A $C^{0,1}$-Itô's Formula for Flows of Semimartingale Distributions

B. Bouchard

CEREMADE, Université Paris Dauphine - PSL

Joint work with Xiaolu Tan and Jixin Wang (Chinese University of Hong Kong)

Motivation

□ Replace $C^{1,2}$-regularity by $C^{0,1}$ when applying Itô's lemma in situations where regularity is difficult to obtain :

- Path-dependent functionals : see Xiaolu's talk.
- McKean-Vlasov optimal control problems.

Motivation

□ Replace $C^{1,2}$-regularity by $C^{0,1}$ when applying Itô's lemma in situations where regularity is difficult to obtain :

- Path-dependent functionals : see Xiaolu's talk.
- McKean-Vlasov optimal control problems.

□ We know that it is possible for functionals on $[0, T] \times \mathbb{R}^d$ associated to classical Markovian problems.

The classical Markovian situation

Weak Dirichlet processes

□ In the Markovian case : works by Russo and his co-authors, using the concept of weak Dirichlet processes and the stochastic calculus by regularization. See in particular Gozzi and Russo [4].

Weak Dirichlet processes

□ In the Markovian case : works by Russo and his co-authors, using the concept of weak Dirichlet processes and the stochastic calculus by regularization. See in particular Gozzi and Russo [4].

Definitions :

- Let X and Y be two real valued càdlàg processes. The co-quadractic variation $[X, Y]$ is defined by

$$[X, Y]_t := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{(s+\varepsilon)\wedge t} - X_s)(Y_{(s+\varepsilon)\wedge t} - Y_s) ds,$$

whenever the limit exists in the sense of u.c.p.

Weak Dirichlet processes

□ In the Markovian case : works by Russo and his co-authors, using the concept of weak Dirichlet processes and the stochastic calculus by regularization. See in particular Gozzi and Russo [4].

Definitions :

- Let X and Y be two real valued càdlàg processes. The co-quadractic variation $[X, Y]$ is defined by

$$[X, Y]_t \; := \; \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{(s+\varepsilon)\wedge t} - X_s)(Y_{(s+\varepsilon)\wedge t} - Y_s) ds,$$

  whenever the limit exists in the sense of u.c.p.

- X has finite quadratic variation, if $[X] := [X, X]$, exists and is finite a.s.

Weak Dirichlet processes

□ In the Markovian case : works by Russo and his co-authors, using the concept of weak Dirichlet processes and the stochastic calculus by regularization. See in particular Gozzi and Russo [4].

Definitions :

- Let X and Y be two real valued càdlàg processes. The co-quadractic variation $[X, Y]$ is defined by

$$[X, Y]_t := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{(s+\varepsilon)\wedge t} - X_s)(Y_{(s+\varepsilon)\wedge t} - Y_s) ds,$$

  whenever the limit exists in the sense of u.c.p.

- X has finite quadratic variation, if $[X] := [X, X]$, exists and is finite a.s.
- A is **orthogonal** if $[A, N] = 0$ for any real valued continuous local martingale N.

Weak Dirichlet processes

☐ In the Markovian case : works by Russo and his co-authors, using the concept of weak Dirichlet processes and the stochastic calculus by regularization. See in particular Gozzi and Russo [4].

Definitions :

- Let X and Y be two real valued càdlàg processes. The co-quadractic variation $[X, Y]$ is defined by

$$[X, Y]_t \; := \; \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \int_0^t (X_{(s+\varepsilon)\wedge t} - X_s)(Y_{(s+\varepsilon)\wedge t} - Y_s) ds,$$

  whenever the limit exists in the sense of u.c.p.
- X has finite quadratic variation, if $[X] := [X, X]$, exists and is finite a.s.
- A is **orthogonal** if $[A, N] = 0$ for any real valued continuous local martingale N.
- X is a **weak Dirichlet process** if $X = X_0 + M + A$, where M is a local martingale and A is orthogonal such that $M_0 = A_0 = 0$.

$\mathbb{C}^{0,1}$-Itô's formula

□ **Theorem (Gozzi and Russo [4]) :** Let $X = X_0 + M + A$ be a continuous weak Dirichlet process with finite quadratic variation, $v \in C^{0,1}([0,T) \times \mathbb{R}^d)$. Then,

$$v(t, X_t) \ = \ v(0, X) \ + \ \int_0^t \partial_x v(s, X_s) dM_s \ + \ \Gamma_t, \quad t \in [0,T),$$

where Γ is a continuous orthogonal process.

$\mathbb{C}^{0,1}$-Itô's formula

□ **Theorem (Gozzi and Russo [4]) :** Let $X = X_0 + M + A$ be a continuous weak Dirichlet process with finite quadratic variation, $v \in C^{0,1}([0,T) \times \mathbb{R}^d)$. Then,

$$v(t, X_t) \ = \ v(0, X) \ + \ \int_0^t \partial_x v(s, X_s) dM_s \ + \ \Gamma_t, \quad t \in [0, T),$$

where Γ is a continuous orthogonal process.

□ **Remark :**

- A version is available for processes with jumps, see Bandini and Russo [1].

$\mathbb{C}^{0,1}$-Itô's formula

□ **Theorem (Gozzi and Russo [4]) :** Let $X = X_0 + M + A$ be a continuous weak Dirichlet process with finite quadratic variation, $v \in C^{0,1}([0,T) \times \mathbb{R}^d)$. Then,

$$v(t, X_t) \ = \ v(0, X) \ + \ \int_0^t \partial_x v(s, X_s) dM_s \ + \ \Gamma_t, \quad t \in [0,T),$$

where Γ is a continuous orthogonal process.

