrt{\mathbb{E}^\circ\big[|\sigma_r^\circ|^2\big]}\;\sup_{s\leq r}\sqrt{\mathbb{E}^\circ\big[(\overline{M}_s^\circ)^2\big]}.$$

Again, under the square integrability conditions in (8), the process $\big(\mathbb{E}^\circ\big[(\overline{M}^\circ_s)^2\big]\big)_{s\ge 0}$ is continuous so that one can localize it by a sequence of stopping times, which can be considered to be $(\tau_n)_{n\ge1}$ w.l.o.g. It follows that, for a sequence of positive constants $(C_n)_{n\ge1}$,

$$\mathbb{E}\Big[\int_0^{t\wedge\tau_n} H_r^2 d[M^\circ]_r\Big] \;\le\; C_n\mathbb{E}\Big[\int_0^t |\sigma^\circ_r|^2 d[M^\circ]_r\Big] < \infty.$$

Together with (13), one can deduce that

$$\int_0^{t+\varepsilon}\frac{1}{\varepsilon}\int_{(r-\varepsilon)_+}^{r\wedge t}\mathbb{E}^\circ\big[(\overline{M}^\circ_r-\overline{M}^\circ_s)\sigma^\circ_r\big]ds\; dM^\circ_r = \int_0^{t+\varepsilon} H^\varepsilon_r dM^\circ_r \longrightarrow 0,\; t\ge 0,\; \text{u.c.p.} \tag{14}$$

Further, it is easy to see that

$$\mathbb{E}^\circ\Big[\int_0^{t+\varepsilon}\frac{r\wedge t-(r-\varepsilon)_+}{\varepsilon}d[\overline{M}^\circ]_r\Big] \;\longrightarrow\; \mathbb{E}^\circ\big[[\overline{M}^\circ]_t\big],\; \text{a.s.}$$

Together with (12) and (14), this leads to

$$\lim_{\varepsilon\to0}\mathbb{E}^\circ\left[\frac{1}{\varepsilon}\int_0^t\Big(\overline{M}^\circ_{s+\varepsilon}-\overline{M}^\circ_s\Big)^2 ds\right] \;=\; \mathbb{E}^\circ\big[[\overline{M}^\circ]_t\big] = \int_0^t\mathbb{E}^\circ\big[(\sigma^\circ_s)^2\big]d[M^\circ]_s,\; t\ge0,\; \text{u.c.p.}$$

This concludes the proof. □

Lemma 2.12. *Let the conditions of Theorem 2.3 and Condition (8) hold. Let $(I^{2,\varepsilon})_{\varepsilon>0}$ be defined as in (10). Then,*

$$I^{2,\varepsilon}_t \;\longrightarrow\; \int_0^t \mathbb{E}^\circ\big[D_mF(s,\cdot,m_s,X_s)\sigma^\circ_s\big](Y_s)d[M^\circ,N]_s,\; t\ge0, \textit{u.c.p., as } \varepsilon\longrightarrow 0.$$

Proof. Recall that

$$I^{2,\varepsilon}_t \;:=\; \int_0^t\big[F(s+\varepsilon,Y_s,m_{s+\varepsilon})-F(s+\varepsilon,Y_s,m_s)\big]\frac{N_{s+\varepsilon}-N_s}{\varepsilon}ds,$$

and that

$$m_s = \mathcal{L}(X_s|\mathcal{G}),\; s\ge0.$$

By the definition of δ_mF and D_mF in (1) and (2), it follows that

$$\begin{aligned}
I^{2,\varepsilon}_t &= \int_0^t\int_0^1\mathbb{E}^\circ\Big[\delta_mF(s+\varepsilon,\cdot,m^{\lambda,\varepsilon}_s,X_{s+\varepsilon})-\delta_mF(s+\varepsilon,\cdot,m^{\lambda,\varepsilon}_s,X_s)\Big](Y_s)\frac{N_{s+\varepsilon}-N_s}{\varepsilon}d\lambda ds\\
&= \int_0^t\int_0^1\int_0^1\mathbb{E}^\circ\Big[D_mF(s+\varepsilon,\cdot,m^{\lambda,\varepsilon}_s,X^{\eta,\varepsilon}_s)(X_{s+\varepsilon}-X_s)\Big](Y_s)\frac{N_{s+\varepsilon}-N_s}{\varepsilon}d\eta d\lambda ds,
\end{aligned}$$

where

$$m^{\lambda,\varepsilon}_s := m_s+\lambda(m_{s+\varepsilon}-m_s) \text{ and } X^{\eta,\varepsilon}_s := X_s+\eta(X_{s+\varepsilon}-X_s).$$

Let us write

$$I^{2,\varepsilon}_t \;=\; J^{1,\varepsilon}_t + J^{2,\varepsilon}_t,\; t\ge0,$$

where

$$J^{1,\varepsilon}_t := \int_0^t\int_0^1\int_0^1\mathbb{E}^\circ\Big[\Delta_mF(s,s+\varepsilon,\cdot,\lambda,\eta)\;(X_{s+\varepsilon}-X_s)\Big](Y_s)\frac{N_{s+\varepsilon}-N_s}{\varepsilon}d\eta d\lambda ds,$$

with

$$\Delta_m F(s,t,y,\lambda,\eta) := D_m F\big(t,y,m_s+\lambda(m_t-m_s), X_s+\eta(X_t-X_s)\big) - D_m F\big(t,y,m_s,X_s\big),$$

and

$$J_t^{2,\varepsilon} := \int_0^t \mathbb{E}^\circ\Big[D_m F(s+\varepsilon,\cdot,m_s,X_s)(X_{s+\varepsilon}-X_s)\Big](Y_s)\frac{N_{s+\varepsilon}-N_s}{\varepsilon}ds.$$

(i) To study the limit of $J_t^{1,\varepsilon}$, we notice that the map $(s,t,y)\longmapsto \Delta_m F(s,t,y,\lambda,\eta)$ is continuous. Hence, by Condition (8),

$$(s,t,y)\longmapsto \int_0^1\int_0^1 \mathbb{E}^\circ\Big[\big(\Delta_m F(s,t,y,\lambda,\eta)\big)^2\Big]d\eta d\lambda$$

is also continuous and hence uniformly continuous on compact sets. In particular, one has

$$\lim_{\varepsilon\to 0}\Delta_t(\varepsilon)=0, \text{ a.s., with } \Delta_t(\varepsilon) := \sup_{s\le t}\int_0^1\int_0^1 \mathbb{E}^\circ\Big[\big(\Delta_m F(s,s+\varepsilon,\cdot,\lambda,\eta)\big)^2\Big](Y_s)d\eta d\lambda.$$

Further, by Cauchy-Schwarz inequality,

$$\sup_{s\le t}\big|J_s^{1,\varepsilon}\big| \;\le\; \sqrt{\Delta_t(\varepsilon)}\,\sqrt{\frac{1}{\varepsilon}\int_0^t \mathbb{E}^\circ\Big[\big(X_{s+\varepsilon}-X_s\big)^2\Big]ds}\,\sqrt{\frac{1}{\varepsilon}\int_0^t (N_{s+\varepsilon}-N_s)^2 ds}.$$