□ **Remark :**

- A version is available for processes with jumps, see Bandini and Russo [1].
- If $v(\cdot, X)$ is a martingale, then $\Gamma \equiv 0$ (e.g. replication price in finance, value function of an optimal control problem along the optimal path,...)

$\mathbb{C}^{0,1}$-Itô's formula

☐ **Theorem (Gozzi and Russo [4]) :** Let $X = X_0 + M + A$ be a continuous weak Dirichlet process with finite quadratic variation, $v \in C^{0,1}([0,T) \times \mathbb{R}^d)$. Then,

$$v(t, X_t) \;=\; v(0, X) \;+\; \int_0^t \partial_x v(s, X_s) dM_s \;+\; \Gamma_t, \quad t \in [0, T),$$

where Γ is a continuous orthogonal process.

☐ **Remark :**

- A version is available for processes with jumps, see Bandini and Russo [1].
- If $v(\cdot, X)$ is a martingale, then $\Gamma \equiv 0$ (e.g. replication price in finance, value function of an optimal control problem along the optimal path,...)
- Can be extended to path-dependent functionals using the notion of Dupire's derivatives, see B., Loeper and Tan [2].

C^1-Itô's formula for flows of semimartingale distributions

The setting

□ Consider a continuous semimartingale on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, satisfying the usual conditions,

$$X = X_0 + A + M^X, \quad \text{with} \quad M^X = M + \int_0^\cdot \sigma_s^\circ dM_s^\circ.$$

Define $\mathcal{G}^\circ = (\mathcal{G}_t^\circ)_{t\geq 0}$, where $\mathcal{G}_t^\circ := \sigma(M_s^\circ,\ 0 \leq s \leq t)$ and

$$\mathbb{E}^\circ[\xi] := \mathbb{E}[\xi | \mathcal{G}^\circ]$$

The setting

☐ Consider a continuous semimartingale on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, satisfying the usual conditions,

$$X = X_0 + A + M^X, \quad \text{with} \quad M^X = M + \int_0^\cdot \sigma_s^\circ dM_s^\circ.$$

Define $\mathcal{G}^\circ = (\mathcal{G}_t^\circ)_{t\geq 0}$, where $\mathcal{G}_t^\circ := \sigma(M_s^\circ,\ 0 \leq s \leq t)$ and

$$\mathbb{E}^\circ[\xi] := \mathbb{E}[\xi|\mathcal{G}^\circ]$$

☐ Consider a continuous weak Dirichlet process

$$Y = Y_0 + A^Y + M^Y,$$

with $[Y, Y]_T < \infty$.

The setting

☐ **Assumption :**

(i) σ° is $\mathbb{F}$-progressively measurable, and $\exists$ sequence of stopping times $(\tau_n)_{n\geq 1}$ w.r.t. $\mathcal{G}^\circ$ s.t. $\tau_n \uparrow \infty$ a.s. and

$$\mathbb{E}\Big[[M]_{\tau_n\wedge t} + |A|^2_{\tau_n\wedge t} + \int_0^{\tau_n\wedge t} |\sigma^\circ_s|^2 d[M^\circ]_s\Big] < +\infty, \text{ for all } t \geq 0 \text{ and } n \geq 1.$$

(ii) M is orthogonal to N (i.e. $[M, N] = 0$), for all $\mathcal{G}^\circ$-martingales N.

(iii) (H)-hypothesis condition :

$$\mathbb{E}\big[1_D \big| \mathcal{G}^\circ_t\big] = \mathbb{E}\big[1_D \big| \mathcal{G}^\circ\big], \text{ a.s., for all } D \in \mathcal{F}_t,\ t \geq 0.$$

Define the $\mathcal{P}(\mathbb{R}^d)$-valued process

$$m_t := \mathcal{L}(X_t\ |\mathcal{G}^\circ_t) = \mathcal{L}(X_t\ |\mathcal{G}^\circ), \quad t \geq 0.$$

Derivative with respect to the measure

□ Given $F : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$, let $\delta F/\delta m : \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}$, be s.t.

$$F(m') - F(m) = \int_0^1 \int_{\mathbb{R}^d} \frac{\delta F}{\delta m}(tm' + (1-t)m, x)\,[m' - m]\,(dx)dt$$

and set

$$D_m F(m, x) := \partial_x \frac{\delta F}{\delta m}(m, x).$$

Main result

□ We consider $F \in C^{0,1,1}(\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ such that the following holds

Main result

□ We consider $F \in C^{0,1,1}(\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ such that the following holds

□ **Assumption :** $\forall\ n \geq 1,\ T > 0$ and compact $K \subset \mathbb{R}^d$, $\exists\ C > 0$ s.t.

$$\mathbb{E}^{\circ}\Big[\big(D_m F(r, y, m_s^{n,\lambda,t}, X_s^{n,\eta,t})\big)^2\Big] \leq C,\ \text{a.s.},$$
$$\forall\ (r, s, t) \in [0, 2T] \times [0, t] \times [0, T], (\lambda, \eta, y) \in [0, 1]^2 \times K,$$

where $m_s^{n,\lambda,t} := (1-\lambda) m_{\tau_n \wedge s} + \lambda m_{\tau_n \wedge t}$, $X_s^{n,\eta,t} := (1-\eta) X_{\tau_n \wedge s} + \eta X_{\tau_n \wedge t}$.

Main result

□ We consider $F \in C^{0,1,1}(\mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d))$ such that the following holds

□ **Assumption :** $\forall\ n \geq 1,\ T > 0$ and compact $K \subset \mathbb{R}^d$, $\exists\ C > 0$ s.t.

$$\mathbb{E}^\circ\Big[\big(D_m F(r, y, m_s^{n,\lambda,t}, X_s^{n,\eta,t})\big)^2\Big] \leq C,\ \text{ a.s.},$$
$$\forall\ (r, s, t) \in [0, 2T] \times [0, t] \times [0, T], (\lambda, \eta, y) \in [0, 1]^2 \times K,$$

where $m_s^{n,\lambda,t} := (1-\lambda) m_{\tau_n \wedge s} + \lambda m_{\tau_n \wedge t}$, $X_s^{n,\eta,t} := (1-\eta) X_{\tau_n \wedge s} + \eta X_{\tau_n \wedge t}$.

□ **Theorem :** $\exists$ a continuous orthogonal process Γ such that

$$F(t, Y_t, m_t) = F(0, Y_0, m_0) + \int_0^t \partial_y F(s, Y_s, m_s)\ dM_s^Y$$
$$+ \int_0^t \mathbb{E}^\circ\big[D_m F(s, \cdot, m_s, X_s)\sigma_s^\circ\big](Y_s) dM_s^\circ + \Gamma_t,\ \ t \geq 0.$$

Sketch of proof

- We restrict to $F(t, m, y) = F(m)$.

Sketch of proof

- ☐ We restrict to $F(t, m, y) = F(m)$.
- ☐ Define

$$\Gamma_t \ := \ F(m_t) - \int_0^t \mathbb{E}^\circ \left[D_m F(m_s, X_s) \sigma_s^\circ \right] dM_s^\circ .$$

Sketch of proof

- ☐ We restrict to $F(t, m, y) = F(m)$.
- ☐ Define

$$\Gamma_t \;:=\; F(m_t) - \int_0^t \mathbb{E}^\circ\big[D_m F(m_s, X_s)\sigma_s^\circ\big] dM_s^\circ.$$

We need to show that, for any continuous martingale N,

$$\frac{1}{\varepsilon}\int_0^t \big[\Gamma_{s+\varepsilon} - \Gamma_s\big](N_{s+\varepsilon} - N_s)ds \;\longrightarrow\; 0,$$

Sketch of proof

- We restrict to $F(t, m, y) = F(m)$.
- Define

$$\Gamma_t \;:=\; F(m_t) - \int_0^t \mathbb{E}^\circ\big[D_m F(m_s, X_s)\sigma_s^\circ\big] dM_s^\circ.$$

We need to show that, for any continuous martingale N,

$$\frac{1}{\varepsilon}\int_0^t \big[\Gamma_{s+\varepsilon} - \Gamma_s\big](N_{s+\varepsilon} - N_s)ds \;\longrightarrow\; 0,$$

or equivalently

$$I_t^\varepsilon := \frac{1}{\varepsilon}\int_0^t \big[F(m_{s+\varepsilon}) - F(m_s)\big](N_{s+\varepsilon} - N_s)ds \;\longrightarrow\; I_t,$$

where

$$I_t := \int_0^t \mathbb{E}^\circ\Big[D_m F(m_s, X_s)\sigma_s^\circ\Big] d[M^\circ, N]_s.$$

By definition of $D_m F$,

$$\int_0^t \left[F(m_{s+\varepsilon}) - F(m_s)\right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds$$
$$= \int_0^t \int_0^1 \int_0^1 \mathbb{E}^\circ \Big[D_m F(m_s^{\lambda,\varepsilon}, X_s^{\eta,\varepsilon})(X_{s+\varepsilon} - X_s) \Big] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} d\eta d\lambda ds,$$

where $m_s^{\lambda,\varepsilon} := m_s + \lambda(m_{s+\varepsilon} - m_s)$ and $X_s^{\eta,\varepsilon} := X_s + \eta(X_{s+\varepsilon} - X_s)$.

By definition of $D_m F$,

$$\int_0^t \left[F(m_{s+\varepsilon}) - F(m_s)\right] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds$$
$$= \int_0^t \int_0^1 \int_0^1 \mathbb{E}^\circ \Big[D_m F(m_s^{\lambda,\varepsilon}, X_s^{\eta,\varepsilon})(X_{s+\varepsilon} - X_s)\Big] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} d\eta d\lambda ds,$$

where $m_s^{\lambda,\varepsilon} := m_s + \lambda(m_{s+\varepsilon} - m_s)$ and $X_s^{\eta,\varepsilon} := X_s + \eta(X_{s+\varepsilon} - X_s)$.

We can show that $\lim_{\varepsilon \to 0} I_t^\varepsilon = \lim_{\varepsilon \to 0} J_t^\varepsilon$, where

$$J_t^\varepsilon := \int_0^t \mathbb{E}^\circ \Big[D_m F(m_s, X_s)(X_{s+\varepsilon} - X_s)\Big] \frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds.$$

We then write

$$J_t^{\varepsilon} = J_t^{1,\varepsilon} + J_t^{2,\varepsilon} + J_t^{3,\varepsilon},$$

where

$$J_t^{1,\varepsilon} := \int_0^t \mathbb{E}^\circ\big[D_m F(m_s, X_s)(A_{s+\varepsilon} - A_s)\big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds,$$

$$J_t^{2,\varepsilon} := \int_0^t \mathbb{E}^\circ\big[D_m F(m_s, X_s)(M_{s+\varepsilon} - M_s)\big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds,$$

$$J_t^{3,\varepsilon} := \int_0^t \mathbb{E}^\circ\Big[D_m F(m_s, X_s)\int_s^{s+\varepsilon} \sigma_r^\circ dM_r^\circ\Big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds.$$