Since the limit in probability of

$$\frac{1}{\varepsilon}\int_0^t (N_{s+\varepsilon}-N_s)^2 ds \text{ and } \frac{1}{\varepsilon}\int_0^t \mathbb{E}^\circ\Big[(X_{s+\varepsilon}-X_s)^2\Big]ds$$

are both finite a.s., by the fact that N has finite quadratic variation and by Lemma 2.11 for the right-hand side term, we hence conclude that

$$J_t^{1,\varepsilon}\longrightarrow 0,\; t\ge 0,\; \text{u.c.p. as } \varepsilon\longrightarrow 0.$$

(ii) We next consider $J_t^{2,\varepsilon}$, and write it as

$$J_t^{2,\varepsilon} \;=\; J_t^{21,\varepsilon}+J_t^{22,\varepsilon}+J_t^{23,\varepsilon},$$

where

$$J_t^{21,\varepsilon} := \int_0^t \mathbb{E}^\circ\big[D_m F(s+\varepsilon,\cdot,m_s,X_s)(A_{s+\varepsilon}-A_s)\big](Y_s)\frac{N_{s+\varepsilon}-N_s}{\varepsilon}ds,$$

$$J_t^{22,\varepsilon} := \int_0^t \mathbb{E}^\circ\big[D_m F(s+\varepsilon,\cdot,m_s,X_s)(M_{s+\varepsilon}-M_s)\big](Y_s)\frac{N_{s+\varepsilon}-N_s}{\varepsilon}ds,$$

and

$$J_t^{23,\varepsilon} := \int_0^t \mathbb{E}^\circ\Big[D_m F(s+\varepsilon,\cdot,m_s,X_s)\int_s^{s+\varepsilon}\sigma_r^\circ dM_r^\circ\Big](Y_s)\frac{N_{s+\varepsilon}-N_s}{\varepsilon}ds.$$

First, observe that, under Condition (8), one has

$$\mathbb{E}^\circ\Big[\Big(D_m F\Big(r,y,m_s^{\lambda,t},X_s^{\eta,t}\Big)\Big)^2\Big]\le C,$$

for some constant $C > 0$, and

$$\left|J_t^{21,\varepsilon}\right| \;\leq\; \sqrt{C}\sqrt{\frac{1}{\varepsilon}\int_0^t \mathbb{E}^\circ[(A_{s+\varepsilon}-A_s)^2]ds}\;\sqrt{\frac{1}{\varepsilon}\int_0^t (N_{s+\varepsilon}-N_s)^2 ds} \longrightarrow 0, \text{ u.c.p.},$$

by Lemma 2.11.

Next, by Lemma 2.10,

$$\mathbb{E}^\circ\Big[D_m F(s+\varepsilon, y, m_s, X_s)\int_s^{s+\varepsilon} dM_r\Big] = 0, \text{ a.s. for all } s \geq 0.$$

Thus,

$$J_t^{22,\varepsilon} = 0, \text{ for all } t \geq 0, \text{ a.s.}$$

Finally, we study the limit of $J_t^{23,\varepsilon}$. Define

$$\psi(s,r,\varepsilon) := \mathbb{E}^\circ\Big[D_m F(s+\varepsilon,\cdot,m_s,X_s)\sigma_r^\circ\Big](Y_s) - \mathbb{E}^\circ\Big[D_m F(r,\cdot,m_r,X_r)\sigma_r^\circ\Big](Y_r).$$

We claim that

$$\lim_{\varepsilon\to 0}\int_0^t\int_s^{s+\varepsilon}\psi(s,r,\varepsilon)dM_r^\circ \frac{N_{s+\varepsilon}-N_s}{\varepsilon}ds = 0, \; t\geq 0, \; u.c.p., \text{ as } \varepsilon \longrightarrow 0. \tag{15}$$

Then, setting $D_t := \int_0^t \mathbb{E}^\circ\Big[D_m F(r,\cdot,m_r,X_r)\sigma_r^\circ\Big](Y_r)dM_r^\circ$, we can apply [19, Proposition 1.1] to deduce that

$$\begin{aligned}
\lim_{\epsilon\to 0} J_t^{23,\epsilon} &= \lim_{\epsilon\to 0}\int_0^t\int_s^{s+\epsilon}\mathbb{E}^\circ\Big[D_m F(r,\cdot,m_r,X_r)\sigma_r^\circ\Big](Y_r)dM_r^\circ\frac{N_{s+\varepsilon}-N_s}{\varepsilon}ds \\
&= \lim_{\epsilon\to 0}\int_0^t (D_{s+\epsilon}-D_s)\frac{N_{s+\varepsilon}-N_s}{\varepsilon}ds \\
&= [D,N]_t \\
&= \int_0^t \mathbb{E}^\circ\Big[D_m F(r,\cdot,m_r,X_r)\sigma_r^\circ\Big](Y_r)d[M^\circ,N]_r,
\end{aligned}$$

where the last equality follows from the property of the classical covariation.

To conclude, it remains to prove our claim (15). By Cauchy-Schwarz inequality,

$$\begin{aligned}
&\left|J_t^{23,\varepsilon} - \int_0^t \mathbb{E}^\circ\Big[\int_s^{s+\varepsilon} D_m F(r,\cdot,m_r,X_r)\sigma_r^\circ dM_r^\circ\Big](Y_r)\frac{N_{s+\varepsilon}-N_s}{\varepsilon}ds\right| \\
= &\left|\int_0^t\int_s^{s+\varepsilon}\psi(s,r,\varepsilon)dM_r^\circ\frac{N_{s+\varepsilon}-N_s}{\varepsilon}ds\right| \\
\leq &\sqrt{\frac{1}{\varepsilon}\int_0^t\left(\int_s^{s+\varepsilon}\psi(s,r,\varepsilon)dM_r^\circ\right)^2 ds}\sqrt{\frac{1}{\varepsilon}\int_0^t (N_{s+\varepsilon}-N_s)^2 ds.}
\end{aligned}$$

Thus, it suffices to show that

$$\frac{1}{\varepsilon}\int_0^t \mathbb{E}[(\int_s^{s+\varepsilon}\psi(s,r,\varepsilon)dM_r^\circ)^2]ds \longrightarrow 0, \text{ as } \varepsilon \longrightarrow 0. \tag{16}$$

By Ito's isometry,

$$\frac{1}{\varepsilon}\int_0^t \mathbb{E}\Big[\Big(\int_s^{s+\varepsilon}\psi(s,r,\varepsilon)dM_r^\circ\Big)^2\Big]ds \;=\; \mathbb{E}\Big[\int_0^t \frac{1}{\varepsilon}\int_s^{s+\varepsilon}\psi(s,r,\varepsilon)^2 d[M^\circ]_r ds\Big]$$
$$= \mathbb{E}\Big[\int_0^{t+\varepsilon}\frac{1}{\varepsilon}\int_{(r-\varepsilon)_+}^r \psi(s,r,\varepsilon)^2 ds d[M^\circ]_r\Big].$$

Also, $\frac{1}{\varepsilon}\int_{(r-\varepsilon)_+}^r \psi(s,r,\varepsilon)^2 ds \longrightarrow 0$ as $\varepsilon \longrightarrow 0$, and

$$\frac{1}{\varepsilon}\int_{(r-\varepsilon)_+}^r \psi(s,r,\varepsilon)^2 ds \;\le\; 4C\mathbb{E}^\circ\big[(\sigma_r^\circ)^2\big],$$

and

$$\mathbb{E}\Big[\int_0^{2t} 4C\mathbb{E}^\circ\big[(\sigma_r^\circ)^2\big]d[M^\circ]_r\Big] < +\infty,$$

both thanks to Condition (8). Therefore, it follows from the dominated convergence theorem that (16) holds, and we hence conclude the proof. □

3 A verification theorem for a class of McKean-Vlasov optimal control problems

Let $\Omega^0 = \Omega^1 := \mathcal{C}([0,T],\mathbb{R}^d)$ be the canonical spaces of $\mathbb{R}^d$-valued continuous paths on $[0,T]$, where the canonical process on Ω^0 is denoted by X^0, and the one on Ω^1 is denoted by W. Under the uniform convergence topology, we define $\mathcal{F}^0 := \mathcal{B}(\Omega^0)$ and $\mathcal{F}^1 := \mathcal{B}(\Omega^1)$ as the Borel σ-field of respectively Ω^0 and Ω^1. On Ω^0 (resp. Ω^1), we define $\mathbb{F}^0$ (resp. $\mathbb{F}^1$) as the canonical filtration generated by X^0 (resp. W), and equip $(\Omega^0,\mathcal{F}^0)$ (resp. $(\Omega^1,\mathcal{F}^1)$) with the Wiener measure $\mathbb{P}_0^0$ (resp. $\mathbb{P}_0^1$). Let U be a bounded Borel subset of $\mathbb{R}^d$, and $\mathcal{U}^0$ denote the collection of all $\mathbb{F}^0$-progressively measurable process $\nu : [0,T]\times\Omega^0 \longrightarrow \mathrm{U}$. Then, for each initial condition $(t,\mathbf{x}^0)\in[0,T]\times\Omega^0$, we consider a collection $\mathcal{P}_W^0(t,\mathbf{x}^0)$ of probability measures on Ω^0:

$$\mathcal{P}_W^0(t,\mathbf{x}^0) := \Big\{\mathbb{P}^0\in\mathcal{P}(\Omega^0) \;:\; X_s^0 = \mathbf{x}_t^0 + \int_t^s \nu_r^{\mathbb{P}^0}dr + \int_t^s dW_r^{\mathbb{P}^0},\; s\in[t,T],\; \mathbb{P}^0\text{-a.s.},$$
$$\mathbb{P}^0[X_{t\wedge\cdot}^0 = \mathbf{x}_{t\wedge\cdot}^0] = 1,\; \text{where } \nu^{\mathbb{P}^0}\in\mathcal{U}^0,\; W^{\mathbb{P}^0} \text{ is a } (\mathbb{P}^0,\mathbb{F}^0)\text{-Brownian motion}\Big\}.$$

Next, let $\Omega := \Omega^0\times\Omega^1$, $\mathcal{F} := \mathcal{F}^0\otimes\mathcal{F}^1 = \mathcal{B}(\Omega)$, and

$$\mathcal{P}_W(t,\mathbf{x}^0) \;:=\; \big\{\mathbb{P} = \mathbb{P}^0\times\mathbb{P}_0^1 \;:\mathbb{P}^0\in\mathcal{P}_W^0(t,\mathbf{x}^0)\big\}.$$

We also consider the canonical space $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$, equipped with the canonical element ξ.

We are given a bounded[1] measurable coefficient $(\sigma,\sigma_0) : \mathbb{R}_+\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{M}^d\times\mathbb{M}^d$ with $\mathbb{M}^d$ denoting the collection of all $d\times d$ matrix. Hereafter, we assume that $(\sigma,\sigma_0)(t,\cdot)$ is Lipschitz continuous, uniformly in $t\le T$. Then, for all $t\in[0,T]$, $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ and $\mathbb{P}\in\mathcal{P}_W(t,\mathbf{x}^0)$, on the space $\mathbb{R}^d\times\Omega$ with canonical element (ξ,X^0,W), we consider the McKean-Vlasov SDE

$$X_s^{t,\mu,\mathbb{P}} = \xi + \int_t^s \sigma_0(r,X_r^{t,\mu,\mathbb{P}},\rho_r^{t,\mu,\mathbb{P}})dX_r^0 + \int_t^s \sigma(r,X_r^{t,\mu,\mathbb{P}},\rho_r^{t,\mu,\mathbb{P}})dW_r,\; \mu\times\mathbb{P}\text{-a.s.},$$

[1] As usual, boundedness could be replaced by a suitable growth condition, see for instance [9, Assumption 2.8].

with $\rho_r^{t,\mu,\mathbb{P}} = \mathcal{L}^{\mu\times\mathbb{P}}(X_r^{t,\mu,\mathbb{P}}|\mathcal{F}_r^{X^0})$, where $\mathcal{F}_r^{X^0} := \sigma(X_s^0 \ : s \in [0,r])$. Notice that the above McKean-Vlasov SDE has a unique strong solution (see e.g. [9, Appendix A]).

Finally, let us define an enlarged canonical space

$$\overline{\Omega} := \Omega \times \mathcal{C}([0,T],\mathbb{R}^d) \times \mathcal{C}([0,T],\mathcal{P}_2(\mathbb{R}^d)),$$

with canonical process (X^0, W, X, ρ), and

$$\overline{\mathcal{P}}_W(t,\mu) \ := \ \Big\{\overline{\mathbb{P}} := (\mu\times\mathbb{P})\circ\big(X^0, W, X^{t,\mu,\mathbb{P}}, \rho^{t,\mu,\mathbb{P}}\big)^{-1} \ : \mathbb{P}\in\mathcal{P}_W(t,\mathbf{x}^0), \ \mathbf{x}^0\in\Omega^0\Big\}.$$

We denote by $\overline{\mathbb{F}}^{X^0} = (\overline{\mathcal{F}}_t^{X^0})_{0\le t\le T}$ the filtration generated by X^0 on the enlarged canonical space, and denote by $\overline{\mathbb{P}}_0$ the probability measure on $\overline{\Omega}$ under which the canonical process X^0 is a Brownian motion. Notice that, for $\overline{\mathbb{P}} = \mu\times\mathbb{P}^0\times\mathbb{P}_0^1 \in \overline{\mathcal{P}}_W(t,\mu)$, the part $\mathbb{P}_0^1$ is the fixed Wiener measure, and $\mathbb{P}^0$ belongs to $\mathcal{P}_W^0(t,\mathbf{x}^0)$ under which the canonical process X^0 is a diffusion process with drift $\nu^{\mathbb{P}^0}$. By abuse of notation, we denote by $\nu^{\overline{\mathbb{P}}}$ the corresponding drift process of X^0 on $\overline{\Omega}$ and by $W^{\overline{\mathbb{P}}}$ the corresponding $(\mathbb{P}^0,\mathbb{F}^0)$-Brownian motion part of X^0 on $\overline{\Omega}$, i.e.

$$X_s^0 = X_t^0 + \int_t^s \nu_r^{\overline{\mathbb{P}}} dr + \int_t^s dW_r^{\overline{\mathbb{P}}}, \ s\in[t,T], \ \overline{\mathbb{P}}\text{-a.s.}$$

Remark 3.1. *A probability measure $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,\mu)$ describes the distribution of the controlled McKean-Vlasov process with initial distribution μ at initial time t, and control $\nu^{\overline{\mathbb{P}}}$. Since controls take values in the bounded set* U*, given a fixed $\mu_0\in\mathcal{P}_2(\mathbb{R}^d)$, all the probability measures in the set $\overline{\mathcal{P}}_W(t,\mu_0)$ are equivalent by Girsanov's theorem.*

Let $L : [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times \mathrm{U} \longrightarrow \mathbb{R}$ and $g : \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$ be measurable functions, to which we associate the value function of the McKean-Vlasov control problem through

$$V(t,\mu) \ := \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(t,\mu)} J\left(t,\overline{\mathbb{P}}\right), \ \ \text{with } J(t,\overline{\mathbb{P}}) := \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^T L\big(s,\rho_s,\nu_s^{\overline{\mathbb{P}}}\big)ds + g(\rho_T)\Big]. \qquad (17)$$

Here again, we assume for simplicity that L and g are bounded.