We then write

$$J_t^{\varepsilon} \;=\; J_t^{1,\varepsilon} + J_t^{2,\varepsilon} + J_t^{3,\varepsilon},$$

where

$$J_t^{1,\varepsilon} := \int_0^t \mathbb{E}^{\circ}\big[D_m F(m_s, X_s)(A_{s+\varepsilon} - A_s)\big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds,$$

$$J_t^{2,\varepsilon} := \int_0^t \mathbb{E}^{\circ}\big[D_m F(m_s, X_s)(M_{s+\varepsilon} - M_s)\big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds,$$

$$J_t^{3,\varepsilon} := \int_0^t \mathbb{E}^{\circ}\Big[D_m F(m_s, X_s) \int_s^{s+\varepsilon} \sigma_r^{\circ} dM_r^{\circ}\Big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon} ds.$$

Then $J_t^{1,\varepsilon} \longrightarrow 0$, $J_t^{2,\varepsilon} \longrightarrow 0$, u.c.p.,

We then write

$$J_t^{\varepsilon} \;=\; J_t^{1,\varepsilon} + J_t^{2,\varepsilon} + J_t^{3,\varepsilon},$$

where

$$J_t^{1,\varepsilon} := \int_0^t \mathbb{E}^\circ\big[D_m F(m_s, X_s)(A_{s+\varepsilon} - A_s)\big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon}ds,$$

$$J_t^{2,\varepsilon} := \int_0^t \mathbb{E}^\circ\big[D_m F(m_s, X_s)(M_{s+\varepsilon} - M_s)\big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon}ds,$$

$$J_t^{3,\varepsilon} := \int_0^t \mathbb{E}^\circ\Big[D_m F(m_s, X_s)\int_s^{s+\varepsilon}\sigma_r^\circ dM_r^\circ\Big]\frac{N_{s+\varepsilon} - N_s}{\varepsilon}ds.$$

Then $J_t^{1,\varepsilon} \longrightarrow 0$, $J_t^{2,\varepsilon} \longrightarrow 0$, u.c.p., and

$$\begin{aligned}\lim_{\varepsilon\to 0} J_t^{3,\varepsilon} \;&=\; \lim_{\varepsilon\to 0}\int_0^t\int_s^{s+\varepsilon}\mathbb{E}^\circ\Big[D_m F(m_r, X_r)\sigma_r^\circ\Big]dM_r^\circ\,\frac{N_{s+\varepsilon} - N_s}{\varepsilon}ds \\ &= \int_0^t \mathbb{E}^\circ\Big[D_m F(m_r, X_r)\sigma_r^\circ\Big]d[M^\circ, N]_r.\end{aligned}$$

A verification theorem for a class of McKean-Vlasov optimal control problems

A class of McKean-Vlasov optimal control problems

□ Let $\Omega^0 = \Omega^1 := \mathcal{C}([0, T], \mathbb{R}^d)$ with canonical process X^0 and W, canonical filtrations $\mathbb{F}^0$ and $\mathbb{F}^1$, and Wiener measures $\mathbb{P}_0^0$ and $\mathbb{P}_0^1$.

A class of McKean-Vlasov optimal control problems

□ Let $\Omega^0 = \Omega^1 := \mathcal{C}([0, T], \mathbb{R}^d)$ with canonical process X^0 and W, canonical filtrations $\mathbb{F}^0$ and $\mathbb{F}^1$, and Wiener measures $\mathbb{P}_0^0$ and $\mathbb{P}_0^1$.

□ Let $\mathcal{A}^0$ denote the collection of $\mathbb{F}^0$-progressively measurable process $\alpha : [0, T] \times \Omega^0 \longrightarrow \mathrm{A}$, bounded.

A class of McKean-Vlasov optimal control problems

☐ Let $\Omega^0 = \Omega^1 := \mathcal{C}([0, T], \mathbb{R}^d)$ with canonical process X^0 and W, canonical filtrations $\mathbb{F}^0$ and $\mathbb{F}^1$, and Wiener measures $\mathbb{P}^0_0$ and $\mathbb{P}^1_0$.

☐ Let $\mathcal{A}^0$ denote the collection of $\mathbb{F}^0$-progressively measurable process $\alpha : [0, T] \times \Omega^0 \longrightarrow \mathrm{A}$, bounded.

☐ Define

$$\begin{aligned}\mathcal{P}^0_W(t, \mathrm{x}^0) := \Big\{\mathbb{P}^0 \in \mathcal{P}(\Omega^0) : & X^0 = \mathrm{x}^0_t + \int_t^{\cdot} \alpha_r^{\mathbb{P}^0} dr + \int_t^{\cdot} dW_r^{\mathbb{P}^0}, \mathbb{P}^0\text{-a.s.} \\ & \mathbb{P}^0[X^0_{t\wedge\cdot} = \mathrm{x}^0_{t\wedge\cdot}] = 1, \text{ where } \alpha^{\mathbb{P}^0} \in \mathcal{A}^0 \\ & \text{and } W^{\mathbb{P}^0} \text{ is a } (\mathbb{P}^0, \mathbb{F}^0)\text{-Brownian motion}\Big\}.\end{aligned}$$

A class of McKean-Vlasov optimal control problems

□ Let $\Omega^0 = \Omega^1 := \mathcal{C}([0,T],\mathbb{R}^d)$ with canonical process X^0 and W, canonical filtrations $\mathbb{F}^0$ and $\mathbb{F}^1$, and Wiener measures $\mathbb{P}_0^0$ and $\mathbb{P}_0^1$.

□ Let $\mathcal{A}^0$ denote the collection of $\mathbb{F}^0$-progressively measurable process $\alpha : [0,T] \times \Omega^0 \longrightarrow \mathrm{A}$, bounded.