Given a probability measure $\mu\in\mathcal{P}(\mathbb{R}^d)$, and (measurable) functions $\varphi : \mathbb{R}^d \longrightarrow \mathbb{R}$ and $\psi : \mathbb{R}^d\times\mathbb{R}^d \longrightarrow \mathbb{R}$, we denote (whenever the integrals are well-defined)

$$\mu(\varphi) := \int_{\mathbb{R}^d}\varphi(x)\mu(dx), \quad \mu\otimes\mu(\psi) := \int_{\mathbb{R}^d\times\mathbb{R}^d}\psi\left(x,x'\right)\mu(dx)\mu\left(dx'\right).$$

Let us also define

$$\mathcal{K} := \Big\{\phi : [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d\to\mathbb{R}^d \ : \ \phi \text{ is bounded and Borel measurable}\Big\}.$$

Remark 3.2. *The above McKean-Vlasov optimal control problem satisfies the following dynamic programming principle: for $(t,\mu)\in[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$ and any $\overline{\mathbb{F}}^{X^0}$-stopping time τ taking values in $[t,T]$, one has*

$$V(t,\mu) \ = \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(t,\mu)} \mathbb{E}^{\overline{\mathbb{P}}}\left[\int_t^\tau L(s,\rho_s,\nu_s^{\overline{\mathbb{P}}})ds + V(\tau,\rho_\tau)\right].$$

Under general conditions, the value function V can be proved to be the unique (viscosity) solution to the HJB master equation (see e.g. Pham and Wei [17]):

$$\begin{cases} -\partial_t V(t,\mu) - \mathbb{L}V(t,\mu) - H(t,\mu,D_m V) = 0, \quad (t,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d), \\ \qquad\qquad\qquad\qquad\qquad\qquad V(T,\mu) = g(\mu), \quad \mu\in\mathcal{P}_2(\mathbb{R}^d), \end{cases}$$

where $\mathbb{L}$ is the operator defined for $\phi \in C^2(\mathcal{P}_2(\mathbb{R}^d))$ as

$$\begin{aligned} \mathbb{L}\phi(t,\mu) \;:=\; & \mu\Big(\operatorname{tr}(\partial_x D_m\phi(\mu,x)(\sigma\sigma^{\mathrm{T}} + \sigma_0\sigma_0^{\mathrm{T}})(t,x,\mu))\Big) \\ & +\; \mu\otimes\mu\Big(\frac{1}{2}\operatorname{tr}(D_m^2\phi(\mu)(x,x')\sigma_0(t,x,\mu)\sigma_0^{\mathrm{T}}(t,x',\mu))\Big), \end{aligned}$$

and $H : [0,T]\times\mathcal{P}_2(\mathbb{R}^d)\times\mathcal{K} \longrightarrow \mathbb{R}$ is the Hamiltonian defined by

$$H(t,\mu,p) \;:=\; \sup_{u\in\mathrm{U}}\left(L(t,\mu,u) + u\mu\left(\sigma_0(t,\cdot,\mu)p(t,\mu,\cdot)\right)\right). \tag{18}$$

From now on we fix an initial distribution $m_0 \in \mathcal{P}_2(\mathbb{R}^d)$, and study the optimal control problem $V(0,m_0)$. Our verification argument goes together with a duality result, (19) below, in which the two dual problems are defined as:

$$\begin{aligned} D_1 := \inf\Big\{ v_0\in\mathbb{R}:\; & v_0 + \int_0^T \rho_t\left(\sigma_0(t,\cdot,\rho_t)\phi(t,\rho_t,\cdot)\right)dX_t^0 \\ & \geq\; g(\rho_T) + \int_0^T\Big(L(t,\rho_t,\nu_t^{\overline{\mathbb{P}}}) + \nu_t^{\overline{\mathbb{P}}}\rho_t(\sigma_0(t,\cdot,\rho_t)\phi(t,\rho_t,\cdot))\Big)dt, \\ & \overline{\mathbb{P}}\text{-a.s. for all } \overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,m_0), \text{ for some } \phi\in\mathcal{K}\Big\}, \end{aligned}$$

and

$$\begin{aligned} D_2 := \inf\Big\{ v_0\in\mathbb{R}:\; \exists\phi\in\mathcal{K} \text{ s.t. } & v_0 + \int_0^T \rho_t\left(\sigma_0(t,\cdot,\rho_t)\phi(t,\rho_t,\cdot)\right)dX_t^0 \\ & \geq\; g(\rho_T) + \int_0^T H(t,\rho_t,\phi)dt,\; \overline{\mathbb{P}}_0\text{-a.s.}\Big\}. \end{aligned}$$

Similar to (6), we say that a function $F : [0,T]\times\mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathbb{R}$ belongs to $C^{0,1}([0,T)\times\mathcal{P}_2(\mathbb{R}^d))$ if F and D_mF are both (jointly) continuous on $[0,T)\times\mathcal{P}_2(\mathbb{R}^d)$, where $D_mF : [0,T)\times\mathcal{P}_2(\mathbb{R}^d)\times\mathbb{R}^d \longrightarrow \mathbb{R}^d$ is the partial derivative in the sense that $(m,x)\mapsto D_mF(t,m,x)$ is the derivative of $m\mapsto F(t,m)$ as defined in (2).

Theorem 3.1. *Assume that the value function V is continuous on $[0,T]\times\mathcal{P}_2(\mathbb{R}^d)$, belongs to $C^{0,1}([0,T)\times\mathcal{P}_2(\mathbb{R}^d))$ and that D_mV is uniformly bounded on $[0,T)\times\mathcal{P}_2(\mathbb{R}^d)$.*
(i) Then, one has the duality

$$V(0,m_0) = D_1 = D_2. \tag{19}$$

(ii) Assume that $\hat{\mathrm{u}} : [0,T)\times\mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathrm{U}$ is a Borel measurable function such that, for all $(s,\mu)\in[0,T)\times\mathcal{P}_2(\mathbb{R}^d)$,

$$H(s,\mu,D_mV) = L(s,\mu,\hat{\mathrm{u}}(s,\mu)) + \hat{\mathrm{u}}(s,\mu)\mu\left(\sigma_0(s,\cdot,\mu)D_mV(s,\mu,\cdot)\right). \tag{20}$$

Then, there exists $\widehat{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$ such that $\nu^{\widehat{\mathbb{P}}} = \hat{\mathrm{u}}(\cdot, \rho)$, $d\widehat{\mathbb{P}} \times dt$ a.e., and $\widehat{\mathbb{P}}$ is an optimal solution to the control problem $V(0, m_0)$, i.e.

$$J(0, \widehat{\mathbb{P}}) \;=\; V(0, m_0) \;=\; \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)} J(0, \overline{\mathbb{P}}),$$

and

$$V(0, m_0) + \int_0^T \rho_t \left(\sigma_0(t, \cdot, \rho_t) D_m V(t, \rho_t, \cdot)\right) dX_t^0 = g(\rho_T) + \int_0^T H(t, \rho_t, D_m V) dt, \;\; \hat{\mathbb{P}}\text{-a.s.}$$

Before to provide the proof of the above, let us comment these results.

Remark 3.3. (i) *In Theorem 3.1, we assume that $D_m V$ is bounded for simplicity in order to apply Theorem 2.3. The same results still hold if this boundedness condition is replaced by a local integrability condition as in Assumption 2.2.*

(ii) *When U is compact, and L is upper semi-continuous, one can choose a Borel version of the optimizer $\hat{\mathrm{u}}$ as required in Theorem 3.1, see e.g. [1, Proposition 7.33, p.153].*

(iii) *The duality result (19) is in the spirit of duality results in mathematical finance and optimal transport, see e.g. [2] for an abstract formulation in optimal control. The novelty here is that it goes together with the proof of the verification argument and appeals directly to a functional class of controls, which is made possible because we know a priori that $D_m V$ is well-defined so that we can identify the optimal control by means of our Itô's formula for C^1-functionals.*

(iv) *In the literature of mean-field control or mean-field games, it is usually difficult to check that the value function is C^2 (see e.g. [6, 13] for examples). The C^1-regularity as required in the above will be clearly easier to prove. We provide in Example 3.6 below an example of McKean-Vlasov control problem, in which the C^1-regularity can be obtained by using purely probabilistic arguments.*

(v) *The control problem in (17) is a pure McKean-Vlasov control problem, so that the value function is in the form $V(t, \rho_t)$. One could also study mixed control problems by considering a controlled diffusion process Y in addition to X, and a reward function in the form $g(\rho_T^X, Y_T)$, so that the value of the control problem would be in the form $V(t, \rho_t^X, Y_t)$. The same arguments as below would lead to a similar verification result. We stay in this pure McKean-Vlasov control setting for ease of presentation.*

Remark 3.4. *The reason for considering a weak formulation of the McKean-Vlasov control problem in (17) comes from our duality type arguments which do not apply to strong formulations. On the other hand, the law induced by a strong control is a weak control, as a probability measure in $\overline{\mathcal{P}}_W(0, m_0)$. Then, given a strong control ν which achieves the optimality in the Hamiltonian as in (20), we obtain an optimal control for the weak formulation (17) by Theorem 3.1, and hence it is also an optimal control for the (more restrictive) strong formulation. More generally speaking, by a direct adaptation of the arguments in Djete, Possamaï and Tan [10, Section 4, Proof of Theorem 3.1], one can prove that, under quite general upper-semicontinuity conditions on L and g, the value functions of the weak and strong formulations are the same, even if an optimal strong control does not exist.*

Proof of Theorem 3.1. (i) Let $(v_0, \phi) \in \mathbb{R} \times \mathcal{K}$ be a couple satisfying the inequality in the definition of D_1, i.e.

$$v_0 + \int_0^T \rho_t \left(\sigma_0(t, \cdot, \rho_t) \phi(t, \rho_t, \cdot)\right) dX_t^0 \geq g(\rho_T) + \int_0^T \Big[L(t, \rho_t, \nu_t^{\overline{\mathbb{P}}}) + \nu_t^{\overline{\mathbb{P}}} \rho_t(\sigma_0(t, \cdot, \rho_t) \phi(t, \rho_t, \cdot)) \Big] dt, \; \overline{\mathbb{P}}\text{-a.s.}$$

for all $\overline{\mathbb{P}} \in \mathcal{P}_W(0, m_0)$. Taking expectation under $\overline{\mathbb{P}}$ on both sides of the above inequality, it leads to

$$v_0 \;\geq\; \mathbb{E}^{\overline{\mathbb{P}}}\Big[g(\rho_T) + \int_0^T L(t, \rho_t, \nu_t^{\overline{\mathbb{P}}})dt\Big], \text{ for all } \overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0).$$

Further, by the definition of H, we notice that, for all $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$,

$$L(t, \rho_t, \nu_t^{\overline{\mathbb{P}}}) + \nu_t^{\overline{\mathbb{P}}} \rho_t(\sigma_0(t, \cdot, \rho_t)\phi(t, \rho_t, \cdot)) \;\leq\; H(t, \rho_t, \phi),\; t \in [0, T].$$

This proves that

$$V(0, m_0) \;\leq\; D_1 \;\leq\; D_2.$$

It remains to prove that $V(0, m_0) \geq D_2$. For each $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$, let us introduce the process $S^{\overline{\mathbb{P}}} = \left(S_t^{\overline{\mathbb{P}}}\right)_{0 \leq t \leq T}$ by

$$S_t^{\overline{\mathbb{P}}} := V(t, \rho_t) + \int_0^t L(s, \rho_s, \nu_s^{\overline{\mathbb{P}}})ds.$$

For all $0 \leq t \leq t+h \leq T$, the dynamic programming principle (see e.g. [9, Theorem 3.2]) implies that

$$\begin{aligned}
S_t^{\overline{\mathbb{P}}}(\omega) \;&=\; \sup_{\tilde{\mathbb{P}} \in \overline{\mathcal{P}}_W(t, \rho_t(\omega))} \mathbb{E}^{\tilde{\mathbb{P}}}\left[\int_t^{t+h} L(s, \rho_s, \nu_s^{\tilde{\mathbb{P}}})ds + V(t+h, \rho_{t+h})\right] + \int_0^t L(s, \rho_s, \nu_s^{\overline{\mathbb{P}}})(\omega)ds \\
&\geq\; \mathbb{E}^{\overline{\mathbb{P}}}\left[\int_t^{t+h} L(s, \rho_s, \nu_s^{\overline{\mathbb{P}}})ds + V(t+h, \rho_{t+h}) \,\Big|\overline{\mathcal{F}}_t^{X^0}\right](\omega) + \int_0^t L(s, \rho_s, \nu_s^{\overline{\mathbb{P}}})(\omega)ds \\
&=\; \mathbb{E}^{\overline{\mathbb{P}}}\left[S_{t+h}^{\overline{\mathbb{P}}} \,\Big|\overline{\mathcal{F}}_t^{X^0}\right](\omega), \qquad (21)
\end{aligned}$$

for $\overline{\mathbb{P}}$-a.e. ω. In other words, $S^{\overline{\mathbb{P}}}$ is a $(\overline{\mathbb{P}}, \overline{\mathbb{F}}^{X^0})$-supermartingale for all $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$. Therefore, for each $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$, one can apply the Doob-Meyer decomposition to obtain a unique $\overline{\mathbb{F}}^{X^0}$-predictable non-decreasing process $A^{\overline{\mathbb{P}}}$ and a $(\overline{\mathbb{P}}, \overline{\mathbb{F}}^{X^0})$-martingale $M^{\overline{\mathbb{P}}}$ such that $A_0^{\overline{\mathbb{P}}} = M_0^{\overline{\mathbb{P}}} = 0$ and

$$V(t, \rho_t) + \int_0^t L(s, \rho_s, \nu_s^{\overline{\mathbb{P}}})ds = V(0, m_0) + M_t^{\overline{\mathbb{P}}} - A_t^{\overline{\mathbb{P}}},\; t \in [0, T],\; \overline{\mathbb{P}}\text{-a.s.}$$