□ Define

$$\mathcal{P}_W^0(t,\mathrm{x}^0) := \Big\{\mathbb{P}^0 \in \mathcal{P}(\Omega^0) : X^0 = \mathrm{x}_t^0 + \int_t^\cdot \alpha_r^{\mathbb{P}^0} dr + \int_t^\cdot dW_r^{\mathbb{P}^0}, \mathbb{P}^0\text{-a.s.}\ \mathbb{P}^0[X^0_{t\wedge\cdot} = \mathrm{x}^0_{t\wedge\cdot}] = 1,\ \text{where } \alpha^{\mathbb{P}^0} \in \mathcal{A}^0 \text{ and } W^{\mathbb{P}^0} \text{ is a } (\mathbb{P}^0,\mathbb{F}^0)\text{-Brownian motion}\Big\}.$$

and

$$\mathcal{P}_W(t,\mathrm{x}^0) := \big\{\mathbb{P} = \mathbb{P}^0 \times \mathbb{P}_0^1 : \mathbb{P}^0 \in \mathcal{P}_W^0(t,\mathrm{x}^0)\big\}.$$

□ For $t \in [0, T]$, $m \in \mathcal{P}_2(\mathbb{R}^d)$ and $\mathbb{P} \in \mathcal{P}_W(t, \mathrm{x}^0)$, we consider the McKean-Vlasov SDE :

$$X_s^{t,\mathbb{P}} = \xi + \int_t^s \sigma_0(r, X_r^{t,\mathbb{P}}, \rho_r^{t,m,\mathbb{P}}) dX_r^0 + \int_t^s \sigma(r, X_r^{t,\mathbb{P}}, \rho_r^{t,m,\mathbb{P}}) dW_r, \; m \times \mathbb{P}\text{-a.s.}$$

with $\rho_r^{t,m,\mathbb{P}} := \mathcal{L}^{m\times\mathbb{P}}(X_r^{t,\mathbb{P}} | \mathcal{F}_r^{X^0})$.

□ For $t \in [0, T]$, $m \in \mathcal{P}_2(\mathbb{R}^d)$ and $\mathbb{P} \in \mathcal{P}_W(t, \mathrm{x}^0)$, we consider the McKean-Vlasov SDE :

$$X_s^{t,\mathbb{P}} = \xi + \int_t^s \sigma_0(r, X_r^{t,\mathbb{P}}, \rho_r^{t,m,\mathbb{P}}) dX_r^0 + \int_t^s \sigma(r, X_r^{t,\mathbb{P}}, \rho_r^{t,m,\mathbb{P}}) dW_r, \ m \times \mathbb{P}\text{-a.s.}$$

with $\rho_r^{t,m,\mathbb{P}} := \mathcal{L}^{m\times\mathbb{P}}(X_r^{t,\mathbb{P}} | \mathcal{F}_r^{X^0})$.

□ Controlled laws of the canonical process (X^0, W, X, ρ) are in

$$\overline{\mathcal{P}}_W(t, m) := \Big\{ (m \times \mathbb{P}) \circ \big(X^0, W, X^{t,\mathbb{P}}, \rho^{t,m,\mathbb{P}} \big)^{-1} \ : \mathbb{P} \in \mathcal{P}_W(t, \mathrm{x}^0), \ \mathrm{x}^0 \in \Omega^0 \Big\}.$$

□ The value function of the McKean-Vlasov control problem is :

$$V(t,m) := \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,m)} J\left(t, \overline{\mathbb{P}}\right), \text{ with } J(t,\overline{\mathbb{P}}) := \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^T L\big(s, \rho_s, \alpha_s^{\overline{\mathbb{P}}}\big) ds + g(\rho_T)\Big].$$

□ The value function of the McKean-Vlasov control problem is :

$$V(t,m) \ := \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(t,m)} J\left(t, \overline{\mathbb{P}}\right), \ \text{with } J(t,\overline{\mathbb{P}}) := \mathbb{E}^{\overline{\mathbb{P}}}\Big[\int_t^T L\big(s,\rho_s,\alpha_s^{\overline{\mathbb{P}}}\big)ds + g(\rho_T)\Big].$$

□ Define

$$\mathcal{K} := \Big\{\phi : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d \ : \ \phi \text{ is bounded and Borel measurable}\Big\},$$

and $H : [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \times \mathcal{K} \longrightarrow \mathbb{R}$, the Hamiltonian, defined by

$$H(t,m,p) := \max_{a \in \mathrm{A}} h(t,m,p,a),$$

$$h(t,m,p,a) := L(t,m,a) + a \int (\sigma_0 p)(t,m,y) m(dy).$$

Dual problems

- From now on, we fix the initial law to be m_0.

Dual problems

□ From now on, we fix the initial law to be m_0.

□ We introduce the dual problems :
(i) D_1 is the infimum over v_1 s.t.

$$v_1 + \int_0^T \int (\sigma_0\phi)(t,\rho_t,y)\rho_t(dy)dX_t^0 \geq \; g(\rho_T) + \int_0^T h(t,\rho_t,\phi,\alpha_t^{\overline{\mathbb{P}}})dt, \; \overline{\mathbb{P}}\text{ -a.s.}$$

for all $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)$, for some $\phi \in \mathcal{K}$.

Dual problems

☐ From now on, we fix the initial law to be m_0.

☐ We introduce the dual problems :
(i) D_1 is the infimum over v_1 s.t.

$$v_1 + \int_0^T \int (\sigma_0 \phi)(t, \rho_t, y)\rho_t(dy) dX_t^0 \geq \; g(\rho_T) + \int_0^T h(t, \rho_t, \phi, \alpha_t^{\overline{\mathbb{P}}}) dt, \; \overline{\mathbb{P}} \text{ -a.s.}$$

for all $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$, for some $\phi \in \mathcal{K}$.