At the same time, as $V \in C^{0,1}$, we apply our C^1-Itô's formula in Theorem 2.3 to deduce another unique decomposition

$$V(t, \rho_t) = V(0, m_0) + \int_0^t \rho_s\big(\sigma_0(s, \cdot, \rho_s)D_m V(s, \rho_s, \cdot)\big)dW_s^{\overline{\mathbb{P}}} + \Gamma_t^{\overline{\mathbb{P}}},\; \overline{\mathbb{P}}\text{-a.s.},$$

where $(\Gamma_t^{\overline{\mathbb{P}}})_{0 \leq t \leq T}$ is an orthogonal process. The above two decompositions are unique, so that the two martingale parts should be the same. It follows that

$$V(t, \rho_t) + \int_0^t L(s, \rho_s, \nu_s^{\overline{\mathbb{P}}})ds = V(0, m_0) + \int_0^t \rho_s\left(\sigma_0(s, \cdot, \rho_s)D_m V(s, \rho_s, \cdot)\right)dW_s^{\overline{\mathbb{P}}} - A_t^{\overline{\mathbb{P}}},\; \overline{\mathbb{P}}\text{-a.s.}$$

As $V(T, \rho_T) = g(\rho_T)$ and $A^{\overline{\mathbb{P}}}$ is non-decreasing, this implies

$$\begin{aligned}
&V(0, m_0) + \int_0^T \rho_t\left(\sigma_0(t, \cdot, \rho_t)D_m V(t, \rho_t, \cdot)\right)dX_t^0 \\
&\geq\; g(\rho_T) + \int_0^T \Big(\rho_t\left(\sigma_0(t, \cdot, \rho_t)D_m V(t, \rho_t, \cdot)\right)\nu_t^{\overline{\mathbb{P}}} + L(t, \rho_t, \nu_t^{\overline{\mathbb{P}}})\Big)dt. \qquad (22)
\end{aligned}$$

For each $\varepsilon > 0$, one can then apply the measurable selection arguments, e.g. combine [11, Proposition 2.21] with [1, Lemma 7.27, p.173], to obtain $\nu^\varepsilon \in \mathcal{U}^0$ such that

$$L(s,\rho_s,\nu_s^\varepsilon) + \nu_s^\varepsilon \rho_s(\sigma_0(s,\cdot,\rho_s)D_mV(s,\rho_s,\cdot)) \;\geq\; H(s,\rho_s,\phi) - \varepsilon,\; d\overline{\mathbb{P}}_0 \times dt\text{-a.e.}$$

One can then construct a probability measure $\overline{\mathbb{P}}^\varepsilon$ such that $\nu^{\overline{\mathbb{P}}^\varepsilon} = \nu^\varepsilon$. As all probability measures in $\overline{\mathcal{P}}_W(0,m_0)$ are equivalent, together with (22), this shows that

$$T\varepsilon + V(0,m_0) + \int_0^T \rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right) dX_t^0 \geq\; g(\rho_T) + \int_0^T H(t,\rho_t,D_mV)dt,\; \overline{\mathbb{P}}_0\text{-a.s.}$$

By arbitrariness of $\varepsilon > 0$, $V(0,m_0) \geq D_2$.

(ii) Assume now that there exists a Borel measurable map $\hat{\mathrm{u}} : [0,T) \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathrm{U}$ such that

$$H(s,\rho_s,D_mV) = L(s,\rho_s,\hat{\mathrm{u}}(s,\rho_s)) + \hat{\mathrm{u}}(s,\rho_s)\rho_s\left(\sigma_0(s,\cdot,\rho_s)D_mV(s,\rho_s,\cdot)\right),\; s \in [0,T].$$

We can then construct a probability measure $\widehat{\mathbb{P}}$ such that $(\nu_t^{\widehat{\mathbb{P}}})_{0\leq t<T} = (\hat{\mathrm{u}}(t,\rho_t))_{0\leq t<T}$, $\widehat{\mathbb{P}}$-a.s. To show that $\widehat{\mathbb{P}}$ is an optimal solution to the control problem $V(0,m_0)$, we appeal to Lemma 3.5 below to deduce that

$$V(0,m_0) + \int_0^T \rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right) dX_t^0 = g(\rho_T) + \int_0^T H(t,\rho_t,D_mV)dt,\; \hat{\mathbb{P}}\text{-a.s.} \tag{23}$$

We can then compute directly that

$$\begin{aligned} V(0,m_0) \;&=\; g(\rho_T) + \int_0^T H(t,\rho_t,D_mV)dt - \int_0^T \rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right) dX_t^0 \\ &=\; g(\rho_T) + \int_0^T L(s,\rho_s,\hat{\mathrm{u}}(s,\rho_s))ds - \int_0^T \rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right) dW_t^{\widehat{\mathbb{P}}}. \end{aligned}$$

Taking expectation on both sides under $\widehat{\mathbb{P}}$, it follows that

$$V(0,m_0) \;=\; \mathbb{E}^{\widehat{\mathbb{P}}}\left[g(\rho_T) + \int_0^T L(s,\rho_s,\hat{\mathrm{u}}(s,\rho_s))ds\right] \;=\; J(0,\widehat{\mathbb{P}}),$$

i.e. $\widehat{\mathbb{P}}$ is an optimal solution to the control problem $V(0,m_0)$. □

Lemma 3.5. *In the setting of Theorem 3.1 (ii), we have*

$$V(0,m_0) + \int_0^T \rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right) dX_t^0 \;=\; g(\rho_T) + \int_0^T H\big(t,\rho_t,D_mV\big)dt,\; \overline{\mathbb{P}}_0\text{-a.s.}$$

Proof. From the definition of the process $(S_t^{\overline{\mathbb{P}}})_{0\leq t\leq T}$ and its decomposition given in the proof of Theorem 3.1, we have

$$\begin{aligned} \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[S_T^{\overline{\mathbb{P}}}] \;&=\; \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}}\Big[g(\rho_T) + \int_0^T L(t,\rho_t,\nu_t^{\overline{\mathbb{P}}})dt\Big] \\ &=\; V(0,m_0) - \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\overline{\mathbb{P}}}],\; \overline{\mathbb{P}}_0\text{-a.s.} \end{aligned}$$

which implies that $\inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\overline{\mathbb{P}}}] = 0$. Moreover, for $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)$ and $\widehat{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)$ such that $\nu^{\widehat{\mathbb{P}}} = \hat{\mathrm{u}}(\cdot,\rho)$, $d\overline{\mathbb{P}} \times dt$-a.e., we have

$$\begin{aligned}
V(0,m_0) &= g(\rho_T) + \int_0^T \Big[L(t,\rho_t,\nu_t^{\overline{\mathbb{P}}}) + \nu^{\overline{\mathbb{P}}}\rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right)\Big]\,dt + A_T^{\overline{\mathbb{P}}} \\
&\quad - \int_0^T \rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right)dX_t^0 \\
&= g(\rho_T) + \int_0^T \Big[L(t,\rho_t,\nu_t^{\widehat{\mathbb{P}}}) + \nu_t^{\widehat{\mathbb{P}}}\rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right)\Big]\,dt + A_T^{\widehat{\mathbb{P}}} \\
&\quad - \int_0^T \rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right)dX_t^0 \qquad (24) \\
&\geq g(\rho_T) + \int_0^T \Big[L(t,\rho_t,\nu_t^{\overline{\mathbb{P}}}) + \nu_t^{\overline{\mathbb{P}}}\rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right)\Big]\,dt + A_T^{\widehat{\mathbb{P}}} \\
&\quad - \int_0^T \rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right)dX_t^0.
\end{aligned}$$

Combining the above implies that $0 \leq A_T^{\widehat{\mathbb{P}}} \leq A_T^{\overline{\mathbb{P}}}$ a.s. for $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)$, and

$$0 = \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\overline{\mathbb{P}}}] \geq \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\widehat{\mathbb{P}}}] = 0.$$

At the same time, we have by the reverse Hölder's inequality

$$\begin{aligned}
\inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\widehat{\mathbb{P}}}] &= \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}_0}\left[\frac{d\overline{\mathbb{P}}}{d\overline{\mathbb{P}}_0}A_T^{\widehat{\mathbb{P}}}\right] \\
&\geq \inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,m_0)} \frac{\mathbb{E}^{\overline{\mathbb{P}}_0}\left[(A_T^{\widehat{\mathbb{P}}})^{\frac{1}{2}}\right]^2}{\mathbb{E}^{\overline{\mathbb{P}}_0}\left[\left(\frac{d\overline{\mathbb{P}}}{d\overline{\mathbb{P}}_0}\right)^{-1}\right]} \\
&\geq \frac{\mathbb{E}^{\overline{\mathbb{P}}_0}\left[(A_T^{\widehat{\mathbb{P}}})^{\frac{1}{2}}\right]^2}{C},
\end{aligned}$$

where $C > 0$ is a fixed constant such that $\mathbb{E}^{\overline{\mathbb{P}}_0}[(\frac{d\overline{\mathbb{P}}}{d\overline{\mathbb{P}}_0})^{-1}] \leq C$ for $\forall\, \overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)$ whose existence is justified as follows:

$$\begin{aligned}
\left(\frac{d\overline{\mathbb{P}}}{d\overline{\mathbb{P}}_0}\right)^{-1} &= \exp\left(\int_0^T -\nu_t^{\overline{\mathbb{P}}}dW_t^{\overline{\mathbb{P}}_0} + \frac{1}{2}\int_0^T (\nu_t^{\overline{\mathbb{P}}})^2dt\right) \\
&= \exp\left(\int_0^T -\nu_t^{\overline{\mathbb{P}}}dW_t^{\overline{\mathbb{P}}_0} - \frac{1}{2}\int_0^T (\nu_t^{\overline{\mathbb{P}}})^2dt\right)\exp\left(\int_0^T (\nu_t^{\overline{\mathbb{P}}})^2dt\right),
\end{aligned}$$

which implies that

$$\mathbb{E}^{\overline{\mathbb{P}}_0}\left[\left(\frac{d\overline{\mathbb{P}}}{d\overline{\mathbb{P}}_0}\right)^{-1}\right] \leq \exp\left(T\bar{u}^2\right) =: C,$$

with $\bar{u} := \max_{u\in\mathrm{U}}|u|$. Therefore, combining the above shows that $A_T^{\widehat{\mathbb{P}}} = 0, \overline{\mathbb{P}}_0$-a.s., and we can conclude by (24) that

$$V(0,m_0) + \int_0^T \rho_t\left(\sigma_0(t,\cdot,\rho_t)D_mV(t,\rho_t,\cdot)\right)dX_t^0 = g(\rho_T) + \int_0^T H\big(t,\rho_t,D_mV\big)dt,\ \overline{\mathbb{P}}_0\text{-a.s.}$$

□

Example 3.6. *Let $d \geq 1$, $\sigma(\cdot) = \sigma_0(\cdot) \equiv 1$, U be a convex and compact subset of $\mathbb{R}^d$, $L(t,\mu,u) = \bar{L}(u)$ for some function*

$$\bar{L} : \mathrm{U} \longrightarrow \mathbb{R} \text{ , strictly concave,}$$

and $g(\mu) = \bar{g}\big(\mu(\phi)\big)$ with $\bar{g} : \mathbb{R}^d \longrightarrow \mathbb{R}$ (resp. $\phi : \mathbb{R} \longrightarrow \mathbb{R}^d$) in $C_b^1(\mathbb{R}^d)$ (resp. $C_b^1(\mathbb{R})$). In this simple setting, we can re-write the value function as

$$V(t,\mu) \;=\; \sup_{\mathbb{P}^0 \in \mathcal{P}_W^0(t,0)} \overline{J}(t,\mu,\mathbb{P}^0), \tag{25}$$

where

$$\overline{J}(t,\mu,\mathbb{P}^0) \;:=\; \mathbb{E}^{\mathbb{P}^0}\Big[\int_t^T \bar{L}(\nu_s^{\mathbb{P}^0})ds + \bar{g}\Big(\int_{\mathbb{R}^d} \bar{\phi}_t(z + X_T^0)\mu(dz)\Big)\Big], \tag{26}$$

with

$$\bar{\phi}_t(y) \;:=\; \mathbb{E}^{\mathbb{P}_0^1}\big[\phi(y + W_T - W_t)\big],$$

and

$$\mathcal{P}_W^0(t,0) := \Big\{\mathbb{P}^0 \in \mathcal{P}(\Omega^0) \;: X_s^0 = \int_t^{s\vee t} \nu_r^{\mathbb{P}^0} dr + \int_t^s dW_r^{\mathbb{P}^0},\; s \in [0,T],\; \mathbb{P}^0\text{-a.s.}$$
$$\text{where } \nu^{\mathbb{P}^0} \in \mathcal{U}^0,\; W^{\mathbb{P}^0} \text{ is a } (\mathbb{P}^0,\mathbb{F}^0)\text{-Brownian motion}\Big\}.$$

For fixed $\mathbb{P}^0 \in \mathcal{P}_W^0(t,0)$, it is clear that the map $\mu \longmapsto \overline{J}(t,\mu,\mathbb{P}^0)$ is differentiable and

$$D_m\overline{J}(t,\mu,x,\mathbb{P}^0) \;=\; \mathbb{E}^{\mathbb{P}^0}\Big[\bar{g}'\Big(\int_{\mathbb{R}^d} \bar{\phi}_t(z + X_T^0)\mu(dz)\Big)\nabla\bar{\phi}_t\big(x + X_T^0\big)\Big] \tag{27}$$

in which $\bar{g}'$ is the Jacobian of $\bar{g}$ and $\nabla\bar{\phi}_t$ is the gradient of $\bar{\phi}_t$ as a column vector. Clearly, $D_m\overline{J}$ is continuous in all its arguments.

Next, as L depends only on u, we can apply Tan and Touzi [22, Lemma 3.9] to deduce that the map

$$\mathbb{P}^0 \longmapsto \overline{J}(t,\mu,\mathbb{P}^0) \text{ is upper semicontinuous,}$$

while, since U is convex and compact, $\mathcal{P}_W^0(t,0)$ is also convex and compact (for the weak convergence topology). Then, for every fixed (t,μ), there exists an optimizer for the optimal control problem (25).