(ii) D_2 is the infimum over v_2 s.t.

$$v_2 + \int_0^T \int (\sigma_0 \phi)(t, \rho_t, y)\rho_t(dy) dX_t^0 \geq \; g(\rho_T) + \int_0^T H(t, \rho_t, \phi) dt, \; \overline{\mathbb{P}}_0\text{-a.s.}$$

for some $\phi \in \mathcal{K}$, where $\overline{\mathbb{P}}_0$ is a probability measure under which the canonical process X^0 is a Brownian motion.

Dual problems

□ From now on, we fix the initial law to be m_0.

□ We introduce the dual problems :
(i) D_1 is the infimum over v_1 s.t.

$$v_1 + \int_0^T \int (\sigma_0 \phi)(t, \rho_t, y)\rho_t(dy)dX_t^0 \geq \; g(\rho_T) + \int_0^T h(t, \rho_t, \phi, \alpha_t^{\overline{\mathbb{P}}})dt, \; \overline{\mathbb{P}} \text{ -a.s.}$$

for all $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$, for some $\phi \in \mathcal{K}$.

(ii) D_2 is the infimum over v_2 s.t.

$$v_2 + \int_0^T \int (\sigma_0 \phi)(t, \rho_t, y)\rho_t(dy)dX_t^0 \geq \; g(\rho_T) + \int_0^T H(t, \rho_t, \phi)dt, \; \overline{\mathbb{P}}_0\text{-a.s.}$$

for some $\phi \in \mathcal{K}$, where $\overline{\mathbb{P}}_0$ is a probability measure under which the canonical process X^0 is a Brownian motion.

□ It is similar in spirit to B. and Dang [4] : stochastic target formulation of the optimal control problem.

- We have $D_2 \geq D_1$ by definition.

□ We have $D_2 \geq D_1$ by definition.

□ Since

$$X^0 = x_0^0 + \int_0^{\cdot} \alpha_r^{\mathbb{P}^0} dr + \int_0^{\cdot} dW_r^{\mathbb{P}^0}$$

the inequality

$$v_1 + \int_0^T \int (\sigma_0 \phi)(t, \rho_t, y) \rho_t(dy) dX_t^0 \geq \; g(\rho_T) + \int_0^T h(t, \rho_t, \phi, \alpha_t^{\overline{\mathbb{P}}}) dt$$

implies

$$v_1 \geq \mathbb{E}^{\overline{\mathbb{P}}}[g(\rho_T) + \int_0^T L(t, \rho_t, \alpha_t^{\overline{\mathbb{P}}})], \quad \text{for } \overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$$

□ We have $D_2 \geq D_1$ by definition.

□ Since

$$X^0 = x_0^0 + \int_0^{\cdot} \alpha_r^{\mathbb{P}^0} dr + \int_0^{\cdot} dW_r^{\mathbb{P}^0}$$

the inequality

$$v_1 + \int_0^T \int (\sigma_0 \phi)(t, \rho_t, y) \rho_t(dy) dX_t^0 \geq \; g(\rho_T) + \int_0^T h(t, \rho_t, \phi, \alpha_t^{\overline{\mathbb{P}}}) dt$$

implies

$$v_1 \geq \mathbb{E}^{\overline{\mathbb{P}}}[g(\rho_T) + \int_0^T L(t, \rho_t, \alpha_t^{\overline{\mathbb{P}}})], \quad \text{for } \overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$$

and therefore

$$D_2 \geq D_1 \geq V(t, m_0).$$

Duality and verification

□ **Theorem :** Assume that $V \in C^{0,1}([0,T], \mathcal{P}_2(\mathbb{R}^d))$ and that $D_m V$ is uniformly bounded (or locally as above). Then,

$$V(0, m_0) = D_1 = D_2.$$

If in addition $\exists$ a Borel measurable function $\hat{a} : [0,T) \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathrm{A}$ s.t.

$$H(\cdot, m, D_m V) = h(\cdot, m, D_m V, \hat{a}(\cdot, m)),$$

for all $m \in \mathcal{P}_2(\mathbb{R}^d)$. Then, $\exists\ \widehat{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$ s.t. $\alpha^{\widehat{\mathbb{P}}} = \hat{a}(\cdot, \rho_\cdot)$, $d\widehat{\mathbb{P}} \times dt$ a.e. and $\widehat{\mathbb{P}}$ is optimal for $V(0, m_0)$.

Duality and verification

□ **Theorem :** Assume that $V \in C^{0,1}([0,T], \mathcal{P}_2(\mathbb{R}^d))$ and that $D_m V$ is uniformly bounded (or locally as above). Then,

$$V(0, m_0) = D_1 = D_2.$$

If in addition $\exists$ a Borel measurable function $\hat{a} : [0,T) \times \mathcal{P}_2(\mathbb{R}^d) \longrightarrow \mathrm{A}$ s.t.

$$H(\cdot, m, D_m V) = h(\cdot, m, D_m V, \hat{a}(\cdot, m)),$$

for all $m \in \mathcal{P}_2(\mathbb{R}^d)$. Then, $\exists\ \widehat{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$ s.t. $\alpha^{\widehat{\mathbb{P}}} = \hat{a}(\cdot, \rho_\cdot)$, $d\widehat{\mathbb{P}} \times dt$ a.e. and $\widehat{\mathbb{P}}$ is optimal for $V(0, m_0)$.

□ **Remark :** If A is compact, existence of $\hat{a}$ holds if L is upper-semicontinuous.