We now prove that

$$\mathbb{P}^0 \longmapsto \overline{J}(t,\mu,\mathbb{P}^0) \text{ is strictly concave.} \tag{28}$$

Let $\mathbb{P}_1^0,\; \mathbb{P}_2^0 \in \mathcal{P}_W^0(t,0)$ and $\mathbb{P}_3^0 := (\mathbb{P}_1^0 + \mathbb{P}_2^0)/2$. Following the arguments in the proof of [22, Proposition 3.11.(ii) and Lemma 3.15], there exists an enlarged canonical space $\overline{\Omega}^0$ (of $\Omega^0 := \mathcal{C}([0,T],\mathbb{R}^d)$) with canonical process $(X^0,\bar{\nu})$, together with $\overline{\mathbb{P}}_1^0, \overline{\mathbb{P}}_2^0 \in \mathcal{P}(\overline{\Omega}^0)$ such that

$$\mathbb{P}_i^0 \circ (X^0, \nu^{\mathbb{P}_i^0})^{-1} \;=\; \overline{\mathbb{P}}_i^0 \circ (X^0,\bar{\nu})^{-1},\; i = 1,2,$$

so that

$$\mathbb{E}^{\mathbb{P}_i^0}\Big[\int_t^T \bar{L}\big(\nu_s^{\mathbb{P}_i^0}\big)ds\Big] \;=\; \mathbb{E}^{\overline{\mathbb{P}}_i^0}\Big[\int_t^T \bar{L}\big(\bar{\nu}_s\big)ds\Big],\; i = 1,2.$$

Let $\overline{\mathbb{P}}^0_3 := (\overline{\mathbb{P}}^0_1 + \overline{\mathbb{P}}^0_2)/2$ *so that* $\overline{\mathbb{P}}^0_3|_{\Omega^0} = \mathbb{P}^0_3$. *Since* U *is convex, one can apply the classical projection theorem (see e.g. [22, Theorem A.3]) to deduce that* $\mathbb{P}^0_3 \in \mathcal{P}^0_W(t,0)$ *and*

$$\nu_s^{\mathbb{P}^0_3} = \mathbb{E}^{\overline{\mathbb{P}}^0_3}\big[\bar\nu_s | \overline{\mathcal{F}}^{X^0}_s\big],\ \ d\mathbb{P}^0_3 \times ds\text{-a.e.},$$

in which $\overline{\mathcal{F}}^{X^0}_s$ *is the* σ*-field on the enlarged space* $\overline{\Omega}^0$ *generated by* $X^0_{s\wedge\cdot}$*, and we consider* $\mathbb{E}^{\overline{\mathbb{P}}^0_3}\big[\bar\nu_s|\overline{\mathcal{F}}^{X^0}_s\big]$ *as a random variable in* Ω^0. *Moreover,* $\nu_s^{\mathbb{P}^0_3} = \bar\nu_s\ d\overline{\mathbb{P}}^0_3 \times ds$*-a.e. only if* $\mathbb{P}^0_1 = \mathbb{P}^0_2$. *Since* $\bar L$ *is strictly concave, when* $\mathbb{P}^0_1 \neq \mathbb{P}^0_2$*, it follows by Jensen's inequality that*

$$\begin{aligned}
\frac{1}{2}\Big(\mathbb{E}^{\mathbb{P}^0_1}\Big[\int_t^T \bar L\big(\nu_s^{\mathbb{P}^0_1}\big)ds\Big] + \mathbb{E}^{\mathbb{P}^0_2}\Big[\int_t^T \bar L\big(\nu_s^{\mathbb{P}^0_2}\big)ds\Big]\Big) &= \frac{1}{2}\Big(\mathbb{E}^{\overline{\mathbb{P}}^0_1}\Big[\int_t^T \bar L\big(\bar\nu_s\big)ds\Big] + \mathbb{E}^{\overline{\mathbb{P}}^0_2}\Big[\int_t^T \bar L\big(\bar\nu_s\big)ds\Big]\Big)\\
&= \mathbb{E}^{\overline{\mathbb{P}}^0_3}\Big[\int_t^T \bar L\big(\bar\nu_s\big)ds\Big]\\
&< \mathbb{E}^{\overline{\mathbb{P}}^0_3}\Big[\int_t^T \bar L\big(\nu_s^{\mathbb{P}^0_3}\big)ds\Big]\\
&= \mathbb{E}^{\mathbb{P}^0_3}\Big[\int_t^T \bar L\big(\nu_s^{\mathbb{P}^0_3}\big)ds\Big].
\end{aligned}$$

As $\mathbb{P}^0_3 := (\mathbb{P}^0_1 + \mathbb{P}^0_2)/2$*, this implies that*

$$\mathbb{P}^0 \longmapsto \mathbb{E}^{\mathbb{P}^0}\Big[\int_t^T \bar L\big(\nu_s^{\mathbb{P}^0}\big)ds\Big] \text{ is strictly concave.}$$

Since

$$\mathbb{P}^0 \longmapsto \mathbb{E}^{\mathbb{P}^0}\Big[\bar g\Big(\int_{\mathbb{R}^d}\bar\phi_t(z + X^0_T)\mu(dz)\Big)\Big] \text{ is linear,}$$

it follows by the definition of $\overline{J}$ *in* (26) *that* (28) *holds true. Therefore, for every fixed* (t,μ)*, there exists a unique optimizer* $\widehat{\mathbb{P}}^0_{t,\mu}$ *for the optimal control problem* (25). *In particular, one immediately deduces that*

$$(t,\mu) \ \longmapsto\ \widehat{\mathbb{P}}^0_{t,\mu} \text{ is continuous.} \tag{29}$$

Now, we claim that

$$\frac{\delta V}{\delta m}(t,\mu,x) \ = \ \frac{\delta \overline{J}}{\delta m}(t,\mu,x,\widehat{\mathbb{P}}^0_{t,\mu}), \quad \text{for all } (t,\mu,x). \tag{30}$$

Indeed, given $\mu \neq \mu'$*, and with the notation* $\mu_\lambda := \lambda\mu + (1-\lambda)\mu'$*, one has*

$$V(t,\mu) - V(t,\mu') \le \overline{J}(t,\mu,\widehat{\mathbb{P}}_{t,\mu}) - \overline{J}(t,\mu',\widehat{\mathbb{P}}_{t,\mu}) = \int_0^1\int_{\mathbb{R}^d}\frac{\delta\overline{J}}{\delta m}\big(t,\mu_\lambda,x,\widehat{\mathbb{P}}_{t,\mu}\big)(\mu-\mu')(dx)d\lambda,$$

and similarly

$$V(t,\mu) - V(t,\mu') \ge \int_0^1\int_{\mathbb{R}^d}\frac{\delta\overline{J}}{\delta m}\big(t,\mu_\lambda,x,\widehat{\mathbb{P}}_{t,\mu'}\big)(\mu-\mu')(dx)d\lambda.$$

Using the continuity of $(t,\mu) \mapsto \widehat{\mathbb{P}}^0_{t,\mu}$ *in* (29) *and applying the two inequalities just above to* $\mu' = \mu'_\varepsilon := \mu + \varepsilon(\mu'' - \mu)$ *with* $\varepsilon \downarrow 0$*, this is enough to prove* (30). *Taking derivatives on both sides in* (30)*, one has*

$$D_m V(t,\mu,x) \ = \ D_m\overline{J}(t,\mu,x,\widehat{\mathbb{P}}^0_{t,\mu}).$$

Finally, using (29) *together with the continuity of* $D_m\bar J$*, we deduce that* $V \in C^{0,1}([0,T]\times\mathcal{P}_2(\mathbb{R}^d))$.

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