Proof of $D_2 \leq V(t, m_0)$

(a) We know that $S^{\overline{\mathbb{P}}} := V(\cdot, \rho_\cdot) + \int_0^\cdot L(s, \rho_s, \alpha_s^{\overline{\mathbb{P}}})ds$ is a super-martingale under any $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$.

Proof of $D_2 \leq V(t, m_0)$

(a) We know that $S^{\overline{\mathbb{P}}} := V(\cdot, \rho_\cdot) + \int_0^\cdot L(s, \rho_s, \alpha_s^{\overline{\mathbb{P}}})ds$ is a super-martingale under any $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$. Combined with our C^1-Itô's formula, we obtain :

$$S^{\overline{\mathbb{P}}} = V(0, m_0) + \int_0^\cdot \int (\sigma_0 D_m V)(t, \rho_t, y)\rho_t(dy)dW_t^{\overline{\mathbb{P}}} - A^{\overline{\mathbb{P}}}$$

in which $A^{\overline{\mathbb{P}}}$ is non-decreasing.

Proof of $D_2 \leq V(t, m_0)$

(a) We know that $S^{\overline{\mathbb{P}}} := V(\cdot, \rho_\cdot) + \int_0^\cdot L(s, \rho_s, \alpha_s^{\overline{\mathbb{P}}})ds$ is a super-martingale under any $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$. Combined with our C^1-Itô's formula, we obtain :

$$S^{\overline{\mathbb{P}}} = V(0, m_0) + \int_0^\cdot \int (\sigma_0 D_m V)(t, \rho_t, y)\rho_t(dy)dW_t^{\overline{\mathbb{P}}} - A^{\overline{\mathbb{P}}}$$

in which $A^{\overline{\mathbb{P}}}$ is non-decreasing.

(b) Since $V(T, \rho_T) = g(\rho_T)$ and $A^{\overline{\mathbb{P}}}$ is non-decreasing,

$$\begin{aligned} & V(0, m_0) + \int_0^T \int (\sigma_0 D_m V)(t, \rho_t, y)\rho_t(dy)dX_t^0 \\ \geq \ & g(\rho_T) + \int_0^T h(t, \rho_t, D_m V, \alpha_t^{\overline{\mathbb{P}}})dt. \end{aligned}$$

Hence, $V(0, m_0) \geq D_2$ by arbitrariness of $\overline{\mathbb{P}}$.

Proof of the verification argument

Set

$$\ell(t,m) := \int (\sigma_0 D_m V)(t,m,y)m(dy)$$

and note that $(A^{\overline{\mathbb{P}}})_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,\boldsymbol{m_0})}$ in the decomposition

$$S^{\overline{\mathbb{P}}} = V(0,m_0) + \int_0^{\cdot} \ell(t,\rho_t,y)dW_t^{\overline{\mathbb{P}}} - A^{\overline{\mathbb{P}}}$$

satisfies

$$\inf_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_W(0,\boldsymbol{m_0})} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\overline{\mathbb{P}}}] = 0.$$

by classical arguments.

Moreover,

$$\begin{aligned} V(0,m_0) &= g(\rho_T) + \int_0^T h(t,\rho_t,\alpha_t^{\overline{\mathbb{P}}})dt + A_T^{\overline{\mathbb{P}}} - \int_0^T \ell(t,\rho_t)dX_t^0 \\ &= g(\rho_T) + \int_0^T h(t,\rho_t,\alpha_t^{\widehat{\mathbb{P}}})dt + A_T^{\widehat{\mathbb{P}}} - \int_0^T \ell(t,\rho_t)dX_t^0 \\ &\geq g(\rho_T) + \int_0^T h(t,\rho_t,\alpha_t^{\overline{\mathbb{P}}})dt + A_T^{\widehat{\mathbb{P}}} - \int_0^T \ell(t,\rho_t)dX_t^0 \end{aligned}$$

so that $0 \leq A_T^{\widehat{\mathbb{P}}} \leq A_T^{\overline{\mathbb{P}}}$ a.s. for $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)$, and

$$0 = \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\overline{\mathbb{P}}}] \geq \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0,m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\widehat{\mathbb{P}}}] = 0.$$

Moreover,

$$\begin{aligned} V(0, m_0) &= g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_t^{\overline{\mathbb{P}}})dt + A_T^{\overline{\mathbb{P}}} - \int_0^T \ell(t, \rho_t)dX_t^0 \\ &= g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_t^{\widehat{\mathbb{P}}})dt + A_T^{\widehat{\mathbb{P}}} - \int_0^T \ell(t, \rho_t)dX_t^0 \\ &\geq g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_t^{\overline{\mathbb{P}}})dt + A_T^{\widehat{\mathbb{P}}} - \int_0^T \ell(t, \rho_t)dX_t^0 \end{aligned}$$

so that $0 \leq A_T^{\widehat{\mathbb{P}}} \leq A_T^{\overline{\mathbb{P}}}$ a.s. for $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)$, and

$$0 = \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\overline{\mathbb{P}}}] \geq \inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\widehat{\mathbb{P}}}] = 0.$$

We indeed have (using the reverse Hölder's inequality)

$$A_T^{\widehat{\mathbb{P}}} = 0,\ \overline{\mathbb{P}} - \text{a.s.}\ \forall\, \overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0).$$

Then,

$$\begin{aligned} V(0, m_0) &= \mathbb{E}^{\widehat{\mathbb{P}}}\left[g(\rho_T) + \int_0^T h(t, \rho_t, \alpha_t^{\widehat{\mathbb{P}}}) dt - \int_0^T \ell(t, \rho_t) \alpha_t^{\widehat{\mathbb{P}}} dt \right] \\ &= \mathbb{E}^{\widehat{\mathbb{P}}}\left[g(\rho_T) + \int_0^T L(t, \rho_t, \alpha_t^{\widehat{\mathbb{P}}}) dt \right]. \end{aligned}$$

Proof of the verification argument

Set

$$\ell(t, m) := \int (\sigma_0 D_m V)(t, m, y) m(dy)$$

and note that $(A^{\overline{\mathbb{P}}})_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)}$ in the decomposition

$$S^{\overline{\mathbb{P}}} = V(0, m_0) + \int_0^{\cdot} \ell(t, \rho_t, y) dW_t^{\overline{\mathbb{P}}} - A^{\overline{\mathbb{P}}}$$

satisfies

$$\inf_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_W(0, m_0)} \mathbb{E}^{\overline{\mathbb{P}}}[A_T^{\overline{\mathbb{P}}}] = 0.$$

by classical arguments.

Example

- Assume that :
  - $\sigma = \sigma_0 \equiv 1$,
  - A is a convex,
  - $L(t, m, a) = \bar{L}(a)$ is strictly concave.
  - $g(m) = \bar{g}\big(\int \phi(y) m(dy)\big)$ with $\bar{g} : \mathbb{R}^d \to \mathbb{R}$ convex and C_b^1, and $\phi : \mathbb{R} \longrightarrow \mathbb{R}^d$ that is C_b^1.

Example

- Assume that :
  - $\sigma = \sigma_0 \equiv 1$,
  - A is a convex,
  - $L(t, m, a) = \bar{L}(a)$ is strictly concave.
  - $g(m) = \bar{g}\big(\int \phi(y)m(dy)\big)$ with $\bar{g} : \mathbb{R}^d \to \mathbb{R}$ convex and C_b^1, and $\phi : \mathbb{R} \longrightarrow \mathbb{R}^d$ that is C_b^1.

Then, $V \in C^{0,1}([0, T], \mathcal{P}_2(\mathbb{R}^d))$.

Example

- Assume that :
  - $\sigma = \sigma_0 \equiv 1$,
  - A is a convex,
  - $L(t, m, a) = \bar{L}(a)$ is strictly concave.
  - $g(m) = \bar{g}\big(\int \phi(y)m(dy)\big)$ with $\bar{g} : \mathbb{R}^d \to \mathbb{R}$ convex and C_b^1, and $\phi : \mathbb{R} \longrightarrow \mathbb{R}^d$ that is C_b^1.

Then, $V \in C^{0,1}([0, T], \mathcal{P}_2(\mathbb{R}^d))$.

- An optimal control $\widehat{\mathbb{P}}$ exists (is unique) and we have

$$D_m V(0, m_0, x) = \mathbb{E}^{\widehat{\mathbb{P}}}\Big[\bar{g}'\Big(\int_{\mathbb{R}^d} \bar{\phi}(y + X_T^0) m_0(dy)\Big) \nabla \bar{\phi}(x + X_T^0)\Big]$$

where

$$\bar{\phi}(y) := \mathbb{E}^{\mathbb{P}_0^1}[\phi(y + W_T)].$$

Thank you !

Elena Bandini and Francesco Russo.
Weak Dirichlet processes with jumps.
Stochastic Processes and their Applications, 127-12 : 4139-4189, 2017.

Bruno Bouchard, Grégoire Loeper and Xiaolu Tan.
A $C^{0,1}$-functional Itô's formula and its applications in mathematical finance.
Stochastic Processes and their Applications, 148, 299-323, 2022.

Bruno Bouchard, Grégoire Loeper and Xiaolu Tan.
Approximate viscosity solutions of path-dependent PDEs and Dupire's vertical differentiability.
The Annals of Applied Probability, 33.6B : 5781-5809, 2023.

Bruno Bouchard and Ngoc Minh Dang.
Optimal control versus stochastic target problems : an equivalence result.
Systems & control letters, 61(2) : 343-346, 2012.

Bruno Bouchard and Xiaolu Tan.
Understanding the dual formulation for the hedging of path-dependent options with price impact.
The Annals of Applied Probability, 32(3), 1705-1733, 2019.

Bruno Bouchard and Xiaolu Tan.
A quasi-sure optional decomposition and super-hedging result on the Skorokhod space.
Finance and Stochastics, 25(3), 505-528, 2021.

Bruno Bouchard and Maximilien Vallet.
Itô-Dupire's formula for $C^{0,1}$-functionals of cádlàg weak Dirichlet processes.
arXiv preprint arXiv :2110.03406, 2021.

Jean-François Chassagneux, Dan Crisan, and François Delarue.
A probabilistic approach to classical solutions of the master equation for large population equilibria.
American Mathematical Society, 280(1379), 2022.

Rama Cont and David-Antoine Fournié.
Functional itô calculus and stochastic integral representation of martingales.
The Annals of Probability, 41(1) :109–133, 2013.

Bruno Dupire.
Functional Itô calculus.
Portfolio Research Paper, 04, 2009.

Fausto Gozzi and Francesco Russo.
Weak dirichlet processes with a stochastic control perspective.
Stochastic Processes and their Applications, 116(11) :1563–1583, 2006.

Xin Guo, Huyen Pham and Xiaoli Wei.
Itô's formula for flows of measures on semimartingales.
Stochastic Processes and their Applications, 159 :350-90, 2023.

Yuri F. Saporito.
The functional Meyer–Tanaka formula.
Stochastics and Dynamics, 18(04) :1850030, 2018.

Mehdi Talbi, Nizar Touzi and Jianfeng Zhang.
Dynamic programming equation for the mean field optimal stopping problem.
SIAM Journal on Control and Optimization, 61(4), 2140-2164, 2023.